Covering properties of derived models

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Outline

Background

- Weak covering for L
- Determinacy in $L(\mathbb{R}^*_G)$
- The derived model $D(V, \kappa, G)$
- Weak covering for derived models?

Results

- Inaccessible limits of Woodin cardinals
- Weakly compact limits of Woodin cardinals

Questions

Let L denote Gödel's constructible universe.

Weak covering

If $0^\sharp$ does not exist, then L is "close to V" in terms of cardinals and cofinalities:

1. If κ is a singular cardinal, then $(\kappa^+)^L = \kappa^+$. (Jensen)
2. If $\kappa \geq \aleph_2$ is regular, then $\mathrm{cf}\big((\kappa^+)^L\big) \geq \kappa$. (Jensen)
3. If κ is weakly compact, then $(\kappa^+)^L = \kappa^+$. (Kunen)

In cases (2) and (3), we can get parallel results with a model of determinacy (a derived model at κ) in place of L, and a strong axiom of determinacy ($\mathrm{AD}_\mathbb{R}$) in place of $0^\sharp$.

Definition

The *Axiom of Determinacy*, AD, says that for every ω-length two-player game of perfect information on the integers, one of the two players has a winning strategy.

Theorem (Woodin)

The following theories are equiconsistent:

1. ZFC + "there are infinitely many Woodin cardinals"
2. ZF + AD.

We will need some details of the forward direction.

Theorem (Woodin)

Let κ be a limit of Woodin cardinals, let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$, and define

$$\mathbb{R}^*_G = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[G\restriction\alpha]}.$$

Then $L(\mathbb{R}^*_G) \models \mathrm{AD}$.

Remark

From a slightly stronger hypothesis, Woodin obtained AD in the $L(\mathbb{R})$ of V itself.

For the rest of the talk:

- Fix a limit κ of Woodin cardinals
- Fix a V-generic filter $G \subset \mathrm{Col}(\omega, {<}\kappa)$
- Define $\mathbb{R}^*_G = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[G\restriction\alpha]}$.

Remark

If κ is regular (hence inaccessible) then

- $\kappa = \omega_1^{V[G]}$, and
- $\mathbb{R}^*_G = \mathbb{R}^{V[G]}$.

Let's look for models of AD larger than $L(\mathbb{R}^*_G)$.
First we consider a *symmetric model*:

Definition

$$V(\mathbb{R}^*_G) = \mathrm{HOD}^{V[G]}_{V \cup \mathbb{R}^*_G \cup \{\mathbb{R}^*_G\}}.$$

Remark

Whether or not κ is regular, we have

- $\kappa = \omega_1^{V(\mathbb{R}^*_G)}$.
- $\mathbb{R}^*_G = \mathbb{R}^{V(\mathbb{R}^*_G)}$.

Remark

AC fails in $V(\mathbb{R}^*_G)$: we cannot choose a surjection $\omega \to \alpha$ for every $\alpha < \kappa$.

Remark

If κ is regular (hence inaccessible) in V, then in $V(\mathbb{R}^*_G)$ every set of reals is Lebesgue measurable and DC holds. (Solovay)

Remark

AD fails in $V(\mathbb{R}^*_G)$.

Theorem (Woodin)

In $V(\mathbb{R}^*_G)$, there is a largest (under $\subset$) pointclass Γ such that

$$L(\Gamma, \mathbb{R}^*_G) \models \mathsf{AD}^+.$$

(AD^+ is a strengthening of AD that holds in $L(\mathbb{R}^*_G)$).

Definition

The derived model of V at κ by G is

$$D(V, \kappa, G) = L(\Gamma, \mathbb{R}^*_G)$$

for the largest pointclass Γ as above.

Remark

- The derived model $D(V, \kappa, G)$ can satisfy stronger determinacy axioms than $L(\mathbb{R}^*_G)$, such as $\mathrm{AD}_\mathbb{R}$.
- (Just as higher core models can satisfy stronger large cardinal axioms than L, such as the existence of $0^\sharp$.)

