Optimal generic absoluteness results from strong cardinals

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Definition

A statement φ is generically absolute if its truth is unchanged by forcing:

$$V \models \varphi \iff V[g] \models \varphi$$

for every generic extension $V[g]$.

Example

For a tree T of height $\leq \omega$ the statement "T is ill-founded (has an infinite branch)" is generically absolute.

Many generic absoluteness results can be proved via continuous reductions to ill-foundedness of trees.

A brief introduction to trees in descriptive set theory:

- Let T be a function from $\omega^{<\omega}$ to trees of height $< \omega$ such that, if s' extends s, then $T(s')$ end-extends $T(s)$.
- Then T extends to a continuous function from Baire space ω^ω to the space of trees of height $\leq \omega$:

$$T(x) = \bigcup_{n<\omega} T(x \restriction n).$$

- We will abuse notation by calling T itself a tree. An "infinite branch of T" consists of a real $x \in \omega^\omega$ in the first coordinate and an infinite branch of $T(x)$ in the second coordinate.

Definition

We say that a set of reals $A \subset \omega^\omega$ has a tree representation if there is a tree T (equivalently, a tree-valued continuous function T) such that for every real $x \in \omega^\omega$,

$$x \in A \iff T(x) \text{ is ill-founded.}$$

Remark

Every set of reals A has a trivial tree representation where the nodes are constant sequences of elements of A.
These are not useful.

A non-trivial kind of tree representation:

Definition

Trees T and $\tilde{T}$ are α-absolutely complementing if, for every real x in every generic extension by a forcing poset of size less than α,

$$T(x) \text{ is ill-founded} \iff \tilde{T}(x) \text{ is well-founded.}$$

Definition

A set of reals A is α-universally Baire if it is represented by an α-absolutely complemented tree.

Definition

Let $\varphi(x)$ be a formula. A tree representation of φ for posets of size less than α is a tree T such that, in any generic extension by a poset of size less than α, the tree T represents the set of reals $\{x \in \omega^\omega : \varphi(x)\}$.

Remark

If $\varphi(x, y)$ has such a representation (generalized to two variables) then so does the formula $\exists y \in \omega^\omega\, \varphi(x, y)$:

$$\begin{aligned}\exists y \in \omega^\omega\, \varphi(x, y) &\iff \exists y\, T(x, y) \text{ is ill-founded}\\ &\iff T(x) \text{ is ill-founded.}\end{aligned}$$

The following theorems are stated in a slightly unusual way to fit with the "generic absoluteness" theme of the talk.[1]

Theorem (Mostowski)

Σ^1_1 formulas have tree representations for posets of any size.
Therefore $\underset{\sim}{\mathbf{\Sigma}}^1_1$ statements are generically absolute.

Theorem (Shoenfield)

Π^1_1 formulas (and hence Σ^1_2 formulas) have tree representations for posets of any size.
Therefore $\underset{\sim}{\mathbf{\Sigma}}^1_2$ statements are generically absolute.
The proof constructs absolute complements of trees for Σ^1_1 formulas.

[1] Note added April 28, 2014: I have been informed that the original proofs of these absoluteness theorems were not phrased in terms of trees.

For a pointclass (take $\underset{\sim}{\mathbf{\Sigma}}^1_3$ for example) we consider two kinds of generic absoluteness.

Definition

- One-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ says for every Σ^1_3 formula $\varphi(v)$, every real x, and every generic extension $V[g]$,
$$V \models \varphi[x] \iff V[g] \models \varphi[x].$$
- Two-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ says that one-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ holds in every generic extension.

Remark

Upward absoluteness ("$\implies$") is automatic by Shoenfield.

Theorem (Martin–Solovay)

Let κ be a measurable cardinal. Then Π^1_2 formulas (and hence Σ^1_3 formulas) have tree representations for posets of size less than κ. Therefore two-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness holds for posets of size less than κ.

