Scales on Π^2_1 sets

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Notation

ω means $\mathbb{N}$.

It is the least infinite ordinal number.

Notation

$\mathcal{N}$ means the Baire space: ω^ω with the product of the discrete topologies on ω.

- It is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ with the subspace topology.
- Sometimes we call elements of $\mathcal{N}$ (or even $\mathcal{N}^m$) "reals".

Definition

$A \subset \mathcal{N}^m$ is κ-Suslin if $A = p[T]$ for some tree T on $\omega^m \times \kappa$.

- $[T]$ is the set of branches of T
- $p[T]$ is its projection, the set of $x \in \mathcal{N}^m$ such that $(x, f) \in [T]$ for some sequence $f \in \kappa^\omega$.

$A \subset \mathcal{N}^m$ is Suslin if it is κ-Suslin for some ordinal κ.

Example

Consider $\kappa = \omega$. The set $A \subset \mathcal{N}^{m+1}$ is $[T]$ for some tree T on $\omega^m \times \omega$ if and only if it is a closed set.
The ω-Suslin subsets of $\mathcal{N}^m$ are the coordinate projections of closed sets, which are called *analytic* (or $\utilde{\boldsymbol{\Sigma}}^1_1$.)

- The Axiom of Choice, AC, implies that every set of reals is Suslin.
- If CH also holds, then every set of reals is ω_1-Suslin (ω_1 is the least uncountable ordinal number.)

To make it interesting, we either:

- Require the trees to be "definable" in some sense, or
- Assume the Axiom of Determinacy, which contradicts AC.

The Axiom of Determinacy (AD) says that for any $B \subset \mathcal{N}$ if we consider the game $\mathcal{G}_B$,

$$\begin{array}{c|ccccccc} \text{I} & n_0 & & n_2 & & \ldots & \\ \text{II} & & n_1 & & n_3 & & \ldots, \end{array}$$

where II wins if and only if $(n_0, n_1, n_2, \ldots) \in B$, one player or the other has a winning strategy.

- This holds for finite-length games.
- For infinite-length games it contradicts AC.

We consider sets beyond $\mathbf{\Sigma}^1_1$:

Projective hierarchy

$$
\begin{array}{ccccccc}
\mathbf{\Sigma}^1_0 & & \mathbf{\Sigma}^1_1 & & \mathbf{\Sigma}^1_2 & & \cdots \\
& \subset & & \subset & & \subset & \\
\mathbf{\Pi}^1_0 & & \mathbf{\Pi}^1_1 & & \mathbf{\Pi}^1_2 & & \cdots
\end{array}
$$

- $\mathbf{\Sigma}^1_0$ sets are open subsets of $\mathcal{N}^m$.
- $\mathbf{\Pi}^1_n$ sets are complements of $\mathbf{\Sigma}^1_n$ sets.
- $\mathbf{\Sigma}^1_{n+1}$ sets are coordinate projections of $\mathbf{\Pi}^1_n$ sets.

There are "lightface" versions—*e.g.* Π^1_0 sets are $[T]$ for T a *computable* tree ("effectively closed" sets.)

Theorem (Shoenfield)

Every Σ^1_2 set $A \subset \mathcal{N}^m$ is ω_1-Suslin.
Moreover A is the projection of a tree T on $\omega^m \times \omega_1$ such that

- T is definable, and
- T is in the constructible universe L.

Remark

The set of reals

$$\{x \in \mathcal{N} : x \notin L\}$$

is Π^1_2 but is not the projection of a tree $T \in L$ (unless empty.)
One cannot prove in ZFC that it is the projection of any *definable* tree.

With assumptions beyond ZFC, one *can* prove that Π^1_2 sets are also definably Suslin.

Theorem (Martin–Solovay)

If there is a measurable cardinal, then every Π^1_2 set $A \subset \mathcal{N}^m$ is the projection of a definable tree on $\omega^m \times \kappa$.

- The assumption is equivalent to the existence of a countably complete (countably additive) two-valued measure defined on *all* subsets of some set X.
- (Motivated by real analysis but in set theory we consider $\{0, 1\}$-valued measures rather than real-valued measures.)
- AD implies there are many such measures.

Remarks on the proof of the Martin–Solovay theorem:

- We start with a tree T projecting to a Σ^1_2 set and get a tree $\tilde{T}$ projecting to its complement.
- $x \in p[T]$ is witnessed by a branch of the tree $T_x = \{s \in \mathrm{Ord}^{<\omega} : (x \restriction \mathrm{lh}(s), s) \in T\}$.
- What sequence of ordinals witnesses $x \notin p[T]$?
- The rank function on T_x is an ordinal-valued function that we "integrate" using various measures to get ordinals.
- The system of measures that makes this work is called a *weak homogeneity system* for T.

Definition

Let $A \subset \mathcal{N}^m$.

- A norm on A is a function $\varphi : A \to \mathrm{Ord}$.
- A semi-scale on A is a sequence $(\varphi_i : i < \omega)$ of norms on A such that if $\{x_k : k < \omega\} \subset A$, and as $k \to \omega$ we have $x_k \to x$ and each $\varphi_i(x_k)$ is eventually constant, then $x \in A$.

