Absolutely complementing trees and generic absoluteness

Trevor Wilson

University of California, Irvine

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Definition
For a tree $T$ on $\omega \times \text{Ord}$, let $[T] \subset \omega^\omega \times \text{Ord}^\omega$ be the class of branches of $T$ and let $p[T] \subset \omega^\omega$ be its projection:

$$x \in p[T] \iff \exists f \in \text{Ord}^\omega (x, f) \in [T].$$

For every real $x \in \omega^\omega$ the statement “$x \in p[T]$” is generically absolute, meaning that its truth is unchanged by forcing.

Theorem (Shoenfield)
If $\varphi(v)$ is a $\Sigma^1_2$ formula then there is a tree $T$ on $\omega \times \text{Ord}$ such that in every generic extension,

$$p[T] = \{x \in \omega^\omega : \varphi[x]\}.$$

Therefore $\Sigma^1_2$ statements are generically absolute.
Definition

- A $<\lambda$-generic extension is a generic extension by a poset of cardinality less than $\lambda$.
- A tree $T$ is $\lambda$-absolutely complemented if there is $\tilde{T}$ such that in every $<\lambda$-generic extension $p[\tilde{T}] = \omega^\omega \setminus p[T]$.

Definition (Feng–Magidor–Woodin)

A set of reals $A$ is $\lambda$-universally Baire$^1$ if $A = p[T]$ for some $\lambda$-absolutely complemented tree $T$.

$A$ is universally Baire if it is $\lambda$-universally Baire for all $\lambda$.

Remark

Universal Baire property implies many regularity properties.

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$^1$Called $<\lambda$-universally Baire in the original notation.
Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. One-step $\Sigma^1_3$ generic absoluteness holds

2. Every $\Delta^1_2$ set of reals is universally Baire

Moreover,

- (1) and (2) can be forced from a $\Sigma_2$-reflecting cardinal. (This is between “inaccessible” and “Mahlo.”)

- (1) and (2) imply that $\omega_1^V$ is $\Sigma_2$-reflecting in $L$. 
Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. Two-step $\Sigma^1_3$ generic absoluteness (in every generic extension, one-step $\Sigma^1_3$ generic absoluteness holds)
2. In every generic extension, every $\Delta^1_2$ set of reals is universally Baire
3. Every $\Sigma^1_2$ set of reals is universally Baire

Theorem (Woodin $\Rightarrow$, Martin–Solovay $\Leftarrow$)

The following statements are equivalent:

- Two-step $\Sigma^1_3$ generic absoluteness
- Every set has a sharp
Next we consider an similar situation “higher up” in terms of large cardinals and descriptive set theory.

- Let $\lambda$ be a limit of Woodin cardinals.
- Let $uB_\lambda$ be the pointclass of $\lambda$-universally Baire sets.

**Analogy:**

\[
\begin{align*}
\Sigma_2^1 & \sim (\Sigma_1^2)^{uB_\lambda} \\
\Sigma_3^1 & \sim \exists^R (\Pi_1^2)^{uB_\lambda}
\end{align*}
\]
Definition
A statement about a real \(x\) is \(\left(\Sigma^2_1\right)^{uB\lambda}\) if for some formula \(\varphi(\nu)\) it has the form

\[
\exists B \in uB\lambda (HC; \in, B) \models \varphi[x]
\]

(E.g. “\(x\) is in a mouse with a \(uB\lambda\) iteration strategy.”)

Theorem (Woodin)
If \(\lambda\) is a limit of Woodin cardinals and \(\varphi(\nu)\) is a formula, then there is a tree \(T\) such that in every \(<\lambda\)-generic extension,

\[
p[T] = \{x \in \omega^\omega : \exists B \in uB\lambda (HC; \in, B) \models \varphi[x]\}.
\]

Therefore \(\left(\Sigma^2_1\right)^{uB\lambda}\) statements are generically absolute below \(\lambda\).
Definition
A statement is $\exists^R(\Pi^2_1)^{uB_\lambda}$ if for some formula $\varphi(\nu)$ it has the form
\[ \exists x \in \omega^\omega \forall B \in uB_\lambda \,(HC; \in, B) \models \varphi[x]. \]
(E.g. “some real is not in any mouse with a $uB_\lambda$ strategy.”)

