Optimal generic absoluteness results from strong cardinals

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Definition
A statement $\varphi$ is **generically absolute** if its truth is unchanged by forcing:

$$V \models \varphi \iff V[g] \models \varphi$$

for every generic extension $V[g]$.

Example
For a tree $T$ of height $\leq \omega$ the statement “$T$ is ill-founded (has an infinite branch)” is generically absolute.

Many generic absoluteness results can be proved via continuous reductions to ill-foundedness of trees.
A brief introduction to trees in descriptive set theory:

- Let $T$ be a function from $\omega^{<\omega}$ to trees of height $<\omega$ such that, if $s'$ extends $s$, then $T(s')$ end-extends $T(s)$.
- Then $T$ extends to a continuous function from Baire space $\omega^\omega$ to the space of trees of height $\leq \omega$:

$$T(x) = \bigcup_{n<\omega} T(x \upharpoonright n).$$

- We will abuse notation by calling $T$ itself a tree. An “infinite branch of $T$” consists of a real $x \in \omega^\omega$ in the first coordinate and an infinite branch of $T(x)$ in the second coordinate.
Definition
We say that a set of reals $A \subset \omega^\omega$ has a tree representation if there is a tree $T$ (equivalently, a tree-valued continuous function $T$) such that for every real $x \in \omega^\omega$,

$$x \in A \iff T(x) \text{ is ill-founded.}$$

Remark
Every set of reals $A$ has a trivial tree representation where the nodes are constant sequences of elements of $A$. These are not useful.
A non-trivial kind of tree representation:

**Definition**
Trees $T$ and $\tilde{T}$ are $\alpha$-absolutely complementing if, for every real $x$ in every generic extension by a forcing poset of size less than $\alpha$,

$$T(x) \text{ is ill-founded } \iff \tilde{T}(x) \text{ is well-founded.}$$

**Definition**
A set of reals $A$ is $\alpha$-universally Baire if it is represented by an $\alpha$-absolutely complemented tree.
Definition
Let $\varphi(x)$ be a formula. A tree representation of $\varphi$ for posets of size less than $\alpha$ is a tree $T$ such that, in any generic extension by a poset of size less than $\alpha$, the tree $T$ represents the set of reals $\{x \in \omega^\omega : \varphi(x)\}$.

Remark
If $\varphi(x, y)$ has such a representation (generalized to two variables) then so does the formula $\exists y \in \omega^\omega \varphi(x, y)$:

$$\exists y \in \omega^\omega \varphi(x, y) \iff \exists y \ T(x, y) \text{ is ill-founded} \iff T(x) \text{ is ill-founded}.$$
The following theorems are stated in a slightly unusual way to fit with the “generic absoluteness” theme of the talk.¹

**Theorem (Mostowski)**

$\Sigma^1_1$ formulas have tree representations for posets of any size. Therefore $\Sigma^1_1$ statements are generically absolute.

**Theorem (Shoenfield)**

$\Pi^1_1$ formulas (and hence $\Sigma^1_2$ formulas) have tree representations for posets of any size. Therefore $\Sigma^1_2$ statements are generically absolute. The proof constructs absolute complements of trees for $\Sigma^1_1$ formulas.

¹Note added April 28, 2014: I have been informed that the original proofs of these absoluteness theorems were not phrased in terms of trees.
For a pointclass (take $\Sigma^1_3$ for example) we consider two kinds of generic absoluteness.

**Definition**

- **One-step generic absoluteness** for $\Sigma^1_3$ says for every $\Sigma^1_3$ formula $\varphi(v)$, every real $x$, and every generic extension $V[g]$,

  $$V \models \varphi[x] \iff V[g] \models \varphi[x].$$

- **Two-step generic absoluteness** for $\Sigma^1_3$ says that one-step generic absoluteness for $\Sigma^1_3$ holds in every generic extension.

**Remark**

Upward absoluteness ("$\implies$") is automatic by Shoenfield.
Theorem (Martin–Solovay)

Let $\kappa$ be a measurable cardinal. Then $\Pi^1_2$ formulas (and hence $\Sigma^1_3$ formulas) have tree representations for posets of size less than $\kappa$. Therefore two-step $\Sigma^1_3$ generic absoluteness holds for posets of size less than $\kappa$.