"A has a semi-scale" is weaker than "A is closed" just like "A is Suslin" is.

Fact

A has a semi-scale if and only if it is Suslin.

Definition

Let $A \subset \mathcal{N}^m$. A scale on A is a sequence $(\varphi_i : i < \omega)$ of norms on A such that if $\{x_k : k < \omega\} \subset A$, and as $k \to \omega$ we have $x_k \to x$ and each $\varphi_i(x_k)$ is eventually equal to λ_i, then $x \in A$ and $\varphi_i(x) \leq \lambda_i$ for all i.

The additional property is called *lower semi-continuity*.

Fact

A has a scale if and only if it has a semi-scale
(if and only if it is Suslin)

Scales give us optimal uniformization results:

- Suppose $A \subset \mathcal{N}^2$ and for every x there is a y with $(x, y) \in A$. How to choose y?
- If $\vec{\varphi}$ is a scale on A, then some y makes

$$(\varphi_0(x, y), y(0), \varphi_1(x, y), y(1), \ldots)$$

  lexicographically minimal.
- Let $f(x)$ denote this (unique) y.
- For all x we have $(x, f(x)) \in A$.
- The function $x \mapsto f(x) \restriction n$ is definable from $\vec{\varphi} \restriction n$.

Theorem (Novikov–Kondô–Addison)

Every Σ^1_2 set has a Σ^1_2 uniformization.

This theorem was later shown to be a corollary of:

Theorem (Moschovakis)

Every Σ^1_2 set has a Σ^1_2-scale.

The definition of "Σ^1_2-scale" is the one that makes Σ^1_2 uniformization work.

Now we consider Σ^2_1 (rather than Σ^1_2) sets.

- A Σ^2_1 formula $\varphi(x)$ says that there is a *set* of reals A such that the real x has a projective-in-A property.
- Work in $\mathsf{ZF} + \mathsf{AD}^+$, a natural strengthening of AD.

Theorem (Woodin, $\mathsf{ZF} + \mathsf{AD}^+$)

Every Σ^2_1 set is definably Suslin.
In fact, it has a Σ^2_1-scale, so it has a Σ^2_1 uniformization.

Under AD^+ there is an analogy between
Σ^2_1/ordinal-definability and Σ^1_2/constructibility.

Like Π^1_2 sets, Π^2_1 sets are harder to uniformize than their "Σ" counterparts.

- AD implies that for every x there is some real $y \notin OD_x$.
- Under AD^+ the set

$$\{(x, y) \in \mathcal{N}^2 : y \notin \text{OD}_x\}$$

is Π^2_1 but has no uniformization that is definable, or even ordinal-definable from a real:

$$f \in \text{OD}_x \implies f(x) \in \text{OD}_x\,.$$

- Getting scales on Π^1_2 sets required stronger hypothesis than getting scales on Σ^1_2 sets.
- Getting scales on Π^2_1 sets requires a hypothesis beyond AD^+.

In $L(\mathbb{R})$, every set is OD from a real.
Under AD, "some set of reals is not OD from a real" is equivalent to $\theta_0 < \Theta$.

- Θ is the least ordinal not the surjective image of $\mathbb{R}$
- θ_0 is the least ordinal not the surjective image of $\mathbb{R}$ by an OD function.

Theorem (Martin, essentially)

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \Theta$. Then every Π^2_1 set is Suslin. Moreover it has a semi-scale whose norms are ordinal-definable.

Proof is similar to Martin–Solovay:

- Start with tree for the complement (Σ^2_1 set)
- Show it's weakly homogeneous.
- Measures are OD by Kunen, so norms are OD.

For optimal uniformization, we want a *scale*.

Theorem (Jackson)

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \Theta$. Then every Π_1^2 set has a scale whose norms are OD from a real parameter.

Combining Jackson's ideas with Woodin's proof of Martin's theorem, we can obtain

Theorem (W.)

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \Theta$. Then every Π_1^2 set has a scale whose norms are OD.

Key Lemma

Assume $\mathrm{ZF} + \mathrm{AD}^+ + \theta_0 < \Theta$ and let $\kappa = \underset{\sim}{\delta}_1^2$. Given

- A tree T on $\omega \times \kappa$, and
- A measure μ on $\kappa^{<\omega}$,

there is a countable set of measures σ on $\kappa^{<\omega}$ such that for any real x and any continuous witness H to the ill-foundedness of towers from σ concentrating on T_x, $[\mathrm{rank}_{T_x}]_\mu \leq H(\mu)$. This is lower semi-continuity for the norm $\varphi_\mu(x) = [\mathrm{rank}_{T_x}]_\mu$.

- Jackson intersected T with sets of measure 1, introducing a real parameter.
- For the typical σ we can show that this is not necessary.

Corollary

The set $\{(x, y) \in \mathcal{N}^2 : y \notin \mathrm{OD}_x\}$ has a uniformization f such that the first n digits, $f(x) \restriction n$, are OD uniformly in x.

This seems in a sense optimal.