Theorem

Assume that every set has a sharp. Then Π^1_2 (and Σ^1_3) formulas have tree representations for posets of any size. Therefore two-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness holds.

The proof constructs absolute complements of trees for Σ^1_2 formulas.

The converse statement also holds:

Theorem (Woodin)

If two-step $\underset{\sim}{\Sigma}^1_3$ generic absoluteness holds, then every set has a sharp.

Sketch of proof

- If $0^\sharp$ does not exist then $\lambda^{+L} = \lambda^+$ where λ is any singular strong limit cardinal. (The case of $A^\sharp$ is similar.)
- $L|\lambda^{+L}$ is $\Sigma^1_2(x)$ in the codes where the real $x \in V^{\mathrm{Col}(\omega,\lambda)}$ codes $L|\lambda$, so the statement $\lambda^{+L} = \lambda^+$ is $\Pi^1_3(x)$. But it is not generically absolute for $\mathrm{Col}(\omega, \lambda^+)$, a contradiction.

Theorem (Woodin)

If δ is a strong cardinal, then two-step $\utilde{\mathbf{\Sigma}}^1_4$ generic absoluteness holds after forcing with $\mathrm{Col}(\omega, 2^{2^\delta})$.

Lemma (Woodin)

If δ is α-strong as witnessed by $j : V \to M$, T is a tree, and $|V_\alpha| = \alpha$, then after forcing with $\mathrm{Col}(\omega, 2^{2^\delta})$, there is an α-absolute complement $\tilde{T}$ for $j(T)$.

- Given a Σ^1_3 formula $\varphi(x, y)$, let T be a tree representation of φ for posets of size less than κ.
- Then $j(T)$ represents φ for posets of size less than α.
- So $\tilde{T}$ is a tree representation of the Π^1_3 formula $\neg\varphi(x, y)$, or equivalently of the Σ^1_4 formula $\exists y \in \omega^\omega\, \neg\varphi(x, y)$, for posets of size less than α.

Woodin's theorem can be reversed using inner model theory:

Theorem (Hauser)

If two-step $\underset{\sim}{\mathbf{\Sigma}}^1_4$ generic absoluteness holds, then there is an inner model with a strong cardinal.

- If there is an inner model with a Woodin cardinal, great.
- If not, then $\lambda^{+K} = \lambda^+$ where K is the core model and λ is any singular strong limit cardinal.
- Some cardinal $\delta < \lambda$ is $<\lambda$-strong in K; otherwise $K|\lambda^{+K}$ would be $\Sigma^1_3(x)$ in the codes where the real $x \in V^{\mathrm{Col}(\omega,\lambda)}$ codes $K|\lambda$, so the statement $\lambda^{+K} = \lambda^+$ would be $\Pi^1_4(x)$. But it is not generically absolute for $\mathrm{Col}(\omega, \lambda^+)$.
- By a pressing-down argument, some δ is strong in K.

A totally different way to get tree representations for Π^1_3 sets:

Theorem (Moschovakis; corollary of 2nd periodicity)

If $\utilde{\boldsymbol{\Delta}}^1_2$ determinacy holds then every Π^1_3 set has a definable tree representation.

Corollary

If $\utilde{\boldsymbol{\Delta}}^1_2$ determinacy holds in every generic extension, then two-step $\utilde{\boldsymbol{\Sigma}}^1_4$ generic absoluteness holds.

- The hypothesis of the corollary has higher consistency strength than "there is a strong cardinal."
- It holds in V_δ if δ is a Woodin cardinal and there is a measurable cardinal above δ.
- More generally, it holds if every set has an $M_1^\sharp$.

Now back to strong cardinals. We can reduce the number 2^{2^δ} in Woodin's consistency proof of $\utilde{\Sigma}^1_4$ generic absoluteness.

Main theorem (W.)

If δ is a strong cardinal, then two-step $\utilde{\Sigma}^1_4$ generic absoluteness holds after forcing with $\mathrm{Col}(\omega, \delta^+)$.

Main lemma (W.)