Theorem

Assume that every set has a sharp. Then $\Pi^1_2$ (and $\Sigma^1_3$) formulas have tree representations for posets of any size. Therefore two-step $\Sigma^1_3$ generic absoluteness holds. The proof constructs absolute complements of trees for $\Sigma^1_2$ formulas.
The converse statement also holds:

**Theorem (Woodin)**

If two-step $\Sigma^1_3$ generic absoluteness holds, then every set has a sharp.

**Sketch of proof**

- If $0^#$ does not exist then $\lambda^{+L} = \lambda^+$ where $\lambda$ is any singular strong limit cardinal. (The case of $A^#$ is similar.)
- $L|\lambda^{+L}$ is $\Sigma^1_2(x)$ in the codes where the real $x \in V^{Col(\omega,\lambda)}$ codes $L|\lambda$, so the statement $\lambda^{+L} = \lambda^+$ is $\Pi^1_3(x)$. But it is not generically absolute for $Col(\omega, \lambda^+)$, a contradiction.
Theorem (Woodin)
If $\delta$ is a strong cardinal, then two-step $\Sigma^1_4$ generic absoluteness holds after forcing with $\text{Col}(\omega, 2^{2\delta})$.

Lemma (Woodin)
If $\delta$ is $\alpha$-strong as witnessed by $j : V \rightarrow M$, $T$ is a tree, and $|V_\alpha| = \alpha$, then after forcing with $\text{Col}(\omega, 2^{2\delta})$, there is an $\alpha$-absolute complement $\tilde{T}$ for $j(T)$.

$\triangleright$ Given a $\Sigma^1_3$ formula $\varphi(x, y)$, let $T$ be a tree representation of $\varphi$ for posets of size less than $\kappa$.
$\triangleright$ Then $j(T)$ represents $\varphi$ for posets of size less than $\alpha$.
$\triangleright$ So $\tilde{T}$ is a tree representation of the $\Pi^1_3$ formula $\neg\varphi(x, y)$, or equivalently of the $\Sigma^1_4$ formula $\exists y \in \omega \omega \neg\varphi(x, y)$, for posets of size less than $\alpha$. 
Woodin’s theorem can be reversed using inner model theory:

**Theorem (Hauser)**

If two-step $\Sigma^1_4$ generic absoluteness holds, then there is an inner model with a strong cardinal.

- If there is an inner model with a Woodin cardinal, great.
- If not, then $\lambda^+K = \lambda^+$ where $K$ is the core model and $\lambda$ is any singular strong limit cardinal.
- Some cardinal $\delta < \lambda$ is $<\lambda$-strong in $K$; otherwise $K|\lambda^+K$ would be $\Sigma^1_3(x)$ in the codes where the real $x \in V^{Col(\omega,\lambda)}$ codes $K|\lambda$, so the statement $\lambda^+K = \lambda^+$ would be $\Pi^1_4(x)$. But it is not generically absolute for Col$(\omega, \lambda^+)$.  
- By a pressing-down argument, some $\delta$ is strong in $K$. 

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A totally different way to get tree representations for $\Pi^1_3$ sets:

**Theorem (Moschovakis; corollary of 2nd periodicity)**

If $\Delta^1_2$ determinacy holds then every $\Pi^1_3$ set has a definable tree representation.

**Corollary**

If $\Delta^1_2$ determinacy holds in every generic extension, then two-step $\Sigma^1_4$ generic absoluteness holds.

- The hypothesis of the corollary has higher consistency strength than “there is a strong cardinal.”
- It holds in $V_\delta$ if $\delta$ is a Woodin cardinal and there is a measurable cardinal above $\delta$.
- More generally, it holds if every set has an $M^\#$.
Now back to strong cardinals. We can reduce the number $2^{2^\delta}$ in Woodin’s consistency proof of $\Sigma^1_4$ generic absoluteness.

**Main theorem (W.)**

If $\delta$ is a strong cardinal, then two-step $\Sigma^1_4$ generic absoluteness holds after forcing with $\text{Col}(\omega, \delta^+)$. 

**Main lemma (W.)**

If $\delta$ is $\alpha$-strong as witnessed by $j : V \rightarrow M$ and $T$ is a tree, then $j(T)$ becomes $\alpha$-absolutely complemented after collapsing $\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)$ to $\omega$.

- In particular, it suffices to collapse $2^\delta$.
- For “nice” trees it suffices to collapse $\tilde{\delta}^+$. 