Definition

$\mathrm{AD}_\mathbb{R}$ is determinacy for games on $\mathbb{R}$ (instead of $\mathbb{N}$.)

Recall that if $0^\sharp$ does not exist, then L is "close to V."

Question

If $\mathsf{AD}_\mathbb{R}$ does not hold in the derived model $D(V, \kappa, G)$, then is $D(V, \kappa, G)$ "close to $V(\mathbb{R}^*_G)$"?

Remark

- The relevant cardinalities and cofinalities are in the vicinity of κ and κ^+.
- We could say "close to V" instead of "close to $V(\mathbb{R}^*_G)$" because the correspondence between cardinals and cofinalities of V and $V(\mathbb{R}^*_G)$ is straightforward.

A caveat in formulating "close to V" for derived models:

- In $D(V,\kappa,G)$ there is a surjection $\mathbb{R}^*_G \to \omega_2$ (by AD, using the Moschovakis coding lemma.)
- In $V(\mathbb{R}^*_G)$ there is no surjection $\mathbb{R}^*_G \to \omega_2$

Because κ is ω_1 in $D(V,\kappa,G)$ and $V(\mathbb{R}^*_G)$, it follows that:

$$(\kappa^+)^{D(V,\kappa,G)} < \kappa^+.$$

Remark

This also shows that $V(\mathbb{R}^*_G)$ does not satisfy AD.

So it seems $(\kappa^+)^{D(V,\kappa,G)}$ is not the relevant thing to look at.

Definition

Θ is the least ordinal that is not a surjective image of $\mathbb{R}$ (*i.e.* the successor of $\mathbb{R}$ in the sense of surjections.)

Remark

- If AC holds, then $\Theta = \mathfrak{c}^+$.
- If AD holds, then Θ is inaccessible by the coding lemma (in particular $\Theta > \omega_2$).

Look at $\Theta^{D(V,\kappa,G)}$ instead of $(\kappa^+)^{D(V,\kappa,G)}$.

Remark

- $\Theta^{D(V,\kappa,G)} \leq \kappa^+$.
- If $AD_\mathbb{R}$ holds in $D(V, \kappa, G)$ then $\Theta^{D(V,\kappa,G)} < \kappa^+$. (Using the fact $\mathcal{P}(\mathbb{R})^{D(V,\kappa,G)} = \text{Hom}^*_G$.)
- If $AD_\mathbb{R}$ fails in $D(V, \kappa, G)$ then in general we may have $\Theta^{D(V,\kappa,G)} < \kappa^+$ or $\Theta^{D(V,\kappa,G)} = \kappa^+$; in specific cases we will be able to say more.

Analogy:

$$\Theta^{D(V,\kappa,G)} \leftrightsquigarrow (\kappa^+)^L$$

$$AD_\mathbb{R} \text{ fails} \leftrightsquigarrow 0^\sharp \text{ does not exist}$$

Theorem (W.)

Let κ be an inaccessible limit of Woodin cardinals.
Let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$.
If $\mathrm{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, then $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) \geq \kappa$.

Remark

An equivalent conclusion is that $D(V, \kappa, G)$ is closed under ω-sequences of sets of reals in $V(\mathbb{R}^*_G)$.

Remark

If $\mathrm{AD}_{\mathbb{R}}$ holds in $D(V, \kappa, G)$ then $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) = \kappa$,
but for trivial reasons.

Proof sketch:

- We want to show that $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) \geq \kappa$.
- If not, assume WLOG that $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) = \omega$ in V.
- Take hull $X \prec H_{\kappa^+}$ with $X \cap \kappa = \bar{\kappa} < \kappa$ and $X^\omega \subset X$.
- Consider $\pi : M \cong X$, the uncollapse map.
- Extend to $\hat{\pi} : M[\bar{G}] \to H_{\kappa^+}[G]$ where $\bar{G} = G \upharpoonright \bar{\kappa}$.
- Set $\bar{D} = D(M, \bar{\kappa}, \bar{G})$ and $D = D(H_{\kappa^+}, \kappa, G)$.
- $\hat{\pi}[\bar{D}]$ is Wadge-cofinal in D (cofinality is small.)