If δ is α-strong as witnessed by $j : V \to M$ and T is a tree, then $j(T)$ becomes α-absolutely complemented after collapsing $\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)$ to ω.

- In particular, it suffices to collapse 2^δ.
- For "nice" trees it suffices to collapse δ^+.

Sketch of proof of the main lemma (for experts):

- Say δ is α-strong as witnessed by $j : V \to M$ and T is a tree. We want an α-absolute complement for $j(T)$.
- Woodin's argument uses a Martin–Solovay construction from measures in the set

$$j``(\text{measures on } \delta^{<\omega} \text{ induced by } j).$$

- The only clear bound on the number of measures is 2^{2^δ}.
- So instead of measures, we consider the corresponding prewellorderings of the Martin–Solovay semiscale.
- The prewellorderings have $\mathrm{Col}(\omega, {<}\delta)$-names in the set

$$j``(\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)).$$

Sketch of proof of the main theorem:

- Let δ be strong. We want to show that $\underset{\sim}{\boldsymbol{\Sigma}}^1_4$ generic absoluteness holds after forcing with $\mathrm{Col}(\omega, \delta^+)$.
- If every set has an $M_1^\sharp$ then it holds in V, so suppose not.
- Then for a cone of $x \in V_\delta$ the core model $K(x)$ exists and contains the Martin–Solovay tree representations T for Σ^1_3 formulas (by the proof of Σ^1_3 correctness of K.)
- Let $j : V \to M$ have critical point δ. We want to show

$$|\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)| \leq \delta^+. \qquad (*)$$

- Let the real $x \in V^{\mathrm{Col}(\omega,\delta)}$ code V_δ. Then $L(j(T), V_\delta) \subset K(x)^M$ and $K(x)^M \models \mathrm{CH}$, so (*) follows.

Remark

The δ^+ in the theorem is optimal:

- If δ is a strong cardinal, two-step $\underset{\sim}{\mathbf{\Sigma}}^1_4$ generic absoluteness can fail after forcing with $\mathrm{Col}(\omega, \delta)$.
- If some cardinal $\delta_0 < \delta$ is also strong, then it holds (simply because δ_0^+ is collapsed.)
- However, this is essentially the only way for it to hold:

Proposition

If δ is strong and two-step (or just one-step) $\underset{\sim}{\mathbf{\Sigma}}^1_4$ generic absoluteness holds after forcing with $\mathrm{Col}(\omega, \delta)$, then there is an inner model with two strong cardinals.

Proof sketch:

- Assume that δ is strong and one-step $\underset{\sim}{\boldsymbol{\Sigma}}^1_4$ generic absoluteness holds after collapsing only δ.
- If there is an inner model with a Woodin cardinal, great.
- Otherwise, the core model K exists. Because δ is weakly compact, $\delta^{+K} = \delta^+$.
- Some cardinal $\delta_0 < \delta$ is $<\delta$-strong in K; otherwise $K|\delta^{+K}$ would be $\Sigma^1_3(x)$ in the codes where the real $x \in V^{\mathrm{Col}(\omega,\delta)}$ codes $K|\delta$, so the statement $\delta^{+K} = \delta^+$ would be $\Pi^1_4(x)$. But it is not generically absolute for $\mathrm{Col}(\omega, \delta^+)$.
- Finally, δ itself is strong in K (we use Steel's local K^c construction) and so δ_0 is strong in K also.

Question

Can we get optimal results higher in the projective hierarchy?
Let $n > 1$ and assume there are n many strong cardinals $\leq \delta$.

- Two-step $\underset{\sim}{\mathbf{\Sigma}}^1_{n+3}$ generic absoluteness holds after forcing with $\text{Col}(\omega, 2^{2^\delta})$ (Woodin).
- Two-step $\underset{\sim}{\mathbf{\Sigma}}^1_{n+3}$ generic absoluteness holds after forcing with $\text{Col}(\omega, 2^\delta)$.
- It is consistent that $2^\delta = \delta^+$ and two-step $\underset{\sim}{\mathbf{\Sigma}}^1_{n+3}$ generic absoluteness fails after forcing with $\text{Col}(\omega, \delta)$ (*e.g.* in the minimal mouse satisfying the hypothesis.)
- Still open: Must two-step $\underset{\sim}{\mathbf{\Sigma}}^1_{n+3}$ generic absoluteness hold after forcing with $\text{Col}(\omega, \delta^+)$?

