Absolutely complementing trees and generic absoluteness

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Definition

For a tree T on $\omega \times \mathrm{Ord}$, let $[T] \subset \omega^\omega \times \mathrm{Ord}^\omega$ be the class of branches of T and let $p[T] \subset \omega^\omega$ be its projection:

$$x \in p[T] \iff \exists f \in \mathrm{Ord}^\omega \, (x, f) \in [T].$$

For every real $x \in \omega^\omega$ the statement "$x \in p[T]$" is *generically absolute*, meaning that its truth is unchanged by forcing.

Theorem (Shoenfield)

If $\varphi(v)$ is a Σ^1_2 formula then there is a tree T on $\omega \times \mathrm{Ord}$ such that in every generic extension,

$$p[T] = \{x \in \omega^\omega : \varphi[x]\}.$$

Therefore $\boldsymbol{\Sigma}^1_2$ statements are generically absolute.

Definition

- A $<\lambda$-generic extension is a generic extension by a poset of cardinality less than λ.
- A tree T is λ-absolutely complemented if there is $\tilde{T}$ such that in every $<\lambda$-generic extension $p[\tilde{T}] = \omega^\omega \setminus p[T]$.

Definition (Feng–Magidor–Woodin)

A set of reals A is λ-universally Baire[1] if $A = p[T]$ for some λ-absolutely complemented tree T.
A is universally Baire if it is λ-universally Baire for all λ.

Remark

Universal Baire property implies many regularity properties.

[1] Called $<\lambda$-universally Baire in the original notation.

Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. One-step $\underset{\sim}{\Sigma}^1_3$ generic absoluteness holds
2. Every $\underset{\sim}{\Delta}^1_2$ set of reals is universally Baire

Moreover,

- (1) and (2) can be forced from a Σ_2-reflecting cardinal. (This is between "inaccessible" and "Mahlo.")
- (1) and (2) imply that ω_1^V is Σ_2-reflecting in L.

Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. Two-step $\mathbf{\Sigma}^1_3$ generic absoluteness (in every generic extension, one-step $\mathbf{\Sigma}^1_3$ generic absoluteness holds)
2. In every generic extension, every $\mathbf{\Delta}^1_2$ set of reals is universally Baire
3. Every Σ^1_2 set of reals is universally Baire

Theorem (Woodin $\Rightarrow$, Martin–Solovay $\Leftarrow$)

The following statements are equivalent:

- Two-step $\mathbf{\Sigma}^1_3$ generic absoluteness
- Every set has a sharp

Next we consider an similar situation "higher up" in terms of large cardinals and descriptive set theory.

- Let λ be a limit of Woodin cardinals.
- Let uB_λ be the pointclass of λ-universally Baire sets.

Analogy:

$$\Sigma^1_2 \rightsquigarrow (\Sigma^2_1)^{\mathrm{uB}_\lambda}$$
$$\Sigma^1_3 \rightsquigarrow \exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$$

Definition

A statement about a real x is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ if for some formula $\varphi(v)$ it has the form

$$\exists B \in \mathrm{uB}_\lambda \, (\mathrm{HC}; \in, B) \models \varphi[x]$$

(*E.g.* "x is in a mouse with a uB_λ iteration strategy.")

Theorem (Woodin)

If λ is a limit of Woodin cardinals and $\varphi(v)$ is a formula, then there is a tree T such that in every $<\lambda$-generic extension,

$$p[T] = \{x \in \omega^\omega : \exists B \in \mathrm{uB}_\lambda \, (\mathrm{HC}; \in, B) \models \varphi[x]\}.$$

Therefore $(\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ statements are generically absolute below λ.

Definition

A statement is $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ if for some formula $\varphi(v)$ it has the form

$$\exists x \in \omega^\omega \, \forall B \in \mathrm{uB}_\lambda \, (\mathrm{HC}; \in, B) \models \varphi[x].$$

(*E.g.* "some real is not in any mouse with a uB_λ strategy.")

Remark

Generic absoluteness for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ can fail even if λ is a limit of Woodin cardinals, and more:

- It fails for the currently studied canonical models.
- It is not known to follow from any large cardinal hypothesis.

Proposition

For a limit λ of Woodin cardinals, the following statements are equivalent:

1. One-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ
2. Every $(\undertilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire

Moreover, (1) and (2) can be forced from a cardinal that is Σ_2-reflecting up to a limit of Woodins.

(This is between an inaccessible limit of Woodins and a Mahlo limit of Woodins.)

Question 1

Do (1) and (2) imply that ω_1^V is Σ_2-reflecting up to a limit of Woodins in some inner model?

Proposition

For a limit λ of Woodin cardinals, the following statements are equivalent:

1. Two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ
2. In every $<\lambda$-generic extension, every $(\underset{\sim}{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire

Moreover, (1) and (2) follow from

3. Every $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.

Question 2

Are (1), (2), and (3) all equivalent, that is, does (1) $\implies$ (3)?

We will give a partial "yes" answer to Question 2.

Remark

- Woodin's proof that two-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness implies "Σ^1_2 sets are universally Baire" uses Jensen's covering lemma to get sharps as an intermediate step.
- To attempt a proof that two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ implies "$(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ sets are λ-universally Baire" we need a higher covering lemma.
- Our "covering lemma" will bypass the inner model theory step and directly construct a λ-absolute complement for the tree T for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$.

Lemma

Let λ be a measurable cardinal with a normal measure μ. Let T be a tree on $\omega \times \gamma$ for some ordinal γ. Assume that for μ-almost every $\alpha < \lambda$ we have

$$|\mathcal{P}(V_\alpha) \cap L(T, V_\alpha)| = \alpha.$$

Then in some $<\lambda$-generic extension, T is λ-absolutely complemented.

Remark

In our application, T will be the tree for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ and the "failure of covering" will come from $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness applied to "$L[T, x] \cap \mathbb{R}$ is countable" for a generic real x coding V_α.

A partial answer to Question 2:

Theorem

Let λ be a measurable cardinal that is a limit of Woodin cardinals. Assume two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness below λ. Then in some $<\lambda$-generic extension, every $(\Sigma^2_1)^{\mathsf{uB}_\lambda}$ set is λ-universally Baire.

Question 2a

Can we do without measurability of λ?

Question 2b

Can we get every $(\Sigma^2_1)^{\mathsf{uB}_\lambda}$ set λ-universally Baire in V?