Trees and generic absoluteness

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The method of *forcing* in set theory was developed by Paul Cohen in the 1960s to prove (among other things) the independence of the Continuum Hypothesis, $|\mathbb{R}| = \aleph_1$.

- Gödel had already showed that CH is consistent: it holds in the constructible universe L.
- Cohen used the forcing method to produce a model of set theory where CH fails. For example, $|\mathbb{R}|$ can be $\aleph_2$.

Forcing can be used to prove the independence of many propositions in set theory, but this talk focuses on the limitations of the method.

Assuming that we have a model M of ZFC, the forcing method adjoins a new element g to get a larger model $M[g]$ of ZFC, called a *generic extension* of M.

Remark

There is a superficial analogy with field extensions:

- Starting with the field $\mathbb{Q}$ we can adjoin, for example, a square root of 2 to get the field $\mathbb{Q}[\sqrt{2}]$.
- There are limitations on what we can change; for example, we can't change the characteristic of the field.
- Similar limitations apply to forcing.

Definition

A statement φ is generically absolute if its truth is invariant under forcing:

$$V \models \varphi \iff V[g] \models \varphi,$$

for every generic extension $V[g]$.

Example

Arithmetic statements—first-order properties of the structure $(\mathbb{N}; 0, 1, +, \times)$—are generically absolute because natural numbers cannot be added by forcing.

Example

Con(ZFC) is an arithmetic statement (Π^0_1) so it is generically absolute. By Gödel's second incompleteness theorem it is independent of ZFC, but this cannot be proved by forcing.

Example

The twin prime conjecture is an arithmetic statement (Π^0_2) so if it is independent of ZFC—which seems unlikely—this cannot be proved by forcing.

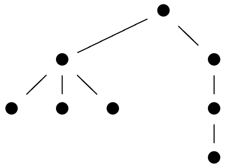
Many open problems from outside of set theory can be phrased as arithmetic statements, eliminating the possiblility of independence proofs using forcing.

Generic absoluteness for some statements beyond arithmetic can be proved using trees.

Definition

A tree (of height $\leq \omega$) is a partial ordering with a maximal element, such that the elements above any given element form a finite chain.

Example

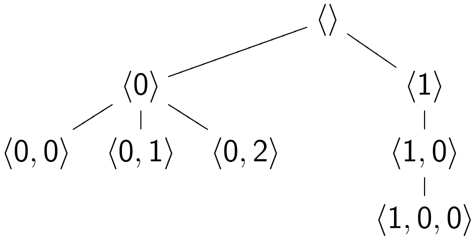


Definition

For a set X, a **tree on X** is a set of finite sequences from X, closed under initial segments and ordered by reverse inclusion.

Example

A tree on the set $X = \{0, 1, 2\}$:



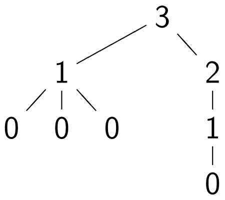
Any tree T is isomorphic to a tree on a cardinal number κ, namely the maximum number of children of any node of T.

Definition

A tree is well-founded if every subset has a minimal element.

Well-foundedness is equivalent to existence of a *rank function* assigning to each node an ordinal rank that is greater than the ranks of its children.

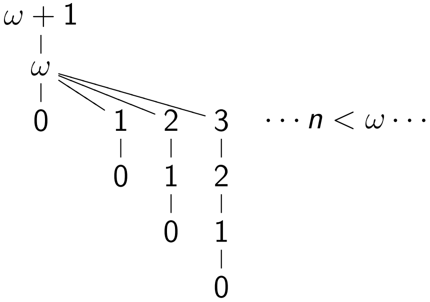
Example



Remark

For infinitely branching well-founded trees the values of the rank function may be transfinite ordinals.

Example

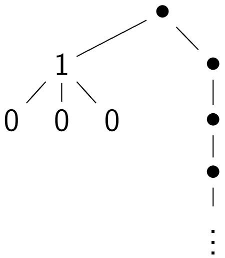


Definition

A tree is ill-founded if it is not well-founded.

Ill-foundedness is equivalent to having an *infinite branch*.

Example



Trees provide an important class of examples of generically absolute statements.

Example

For a tree T the statement "T is well-founded" (or "T is ill-founded") is generically absolute:

- If T is well-founded, then any rank function for T in V is also a rank function for T in any generic extension $V[g]$.
- If T is ill-founded, then any branch of T in V is also a branch of T in any generic extension $V[g]$.

