

A dichotomy for $(\Sigma_1^2)^{\text{Hom}\infty}$ sets of reals, with applications to generic absoluteness

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We begin by introducing some basic notions of descriptive set theory.

- ▶ Descriptive set theory deals with nice sets of reals, understood in terms of complexity.
- ▶ By “nice” we mean in particular less complex than a well-ordering of the reals.
- ▶ Above this level of complexity, the subject would turn into combinatorics of the continuum.

Remark

By convention, we identify the set of reals \mathbb{R} with the set of integer sequences ω^ω .

What are “nice” properties?

- ▶ regularity properties (e.g. property of Baire, Lebesgue measurability, perfect set property)
- ▶ determinacy

What are not?

- ▶ well-orderings of \mathbb{R}
- ▶ uncountable well-orderings

Possible definitions of “nice”:

- ▶ Universally Baire
- ▶ ∞ -Homogeneous (Hom_∞)

Definition

For a tree $T \subset \omega^{<\omega} \times \text{Ord}^{<\omega}$ we write $[T] \subset \omega^\omega \times \text{Ord}^\omega$ for the set of branches of T , and $p[T] \subset \omega^\omega$ for its projection.

Definition (Feng–Magidor–Woodin)

A set of reals A is κ -**universally Baire** if $A = p[T]$ for some pair of trees (T, \tilde{T}) that is κ -**absolutely complementing**:
 $p[\tilde{T}] = \mathbb{R} \setminus p[T]$ in all forcing extensions by posets of size $< \kappa$.

Example

Σ_1^1 and Π_1^1 sets of reals are universally Baire (κ -uB for all κ .)

Remark

Universal Baire-ness is a natural strengthening of the Property of Baire, and it also implies Lebesgue measurability.

Definition (Kechris, Martin)

Let A be a set of reals and κ an uncountable cardinal.

- ▶ A is κ -Homogeneous, or Hom_κ , if there is a continuous function f from \mathbb{R} to sequences of κ -complete measures such that $x \in A \iff f(x)$ is a well-founded tower.
- ▶ A is ∞ -Homogeneous, or Hom_∞ , if it is κ -homogeneous for every κ .

Example (Martin)

If κ is measurable then every $\overset{1}{\underset{\sim}{\Pi}}_1$ set of reals is Hom_κ .

Theorem (Martin)

Hom_κ sets are determined (for $\kappa > \omega$.)

Theorem (Martin–Solovay)

If the set $A \subset \mathbb{R}^{n+1}$ is Hom_κ , then the projection $pA \subset \mathbb{R}^n$ is κ -universally Baire.

Corollary

If κ is measurable, then every Σ_2^1 set is κ -universally Baire.

Their proof shows more:

Theorem (Martin–Solovay)

If κ is measurable, then the Shoenfield tree for Σ_2^1 is κ -absolutely complemented.

Corollary

If κ is a measurable cardinal, then Σ_3^1 statements about reals are absolute between $<\kappa$ -generic extensions.

Theorem (Martin–Steel)

If κ is a limit of Woodin cardinals, then the class of $\text{Hom}_{<\kappa}$ sets is closed under real quantifiers $\forall^{\mathbb{R}}$ and $\exists^{\mathbb{R}}$.

Corollary

If there are infinitely many Woodin cardinals, then all projective sets are determined.

Remark

The hypothesis of infinitely many Woodin cardinals is more than enough for PD—by a theorem of Woodin it is equiconsistent with $\text{AD}^{L(\mathbb{R})}$.

A Woodin cardinal is a large cardinal property between strongs and superstrongs in consistency strength.

- ▶ Every superstrong cardinal is Woodin and is a limit of Woodin cardinals.
- ▶ If there is a Woodin cardinal, then “there is a strong cardinal” is consistent, but does not necessarily hold in V .

Theorem (Martin–Steel + Woodin)

If κ is a limit of Woodin cardinals, a set of reals is κ -universally Baire if and only if it is $\text{Hom}_{<\kappa}$.

If there is a proper class of Woodin cardinals, then Hom_∞ (= universally Baire) is a natural class of “nice” sets of reals.

Remark

Hom_∞ is not necessarily closed under quantification over sets of reals, as we will discuss next.

Definition

A $(\Sigma_1^2)^{\text{Hom}_\infty}$ statement about $x \in \mathbb{R}^n$ says

$$\exists A \in \text{Hom}_\infty (H_{\omega_1}; A, \epsilon) \models \phi[x].$$

Example

For a real x , the following statements are $(\Sigma_1^2)^{\text{Hom}_\infty}$.

- ▶ “ x is $(\Sigma_1^2)^{\text{Hom}_\infty}$ in a countable ordinal”
- ▶ “ x is in a premouse with a Hom_∞ iteration strategy”

Remark

- ▶ A *premouse* is a generalization of “model of $V = L$ ” to accommodate large cardinals.
- ▶ For a premouse to be a canonical inner model (a “mouse”) it is not enough to be wellfounded as with models of $V = L$ —it needs to have an iteration strategy.