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Sketch of proof of the main lemma (for experts):

- Say $\delta$ is $\alpha$-strong as witnessed by $j : V \rightarrow M$ and $T$ is a tree. We want an $\alpha$-absolute complement for $j(T)$.
- Woodin’s argument uses a Martin–Solovay construction from measures in the set $j"(\text{measures on } \delta^{<\omega} \text{ induced by } j)$.
- The only clear bound on the number of measures is $2^{2^{\delta}}$.
- So instead of measures, we consider the corresponding prewellorderings of the Martin–Solovay semiscale.
- The prewellorderings have $\text{Col}(\omega, <\delta)$-names in the set $j"(\mathcal{P}(V_{\delta}) \cap L(j(T), V_{\delta})).$
Sketch of proof of the main theorem:

- Let $\delta$ be strong. We want to show that $\Sigma^1_4$ generic absoluteness holds after forcing with $\text{Col}(\omega, \delta^+)$. 
- If every set has an $M_1^\sharp$ then it holds in $V$, so suppose not. 
- Then for a cone of $x \in V_\delta$ the core model $K(x)$ exists and contains the Martin–Solovay tree representations $T$ for $\Sigma^1_3$ formulas (by the proof of $\Sigma^1_3$ correctness of $K$.) 
- Let $j : V \rightarrow M$ have critical point $\delta$. We want to show 
  $$|\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)| \leq \delta^+. \quad (*)$$
- Let the real $x \in V^{\text{Col}(\omega, \delta)}$ code $V_\delta$. Then $L(j(T), V_\delta) \subset K(x)^M$ and $K(x)^M \models \text{CH}$, so $(*)$ follows.
Remark
The $\delta^+$ in the theorem is optimal:

- If $\delta$ is a strong cardinal, two-step $\Sigma_4^1$ generic absoluteness can fail after forcing with $\text{Col}(\omega, \delta)$.
- If some cardinal $\delta_0 < \delta$ is also strong, then it holds (simply because $\delta_0^+$ is collapsed.)
- However, this is essentially the only way for it to hold:

Proposition
If $\delta$ is strong and two-step (or just one-step) $\Sigma_4^1$ generic absoluteness holds after forcing with $\text{Col}(\omega, \delta)$, then there is an inner model with two strong cardinals.
Proof sketch:

- Assume that $\delta$ is strong and one-step $\Sigma_4^1$ generic absoluteness holds after collapsing only $\delta$.
- If there is an inner model with a Woodin cardinal, great.
- Otherwise, the core model $K$ exists. Because $\delta$ is weakly compact, $\delta^{+K} = \delta^+$.
- Some cardinal $\delta_0 < \delta$ is $<\delta$-strong in $K$; otherwise $K|\delta^{+K}$ would be $\Sigma_3^1(x)$ in the codes where the real $x \in V^{\text{Col}(\omega,\delta)}$ codes $K|\delta$, so the statement $\delta^{+K} = \delta^+$ would be $\Pi_4^1(x)$. But it is not generically absolute for $\text{Col}(\omega,\delta^+)$.  
- Finally, $\delta$ itself is strong in $K$ (we use Steel’s local $K^c$ construction) and so $\delta_0$ is strong in $K$ also.
Question

Can we get optimal results higher in the projective hierarchy? Let $n > 1$ and assume there are $n$ many strong cardinals $\leq \delta$.

- Two-step $\Sigma^1_{n+3}$ generic absoluteness holds after forcing with $\text{Col}(\omega, 2^{2^\delta})$ (Woodin).
- Two-step $\Sigma^1_{n+3}$ generic absoluteness holds after forcing with $\text{Col}(\omega, 2^\delta)$.
- It is consistent that $2^\delta = \delta^+$ and two-step $\Sigma^1_{n+3}$ generic absoluteness fails after forcing with $\text{Col}(\omega, \delta)$ (e.g. in the minimal mouse satisfying the hypothesis.)
- Still open: Must two-step $\Sigma^1_{n+3}$ generic absoluteness hold after forcing with $\text{Col}(\omega, \delta^+)$?
Now we turn to a pointclass beyond the projective hierarchy.

**Definition**

Let $\lambda$ be a cardinal.