Proof sketch (continued):

- In $D(V, \kappa, G)$, if $\mathrm{AD}_{\mathbb{R}}$ fails, then there is a Suslin set of reals $p[T]$ whose complement is not Suslin.
- Assume WLOG that $T \in V$.
- Using that $\hat{\pi}[\bar{D}]$ is Wadge-cofinal in D, show the hull is T-full: every subset of $\mathbb{R}^*_{\bar{G}}$ in $L(T, \mathbb{R}^*_{\bar{G}})$ is in $\bar{D}$.
- Use T-fullness and $\hat{\pi}$ to get a tree T' in $V(\mathbb{R}^*_G)$ such that T and T' project to complements, a contradiction.

So if κ is an inaccessible limit of Woodin cardinals and $\mathrm{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$ then either

1. $\Theta^{D(V,\kappa,G)} = \kappa^+$, or
2. $\mathrm{cf}(\Theta^{D(V,\kappa,G)}) = \kappa$.

Remark

Both cases are possible.

- Case 1 holds if κ is weakly compact (as we will see.)
- Can get case 2 from case 1 by forcing with $\mathrm{Col}(\kappa, \kappa^+)$.

Theorem (W.)

Let κ be a weakly compact limit of Woodin cardinals.
Let G be a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$.
If $\mathrm{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, then $\Theta^{D(V,\kappa,G)} = \kappa^+$.

Remark

The hypothesis is consistent: $\mathrm{AD}_{\mathbb{R}}$ has higher consistency strength than a weakly compact limit of Woodin cardinals.

We can force a failure of covering for the derived model. This does not typically preserve weak compactness. But:

Corollary

If κ is a $\mathrm{Col}(\kappa, \kappa^+)$-indestructibly weakly compact limit of Woodin cardinals and G is a V-generic filter over $\mathrm{Col}(\omega, {<}\kappa)$, then $D(V, \kappa, G) \models \mathrm{AD}_{\mathbb{R}}$.

Remark

A better relative consistency result comes from Jensen–Schimmerling–Schindler–Steel, Stacking mice.

Can we get weak covering in the singular case?

Question

Let κ be a singular limit of Woodin cardinals. If $\mathsf{AD}_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, then must $\Theta^{D(V,\kappa,G)} = \kappa^+$?

This would result in incompactness:

Proposition (W.)

Let κ be a singular limit of Woodin cardinals.
If $\Theta^{D(V,\kappa,G)} = \kappa^+$, then $\square^*_\kappa$ holds after some small forcing.

(The small forcing is only needed if $D(V, \kappa, G) \models \mathsf{LSA}$; perhaps not even then.)

In the inaccessible case, where we do have weak covering, does this result in incompactness?
(Note $\square^*_\kappa$ is trivial at an inaccessible.)

Question

Let κ be an inaccessible limit of Woodin cardinals.
If $AD_{\mathbb{R}}$ fails in $D(V, \kappa, G)$, then

- In the case $\mathrm{cf}(\Theta^{D(V,\lambda,G)}) = \kappa$, must $\square(\kappa)$ hold?
- In the case $\Theta^{D(V,\lambda,G)} = \kappa^+$, must $\square(\kappa^+)$ hold?

Recall that if $\mathrm{AD}_{\mathbb{R}}$ holds, then we have $\Theta^{D(V,\kappa,G)} < \kappa^+$.
Can we still get some kind of weak covering?

Question

Let κ be a limit of Woodin cardinals. Assume that

- κ is singular, or
- κ is weakly compact.

Assume that $\mathrm{AD}_{\mathbb{R}}$ holds in $D(V, \kappa, G)$
(and maybe that some stronger determinacy axiom fails.)
Is the successor of $\Theta^{D(V,\kappa,G)}$ in $\mathrm{HOD}^{D(V,\kappa,G)}$ equal to κ^+?