Some statements beyond arithmetic can be shown to be generically absolute by considering appropriate trees.

Definition

Let $p \in \omega^\omega$ be a real parameter.

- A $\Sigma^1_1(p)$ statement is one of the form

$$\exists x \in \omega^\omega \, \theta(x, p)$$

  where θ is an arithmetic formula.
- Σ^1_1 means $\Sigma^1_1(p)$ for trivial $p = \langle 0, 0, 0, \ldots \rangle$.
- $\underset{\sim}{\mathbf{\Sigma}}^1_1$ means $\Sigma^1_1(p)$ for some $p \in \omega^\omega$.

Remark

Without loss of generality θ may be taken to be Π^0_1.

Given a $\underset{\sim}{\mathbf{\Sigma}}^1_1$ statement φ:

$$\exists x \in \omega^\omega \, \theta(x, p)$$

where θ is a Π^0_1 formula, there is a tree T of finite attempts to build a real $x \in \omega^\omega$ satisfying $\theta(x, p)$.

- T is ill-founded (has an infinite branch x) if and only if the $\underset{\sim}{\mathbf{\Sigma}}^1_1$ statement φ holds.
- Therefore $\underset{\sim}{\mathbf{\Sigma}}^1_1$ statements are generically absolute.

(Due to Mostowski, and holds for all transitive models.)

At the next step of complexity we reach the limit of generic absoluteness that is provable in ZFC.

Definition

A $\underset{\sim}{\mathbf{\Sigma}}^1_2$ statement is one of the form

$$\exists y \in \omega^\omega \, \forall x \in \omega^\omega \, \theta(x, y, p)$$

where θ is an arithmetic formula and p is a real parameter.

Theorem (Shoenfield)

$\underset{\sim}{\mathbf{\Sigma}}^1_2$ statements are generically absolute.

- The theorem is proved by reducing the truth of $\underset{\sim}{\mathbf{\Sigma}}^1_2$ statements to the ill-foundedness of trees on $\aleph_1$.
- Most statements encountered in real analysis can be phrased as $\underset{\sim}{\mathbf{\Sigma}}^1_2$ statements.

Definition

A $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$ statement is one of the form

$$\exists z \in \omega^\omega \, \forall y \in \omega^\omega \, \exists x \in \omega^\omega \, \theta(x, y, z, p)$$

where θ is an arithmetic formula and p is a real parameter.

Remark

The statement

- "there is a non-constructible real"

is a Σ^1_3 statement that can fail to be generically absolute: If $V = L$ then it is false, but it is true in any generic extension that adds a real.

Remark

Although Σ^1_3 generic absoluteness may fail in V, it always holds in some generic extension of V.

Next we consider $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$ generic absoluteness. We can allow real parameters from V or from generic extensions of V.

Definition

- One-step generic absoluteness for $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$ says that every $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$ statement (with parameter $p \in V$) is generically absolute.
- Two-step generic absoluteness for $\underset{\sim}{\boldsymbol{\Sigma}}^1_3$ says that one-step generic absoluteness holds in every generic extension.

To show these principles are consistent we need assumptions beyond ZFC, such as large cardinals (strong axioms of infinity.)

Theorem (Feng–Magidor–Woodin)

The following statements are equiconsistent.

1. There is a Σ_2-reflecting cardinal.
2. One-step $\boldsymbol{\Sigma}^1_3$ generic absoluteness holds.

Theorem (Martin–Solovay $\Rightarrow$, Woodin $\Leftarrow$)

The following statements are *equivalent*:

1. Every set has a sharp.
2. Two-step $\boldsymbol{\Sigma}^1_3$ generic absoluteness holds.

Generic absoluteness is also related to the extent of the *universally Baire* sets of reals.

Definition

A set of reals $A \subset \omega^\omega$ is universally Baire if there are continuous functions $x \mapsto T_x$ and $x \mapsto \tilde{T}_x$ such that

- For every real x in V,

$$\begin{aligned} x \in A &\iff T_x \text{ is ill-founded} \\ &\iff \tilde{T}_x \text{ is well-founded, and} \end{aligned}$$

- For every real x in every generic extension of V,

$$T_x \text{ is ill-founded} \iff \tilde{T}_x \text{ is well-founded.}$$

Example

Every $\underset{\sim}{\mathbf{\Sigma}}^1_1$ set of reals A is universally Baire.

Remark

Universally Baire sets have many regularity properties, *e.g.*

- the Baire property,
- Lebesgue measurability, and
- the Bernstein property.

Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

- One-step $\utilde{\mathbf{\Sigma}}^1_3$ generic absoluteness holds.
- Every $\utilde{\mathbf{\Delta}}^1_2$ set of reals is universally Baire.

The following statements are equivalent:

- Two-step $\utilde{\mathbf{\Sigma}}^1_3$ generic absoluteness holds.
- Every $\utilde{\mathbf{\Sigma}}^1_2$ set of reals is universally Baire.

Next we consider an similar situation, but higher up in terms of large cardinals and descriptive complexity. Assume:

- There is a cardinal λ that is a limit of Woodin cardinals.

We define a local version of "universally Baire":

Definition

Let A be a set of reals and let λ be a cardinal. Then A is λ-universally Baire, written $A \in \mathrm{uB}_\lambda$, if it is universally Baire with respect to generic extensions by posets of size $< \lambda$.

Definition

A $(\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ statement is one of the form

$$\exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \theta(p).$$

A $\exists^{\mathbb{R}}(\underset{\sim}{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ statement is one of the form

$$\exists x \in \omega^\omega\, \forall B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \theta(x, p).$$

Here θ is any formula and p is a real parameter.

Analogy:

- $\underset{\sim}{\Sigma}^1_2 \rightsquigarrow (\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$
- $\underset{\sim}{\Pi}^1_2 \rightsquigarrow (\underset{\sim}{\Pi}^2_1)^{\mathrm{uB}_\lambda}$
- $\underset{\sim}{\Sigma}^1_3 = \exists^{\mathbb{R}}\, \underset{\sim}{\Pi}^1_2 \rightsquigarrow \exists^{\mathbb{R}}(\underset{\sim}{\Pi}^2_1)^{\mathrm{uB}_\lambda}$

An analogue of Shoenfield absoluteness for $\underset{\sim}{\Sigma}^1_2$:

Theorem (Woodin)

If λ is a limit of Woodin cardinals, then $(\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ statements are generically absolute with respect to generic extensions by posets of size $< \lambda$.

Remark

As before, this generic absoluteness can be explained in terms of a continuous reduction to ill-foundedness of trees.

Generic absoluteness for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ is different:

- It is not known to follow from *any* large cardinal hypothesis.
- Many large cardinals can be accomodated in "L-like" inner models where $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness fails, much as Σ^1_3 generic absoluteness fails in L.

Questions

How is generic absoluteness for $\exists^{\mathbb{R}}(\underset{\sim}{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ related to:

- The extent (or closure properties) of the pointclass of λ-universally Baire sets?
- Large cardinals?
- The absoluteness of well-foundedness for trees?

One-step generic absoluteness for $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ is related to closure properties of the uB_λ sets.

Proposition (W.)

For a limit λ of Woodin cardinals, the following statements are equivalent:

1. One-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size $< \lambda$.
2. Every $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.

What is its consistency strength?

Proposition (W.)

Con(1) $\implies$ Con(2), where

1. There is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is Σ_2-reflecting in V_λ.
2. There is a limit λ of Woodin cardinals such that one-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size $< \lambda$.

Open Question 1

Con(2) $\implies$ Con(1)?

Two-step generic absoluteness for $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ is also related to closure properties of the uB_λ sets.

Proposition (W.)

Let λ be a limit of Woodin cardinals. Then (1) $\implies$ (2), where

1. Every $(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.
2. Two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size $< \lambda$.

Open Question 2

(2) $\implies$ (1)?

What is the consistency strength of two-step absoluteness?

Theorem (Woodin)

$\mathrm{Con}(1) \implies \mathrm{Con}(2)$, where

1. There is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is $<\lambda$-strong.
2. There is a limit λ of Woodin cardinals such that two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size $< \lambda$.

Remark

If the answer to Open Question 2 is "yes" then $\mathrm{Con}(2) \implies \mathrm{Con}(1)$.

A partial "yes" answer to Open Question 2:

Proposition (W.)

Let λ be a measurable cardinal that is a limit of Woodin cardinals. If two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size $< \lambda$, then some such generic extension satisfies "every $(\utilde{\mathbf{\Sigma}}^2_1)^{\mathsf{uB}_\lambda}$ set of reals is λ-universally Baire."

From this we get something close to an equiconsistency involving two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness.

Theorem (W.)

Con(2') $\implies$ Con(1), where

1. There is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is $<\lambda$-strong.

2'. There is a measurable cardinal λ that is a limit of Woodin cardinals, and such that two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size $< \lambda$.