Remark

- ▶ If x is in a premouse with a Hom_∞ iteration strategy, then it is $(\Sigma_1^2)^{\text{Hom}_\infty}$ in a countable ordinal.
- ▶ The *Mouse Set Conjecture* says roughly the converse.

Example

There is a $(\Sigma_1^2)^{\text{Hom}_\infty}$ well-ordering of reals appearing in canonical inner models:

- ▶ Define $x < y$ if x is constructed before y in some/all premice with Hom_∞ iteration strategies.

This extends the Σ_2^1 well-ordering of reals in L .

Remark

If V itself is a canonical inner model of a certain type then there is a $(\Sigma_1^2)^{\text{Hom}_\infty}$ well-ordering of \mathbb{R} , so $(\Sigma_1^2)^{\text{Hom}_\infty} \not\subseteq \text{Hom}_\infty$.

Open question

Is every large cardinal axiom consistent with the existence of a $(\Sigma_1^2)^{\text{Hom}_\infty}$ well-ordering of \mathbb{R} ?

Theorem (Woodin)

Let κ be a limit of Woodin cardinals.

- ▶ There is a tree T such that in every $<\kappa$ -generic extension, $p[T]$ is the universal $(\Sigma_1^2)^{\text{Hom}_{<\kappa}}$ set of reals. (Compare to Shoenfield tree for Σ_2^1 .)
- ▶ If V_κ has a strong cardinal δ , then there is a $<\kappa$ -generic extension in which T is κ -absolutely complemented. (Compare to Martin–Solovay tree for Π_2^1 .)

Remark

If the tree T for $(\Sigma_1^2)^{\text{Hom}_{<\kappa}}$ is κ -absolutely complemented, then in every generic extension of V_κ we have $(\Sigma_1^2)^{\text{Hom}_\infty} \subset \text{Hom}_\infty$.

Theorem (W.)

If κ is a measurable limit of Woodin cardinals, then either:

1. In cofinally many $<\kappa$ -generic extensions there is an uncountable $(\Sigma_1^2)^{\text{Hom}<\kappa}$ well-ordering, or
2. The tree for $(\Sigma_1^2)^{\text{Hom}<\kappa}$ is κ -absolutely complemented in some $<\kappa$ -generic extension.

Remark

- ▶ Cases 1 and 2 are mutually exclusive.
- ▶ If V_κ has a strong cardinal then Case 2 must hold.
- ▶ If V_κ has no strong cardinal *and* is a certain type of canonical inner model then Case 1 must hold.

Remark

Suppose Case 2 holds: the tree for $(\Sigma_1^2)^{\text{Hom}<\kappa}$ is κ -absolutely complemented in some $<\kappa$ -generic extension.

- ▶ This is equivalent to saying that the *derived model* at κ , which is always a model of AD, satisfies “every Π_1^2 set is Suslin” (*i.e.* the projection of a tree on $\omega \times \text{Ord}$.)
- ▶ The theory AD + “every Π_1^2 set is Suslin” has high consistency strength: it implies there is an inner model with a cardinal that is strong past a Woodin cardinal.

So we can recover a “trace” of a collapsed $<\kappa$ -strong cardinal.

Proof idea

- ▶ Assume Case 1 fails: for some $<\kappa$ -generic extension, in every further $<\kappa$ -generic extension every $(\Sigma_1^2)^{\text{Hom}_{<\kappa}}$ well-ordering is countable.
- ▶ We want to establish Case 2 by showing that the tree T for $(\Sigma_1^2)^{\text{Hom}_{<\kappa}}$ has a κ -absolute complement \tilde{T} in some $<\kappa$ -generic extension.
- ▶ If κ were 2^{2^κ} -supercompact, this would come from a theorem of Martin and Woodin.
- ▶ In $\text{Ult}(V, \mu)[x_g]$ where μ is on κ and x_g generically codes V_κ , only countably many reals are $(\Sigma_1^2(x_g))^{\text{Hom}_{<j(\kappa)}}$. So there are only κ many partial measures to consider, and measurability of κ is enough (details omitted.)

Now we derive some consequences of this dichotomy related to generic absoluteness.

$(\Sigma_1^2)^{\text{Hom}\infty}$ generic absoluteness (Woodin)

Assume there is a proper class of Woodin cardinals.

- ▶ Let $V[g]$ and $V[g][h]$ be generic extensions and let $x \in \mathbb{R} \cap V[g]$.
- ▶ Then any $(\Sigma_1^2)^{\text{Hom}\infty}$ statement about x holds in $V[g]$ if and only if it holds in $V[g][h]$.

Remark

This is analogous to Shoenfield's Σ_2^1 absoluteness.

What is analogous to Martin–Solovay Σ_3^1 absoluteness?