Now we turn to a pointclass beyond the projective hierarchy.

Definition

Let λ be a cardinal.

- uB_λ is the pointclass of λ-universally Baire sets.
- A formula $\varphi(\vec{v})$ is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ if it has the form

$$\exists B \in \mathrm{uB}_\lambda \, (\mathrm{HC}; \in, B) \models \theta(\vec{v}).$$

- A formula $\varphi(\vec{v})$ is $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ if it has the form

$$\exists u \in \omega^\omega \, \forall B \in \mathrm{uB}_\lambda \, (\mathrm{HC}; \in, B) \models \theta(u, \vec{v}).$$

Example

- The formula $\varphi(v)$ saying "the real v is in a mouse with a uB_λ iteration strategy" is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$.
- The sentence φ saying "there is a real that is not in any mouse with a uB_λ iteration strategy" is $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$.

Theorem (Woodin)

Let λ be a limit of Woodin cardinals.

- Every $(\utilde{\boldsymbol{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ statement is generically absolute for posets of size less than λ.
- Every $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula has a tree representation for posets of size less than λ.

By contrast, generic absoluteness for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathsf{uB}_\lambda}$ is not known to follow from any large cardinal hypothesis. It can be obtained from strong cardinals by forcing, however:

Theorem (Woodin)

Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be $<\lambda$-strong. Then two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness for posets of size less than λ holds after forcing with $\mathrm{Col}(\omega, 2^{2^\delta})$.

Theorem (W.)

Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be $<\lambda$-strong. Then two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness for posets of size less than λ holds after forcing with $\mathrm{Col}(\omega, \delta^+)$.

Proof sketch:

- Let T be Woodin's tree representation of a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula for posets of size less than λ.
- Let $j : V \to M$ witness that δ is α-strong for sufficiently large $\alpha < \lambda$.
- We want to show

$$|\mathcal{P}(V_\delta) \cap L(j(T), V_\delta)| \leq \delta^+. \qquad (*)$$

- Let the real $x \in V^{\mathrm{Col}(\omega,\delta)}$ code V_δ. Then $L(j(T), V_\delta) \subset L(j(T), x)$ and $L(j(T), x) \models \mathrm{CH}$, so (*) follows.
- Here CH comes not from fine structure, but from determinacy ("CH on a Turing cone.")

The theorem is optimal because of the following result:

Proposition (W.)

Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be $<\lambda$-strong. If one-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness for $\mathrm{Col}(\omega, \delta^+)$ holds after forcing with $\mathrm{Col}(\omega, \delta)$, then:

- The derived model at δ satisfies $\mathrm{ZF} + \mathrm{AD}^+ + \theta_0 < \Theta$.
- The derived model at λ satisfies $\mathrm{ZF} + \mathrm{AD}^+ + \theta_1 < \Theta$.

Remark

The theory "$\mathrm{ZF} + \mathrm{AD}^+ + \theta_1 < \Theta$" is equiconsistent with the theory "$\mathrm{ZFC} + \lambda$ is a limit of Woodin cardinals $+$ there are two $<\lambda$-strong cardinals below λ" (I think.)

Question

To what extent does generic absoluteness come from tree representations? More precisely,

1. Assume two-step $\underset{\sim}{\mathbf{\Sigma}}^1_4$ generic absoluteness. Does every Π^1_3 formula have tree representations for arbitrarily large posets?
2. Assume two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness for posets of size less than λ where λ is a limit of Woodin cardinals. Does every $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formula have a tree representation for posets of size less than λ?