Remark
Generic absoluteness for $\exists^R(\Pi^2_1)^{uB_\lambda}$ can fail even if $\lambda$ is a limit of Woodin cardinals, and more:

- It fails for the currently studied canonical models.
- It is not known to follow from any large cardinal hypothesis.
Proposition

For a limit \( \lambda \) of Woodin cardinals, the following statements are equivalent:

1. One-step \( \exists^R (\Pi^2_1)^{uB}\lambda \) generic absoluteness below \( \lambda \)

2. Every \( (\Delta^2_1)^{uB}\lambda \) set of reals is \( \lambda \)-universally Baire

Moreover, (1) and (2) can be forced from a cardinal that is \( \Sigma^2_2 \)-reflecting up to a limit of Woodins.

(This is between an inaccessible limit of Woodins and a Mahlo limit of Woodins.)

Question 1

Do (1) and (2) imply that \( \omega^V_1 \) is \( \Sigma^2_2 \)-reflecting up to a limit of Woodins in some inner model?
Proposition
For a limit $\lambda$ of Woodin cardinals, the following statements are equivalent:

1. Two-step $\exists^R (\Pi^2_1)^{uB_\lambda}$ generic absoluteness below $\lambda$

2. In every $<\lambda$-generic extension, every $(\Delta^2_1)^{uB_\lambda}$ set of reals is $\lambda$-universally Baire

Moreover, (1) and (2) follow from

3. Every $(\Sigma^2_1)^{uB_\lambda}$ set of reals is $\lambda$-universally Baire.

Question 2
Are (1), (2), and (3) all equivalent, that is, does (1) $\implies$ (3)?
We will give a partial “yes” answer to Question 2.

Remark

- Woodin’s proof that two-step $\Sigma^1_3$ generic absoluteness implies “$\Sigma^1_2$ sets are universally Baire” uses Jensen’s covering lemma to get sharps as an intermediate step.

- To attempt a proof that two-step $\exists^R(\Pi^2_1)^{uB_\lambda}$ generic absoluteness below $\lambda$ implies “$(\Sigma^2_1)^{uB_\lambda}$ sets are $\lambda$-universally Baire” we need a higher covering lemma.

- Our “covering lemma” will bypass the inner model theory step and directly construct a $\lambda$-absolute complement for the tree $\mathcal{T}$ for $(\Sigma^2_1)^{uB_\lambda}$. 
**Lemma**

Let $\lambda$ be a measurable cardinal with a normal measure $\mu$. Let $T$ be a tree on $\omega \times \gamma$ for some ordinal $\gamma$. Assume that for $\mu$-almost every $\alpha < \lambda$ we have

$$|\mathcal{P}(V_\alpha) \cap L(T, V_\alpha)| = \alpha.$$  

Then in some $<\lambda$-generic extension, $T$ is $\lambda$-absolutely complemented.

**Remark**

In our application, $T$ will be the tree for $(\Sigma^2_1)^{uB_\lambda}$ and the “failure of covering” will come from $\exists^\mathbb{R}(\Pi^2_1)^{uB_\lambda}$ generic absoluteness applied to “$L[T, x] \cap \mathbb{R}$ is countable” for a generic real $x$ coding $V_\alpha$. 
A partial answer to Question 2:

**Theorem**
Let $\lambda$ be a measurable cardinal that is a limit of Woodin cardinals. Assume two-step $\exists^R (\Pi^2_1)^uB_\lambda$ generic absoluteness below $\lambda$. Then in some $<\lambda$-generic extension, every $(\Sigma^2_1)^uB_\lambda$ set is $\lambda$-universally Baire.

**Question 2a**
Can we do without measurability of $\lambda$?

**Question 2b**
Can we get every $(\Sigma^2_1)^uB_\lambda$ set $\lambda$-universally Baire in $V$?