Covering properties of derived models

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Outline

Theorem (Jensen)

- If κ is a singular cardinal and $(\kappa^+)^L < \kappa^+$, then $0^\sharp$ exists.
- If $\kappa \geq \aleph_2$ is regular and $\mathrm{cf}\left((\kappa^+)^L\right) < \kappa$, then $0^\sharp$ exists.

Theorem (Kunen)

If κ is weakly compact and $(\kappa^+)^L < \kappa^+$, then $0^\sharp$ exists.

Remark

In the regular and weakly compact cases we will get parallel results with derived models in place of L and strong axioms of determinacy in place of $0^\sharp$.

Theorem (Woodin)

The following theories are equiconsistent:

1. ZFC + "there are infinitely many Woodin cardinals"
2. ZF + AD.

More specifically:

Theorem (Woodin)

Let κ be a limit of Woodin cardinals, let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$, and define $\mathbb{R}^*_G = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[G \restriction \alpha]}$. Then $L(\mathbb{R}^*_G) \models \mathrm{AD}$.

Remark

- The existence of infinitely many Woodin cardinals does not imply AD in $L(\mathbb{R})$ of V itself.
- For example, in the least mouse with infinitely many Woodin cardinals, $\mathrm{AD}^{L(\mathbb{R})}$ fails.

Remark

- We can consider models of AD extending $L(\mathbb{R}^*_G)$, such as derived models.
- Larger derived models can satisfy stronger determinacy axioms, for example $\mathrm{AD}_{\mathbb{R}}$, which cannot hold in $L(\mathbb{R}^*_G)$.

Definition

$AD_{\mathbb{R}}$ (a strengthening of AD) says that two-player games on $\mathbb{R}$ (instead of $\mathbb{N}$) of length ω are determined.

Remark

- $AD_{\mathbb{R}}$ has higher consistency strength than AD.
- What we are really interested in is the axiom

  $$\text{AD} + \text{“\textit{every set of reals is Suslin},”}$$

  which is equivalent to $AD_{\mathbb{R}}$ modulo $\text{ZF} + \text{DC}$. (Woodin)
- A set is Suslin if it is the projection of a tree on $\omega \times \text{Ord}$ (Just like analytic sets are projections of trees on $\omega \times \omega$.)

Let κ be a limit of Woodin cardinals and let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$.

Definition

The derived model of V at κ by G, denoted by $D(V, \kappa, G)$, is characterized by the following properties.

1. $L(\mathbb{R}^*_G) \subset D(V, \kappa, G) \subset V(\mathbb{R}^*_G)$
2. $D(V, \kappa, G) \models \mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$
3. It is $\subset$-maximal subject to 1 and 2 (exists by Woodin.)

Remark

AD^+ is a strengthening of AD that holds in $L(\mathbb{R}^*_G)$ (and in all known models of AD, so let's ignore the "+".)

Theorem (W.)

Let κ be an inaccessible limit of Woodin cardinals.
Let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$. Then

$$\mathrm{cf}(\Theta^{D(V,\kappa,G)}) \geq \kappa.$$

(Θ is the least ordinal that is not a surjective image of $\mathbb{R}$.)

Remark

- $\Theta^{D(V,\kappa,G)}$ is analogous to $(\kappa^+)^L$ in weak covering for L.
- $\Theta^{D(V,\kappa,G)}$ does not depend on G.
- If κ is inaccessible, then $\mathbb{R}^{D(V,\kappa,G)} = \mathbb{R}^*_G = \mathbb{R}^{V[G]}$.

To restate using an equivalent version of the conclusion $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) \geq \kappa$:

Corollary

Let κ be an inaccessible limit of Woodin cardinals.
Let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$. Then:

> *In $V[G]$, every countable sequence of sets of reals in $D(V, \kappa, G)$ is in $D(V, \kappa, G)$.*

Remark

In other words, weak covering for $D(V, \kappa, G)$ is not so weak.

If $D(V, \kappa, G) \models \mathrm{AD}_{\mathbb{R}}$ (this case was already known):

- The sets of reals of $D(V, \kappa, G)$ are exactly the Suslin co-Suslin sets of reals in $V(\mathbb{R}^*_G)$. (Woodin)
- (Think of Suslin co-Suslin as a generalization of Borel.)
- Every countable sequence of Suslin co-Suslin sets is coded by a Suslin co-Suslin set, using DC in $V(\mathbb{R}^*_G)$.

If $D(V, \kappa, G) \models \neg\mathrm{AD}_{\mathbb{R}}$:

- Not all sets of reals in $D(V, \kappa, G)$ are Suslin in $V(\mathbb{R}^*_G)$.
- We show that *if covering fails*, then they are.
- The work lies in constructing Suslin representations from failures of covering. (We omit the details in this talk.)

If $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) \geq \kappa$, then either

1. $\Theta^{D(V,\kappa,G)} = \kappa^+$, or
2. $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) = \kappa$.

Remark

- If $\mathrm{AD}_{\mathbb{R}}$ holds in $D(V, \kappa, G)$, then Case 2 holds.
- If $\mathrm{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, both cases are possible.
- Case 1 should hold in the least mouse with an inaccessible limit of Woodin cardinals (I think.)
- Can get Case 2 from Case 1 by forcing with $\mathrm{Col}(\kappa, \kappa^+)$.

Theorem (W.)

Let κ be a weakly compact limit of Woodin cardinals.
Let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$.
If $\mathrm{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, then

$$\Theta^{D(V,\kappa,G)} = \kappa^{+}.$$

Remark

The hypothesis is consistent:

- $\mathrm{AD}_{\mathbb{R}}$ has higher consistency strength than the existence of a weakly compact limit of Woodin cardinals.
- Also, the hypothesis holds in the least mouse with a weakly compact limit of Woodin cardinals.

We can force a failure of covering for the derived model. This does not typically preserve weak compactness. But:

Corollary

If κ is a $\mathrm{Col}(\kappa, \kappa^+)$-indestructibly weakly compact limit of Woodin cardinals and G is a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$, then $D(V, \kappa, G) \models \mathrm{AD}_{\mathbb{R}}$.

Remark

A better relative consistency result comes from Jensen–Schimmerling–Schindler–Steel, Stacking mice.

What if the limit κ of Woodin cardinals is not inaccessible (and is therefore singular)?

Question

Let κ be a singular limit of Woodin cardinals. If $\mathrm{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, then must $\Theta^{D(V,\kappa,G)} = \kappa^+$?

Remark

Failures of covering for derived models at singular cardinals can be obtained from forcing axioms.