- $uB_\lambda$ is the pointclass of $\lambda$-universally Baire sets.
- A formula $\varphi(\vec{v})$ is $(\Sigma^2_1)^{uB_\lambda}$ if it has the form
  \[ \exists B \in uB_\lambda \ (HC; \in, B) \models \theta(\vec{v}). \]
- A formula $\varphi(\vec{v})$ is $\exists^R(\Pi^2_1)^{uB_\lambda}$ if it has the form
  \[ \exists u \in \omega^\omega \ \forall B \in uB_\lambda \ (HC; \in, B) \models \theta(u, \vec{v}). \]
Example

- The formula $\varphi(\nu)$ saying “the real $\nu$ is in a mouse with a $uB^\lambda$ iteration strategy” is $(\Sigma^2_1)^{uB^\lambda}$.
- The sentence $\varphi$ saying “there is a real that is not in any mouse with a $uB^\lambda$ iteration strategy” is $\exists^R (\Pi^2_1)^{uB^\lambda}$.

Theorem (Woodin)

Let $\lambda$ be a limit of Woodin cardinals.

- Every $(\Sigma^2_1)^{uB^\lambda}$ statement is generically absolute for posets of size less than $\lambda$.
- Every $(\Sigma^2_1)^{uB^\lambda}$ formula has a tree representation for posets of size less than $\lambda$. 
By contrast, generic absoluteness for $\exists^R (\Pi^2_1)^{uB_\lambda}$ is not known to follow from any large cardinal hypothesis. It can be obtained from strong cardinals by forcing, however:

**Theorem (Woodin)**

Let $\lambda$ be a limit of Woodin cardinals and let $\delta < \lambda$ be $<$\lambda-\text{strong}$. Then two-step $\exists^R (\Pi^2_1)^{uB_\lambda}$ generic absoluteness for posets of size less than $\lambda$ holds after forcing with $\text{Col}(\omega, 2^{2^\delta})$.

**Theorem (W.)**

Let $\lambda$ be a limit of Woodin cardinals and let $\delta < \lambda$ be $<$\lambda-\text{strong}$. Then two-step $\exists^R (\Pi^2_1)^{uB_\lambda}$ generic absoluteness for posets of size less than $\lambda$ holds after forcing with $\text{Col}(\omega, \delta^+)$. 
Proof sketch:

- Let $T$ be Woodin’s tree representation of a $(\Sigma^2_1)^{uB\lambda}$ formula for posets of size less than $\lambda$.
- Let $j : V \to M$ witness that $\delta$ is $\alpha$-strong for sufficiently large $\alpha < \lambda$.
- We want to show

$$|\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)| \leq \delta^+$.$$(*)

- Let the real $x \in V^{\text{Col}(\omega, \delta)}$ code $V_\delta$. Then $L(j(T), V_\delta) \subseteq L(j(T), x)$ and $L(j(T), x) \models \text{CH}$, so (*) follows.
- Here CH comes not from fine structure, but from determinacy (“CH on a Turing cone.”)
The theorem is optimal because of the following result:

**Proposition (W.)**

Let $\lambda$ be a limit of Woodin cardinals and let $\delta < \lambda$ be $<\lambda$-strong. If one-step $\exists^\mathbb{R} (\prod_1^2)^{uB\lambda}$ generic absoluteness for $\text{Col}(\omega, \delta^+)$ holds after forcing with $\text{Col}(\omega, \delta)$, then:

- The derived model at $\delta$ satisfies $\text{ZF} + \text{AD}^+ + \theta_0 < \Theta$.
- The derived model at $\lambda$ satisfies $\text{ZF} + \text{AD}^+ + \theta_1 < \Theta$.

**Remark**

The theory “$\text{ZF} + \text{AD}^+ + \theta_1 < \Theta$” is equiconsistent with the theory “$\text{ZFC} + \lambda$ is a limit of Woodin cardinals + there are two $<\lambda$-strong cardinals below $\lambda$” (I think.)
Question
To what extent does generic absoluteness come from tree representations? More precisely,

1. Assume two-step $\Sigma^1_4$ generic absoluteness. Does every $\Pi^1_3$ formula have tree representations for arbitrarily large posets?

2. Assume two-step $\exists^R (\Pi^2_1)^{uB\lambda}$ generic absoluteness for posets of size less than $\lambda$ where $\lambda$ is a limit of Woodin cardinals. Does every $(\Pi^2_1)^{uB\lambda}$ formula have a tree representation for posets of size less than $\lambda$?