Definition

A $\forall^{\mathbb{R}}(\Sigma_1^2)^{\text{Hom}_\infty}$ statement says that all reals have a $(\Sigma_1^2)^{\text{Hom}_\infty}$ property.

Example

“All reals are in a canonical inner model” is $\forall^{\mathbb{R}}(\Sigma_1^2)^{\text{Hom}_\infty}$.

Remark

- ▶ Forcing over a canonical inner model to add a Cohen real makes this statement go from true to false.
- ▶ Forcing over L to add a Cohen real makes the $\forall^{\mathbb{R}}\Sigma_2^1$ ($= \Pi_3^1$) statement “all reals are constructible” go from true to false.

Generic absoluteness for $\forall^{\mathbb{R}}(\Sigma_1^2)^{\text{Hom}_\infty}$ holds or fails according to the cases of the dichotomy theorem:

1. An uncountable $(\underline{\Sigma}_1^2)^{\text{Hom}_\infty}$ well-ordering gives a true $\forall^{\mathbb{R}}(\underline{\Sigma}_1^2)^{\text{Hom}_\infty}$ statement that is false after forcing with $\text{Col}(\omega, \mathbb{R})$.¹
2. An absolute complement \tilde{T} for the tree T for $(\Sigma_1^2)^{\text{Hom}_\infty}$ gives $\forall^{\mathbb{R}}(\Sigma_1^2)^{\text{Hom}_\infty}$ generic absoluteness, by the absoluteness of well-foundedness.

¹Correction added June 16, 2013: " $(\underline{\Sigma}_1^2)^{\text{Hom}_\infty}$ well-ordering" should be replaced here and elsewhere with " $(\underline{\Sigma}_1^2)^{\text{Hom}_\infty}$ -good well-ordering."

Corollary

Let κ be a measurable limit of Woodins. Then the following statements are equivalent:

- (a) In some $<\kappa$ -generic extension, $\forall^{\mathbb{R}} (\Sigma_1^2)^{\text{Hom}_{<\kappa}}$ generic absoluteness holds between further $<\kappa$ -generic extensions
- (b) In some $<\kappa$ -generic extension, the tree for $(\Sigma_1^2)^{\text{Hom}_{<\kappa}}$ is κ -absolutely complemented.
- (c) The derived model at κ satisfies “every Π_1^2 set is Suslin.”

Compare:

Theorem (Martin–Solovay + Woodin)

The following statements are equivalent.

- (a) Σ_3^1 generic absoluteness.
- (b) The Shoenfield tree for Σ_2^1 is absolutely complemented.
- (c) Every set has a sharp.

Remark

In this talk “generic absoluteness” always means *two-step* generic absoluteness.

We can get even more consistency strength (in the form of strong determinacy axioms) from the generic absoluteness provided by the following theorem:

Theorem (Woodin)

Assume there is a proper class of Woodin cardinals. If there is a supercompact cardinal δ , then there is a forcing extension in which

- ▶ The theory of $L(\text{Hom}_\infty, \mathbb{R})$ is generically absolute for further forcing extensions, and
- ▶ $L(\text{Hom}_\infty, \mathbb{R}) \models \text{AD} + \text{DC} + \text{“every set of reals is Suslin.”}$

Using the dichotomy, we can get a partial reversal:

Theorem (W.)

Let κ be a measurable limit of Woodin cardinals.

If the theory of $L(\text{Hom}_{<\kappa}, \mathbb{R})$ is $<\kappa$ -generically absolute, then $L(\text{Hom}_{<\kappa}, \mathbb{R}) \models \text{AD} + \text{DC} + \text{“every set of reals is Suslin.”}$

Remark

The proof uses a relativization of above results for $(\Sigma_1^2)^{\text{Hom}_{<\kappa}}$, and the equivalence of the following statements for the derived model $L(\text{Hom}^*, \mathbb{R}^*)$.

- ▶ Every set of reals is Suslin.
- ▶ Every $\Pi_1^2(A)$ set of reals is Suslin for every $A \in \text{Hom}^*$.

Remark

The partial reversal leaves a very big gap.

Ranked in increasing order of consistency strength:

- ▶ “there is a proper class of Woodins”
- ▶ “there is a measurable limit of Woodins”
- ▶ ZF + AD + DC + “every set of reals is Suslin”
- ▶ (very big gap)
- ▶ “there is a supercompact cardinal.”

Questions

- ▶ Is the measurability of κ really necessary for anything?
- ▶ To get κ -absolute complementation of the tree for $(\Sigma_1^2)^{\text{Hom} < \kappa}$ from $\forall^{\mathbb{R}}(\Sigma_1^2)^{\text{Hom} < \kappa}$ generic absoluteness, must we go to a forcing extension or does it hold already in V ?
- ▶ How much more consistency strength can we get from generic absoluteness of the theory of $L(\text{Hom}_{< \kappa}, \mathbb{R})$?