Determinacy models and good scales at singular cardinals

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Remark

After submitting the title and abstract for this talk, I noticed that the hypothesis of determinacy could be weakened to countable choice for reals, and the conclusion of the existence of good scales could be strengthened in various ways. A better title for the talk would be:

> *Countable choice and combinatorial incompactness principles at singular cardinals.*

The material about (very) good scales is in an appendix.

The ordinal numbers

$$0, 1, 2, 3 \ldots, \omega, \omega + 1, \ldots, \omega + \omega, \omega + \omega + 1, \ldots$$

measure the lengths of well-orderings.

Example

- ω is the order type of the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- $\omega + 1$ is the order type of the set $\{0, 1/2, 3/4, 7/8, \ldots 1\}$.
- $\omega + \omega$ is the order type of the set $\{0, 1/2, 3/4, 7/8, \ldots 1, 2, 3, \ldots\}$.
- $\omega + \omega + 1$ is the order type of the set $\{0, 1/2, 3/4, 7/8, \ldots 1, 1 + 1/2, 1 + 3/4, 1 + 7/8, \ldots 2\}$.

At some point we run out of room in $\mathbb{R}$ (but we can still represent ordinals by well-orderings of more general sets.)

Definition

ω_1 is the least uncountable ordinal.

- ω_1 is a *cardinal*: it does not admit a bijection with any smaller ordinal.
- $\omega_1 = \omega^+$ (*cardinal successor*.)

Definition

$\omega_2 = \omega_1^+$, $\omega_3 = \omega_2^+$, ...

Notation

ω_n is also called $\aleph_n$, the n^{th} uncountable cardinal.

Definition

At the first limit step in the $\aleph$-sequence, define:

- $\aleph_\omega = \sup_{n<\omega} \aleph_n$. Equivalently,
- $\aleph_\omega = \left|\bigcup_{n\in\mathbb{N}} S_n\right|$ where $|S_n| = \aleph_n$.

Remark

We can go further: $\aleph_{\omega+1} = \aleph_\omega^+$, $\aleph_{\omega+2} = \aleph_{\omega+1}^+$, ...

Definition

A cardinal κ is:

- regular if $\sup_{i<\xi} \kappa_i < \kappa$ whenever $\xi < \kappa$ and $\kappa_i < \kappa$
- singular if it is not regular
- countable cofinality if $\kappa = \sup_{i<\omega} \kappa_i$ where $\kappa_i < \kappa$

Example

- $\aleph_0$ is regular: a finite union of finite sets is finite
- $\aleph_1$ is regular: a countable union of countable sets is countable, assuming the Axiom of Countable Choice
- $\aleph_2$, $\aleph_3$,... are regular, assuming AC
- $\aleph_\omega$ is singular of countable cofinality

Remark

- If κ is a singular cardinal of countable cofinality, then κ is a countable union of sets of size $< \kappa$.
- If $\alpha < \kappa^+$ then $|\alpha| \leq \kappa$, so α is also a countable union of sets of size $< \kappa$. The following definition records this.

Definition (Viale[1])

Let κ be a singular cardinal of countable cofinality.
A covering matrix for κ^+ assigns to each ordinal $\alpha < \kappa^+$ an increasing sequence of subsets $(K_\alpha(i) : i \in \mathbb{N})$ such that

- $\alpha = \bigcup_{i \in \mathbb{N}} K_\alpha(i)$
- $|K_\alpha(i)| < \kappa$ for all $i \in \mathbb{N}$.

[1] Note added Nov. 18, 2014: I have been informed that definitions similar to this one and the next one were considered previously by Jensen.

Definition (Viale)

Let κ be a singular cardinal of countable cofinality. A covering matrix $(K_\alpha(i) : \alpha < \kappa^+, i \in \mathbb{N})$ for κ^+ is coherent if whenever $\alpha < \beta < \kappa^+$ the sequences of subsets of α given by

- $(K_\alpha(i) : i \in \mathbb{N})$ and
- $(K_\beta(i) \cap \alpha : i \in \mathbb{N})$

are *cofinally interleaved* with respect to inclusion: every set in one sequence is contained in some set in the other sequence.

Remark

The existence of coherent covering matrices for successors of singular cardinals is independent of ZFC!

We consider different models of ZFC:

Example

- Gödel's constructible universe L is "thin." It only contains the sets that "need to exist."
- Models of the *Proper Forcing Axiom* PFA are very "fat."

They have different properties:

Example

- $V = L$ implies $|\mathbb{R}| = \aleph_1$.
- PFA implies $|\mathbb{R}| = \aleph_2$.

Remark

$V = L$ and PFA have opposite combinatorial effects:

- $V = L$ implies incompactness principles such as $\square_\kappa$.
- PFA implies compactness principles such as $\neg\square_\kappa$.

The existence of a coherent covering matrix is a kind of incompactness principle, and in fact we have:

Theorem (Viale)

Let κ be a singular cardinal of countable cofinality.

- $V = L$ implies there is a coherent covering matrix for κ^+.
- PFA implies there is no coherent covering matrix for κ^+.

Besides the $V = L$ construction, we have this one:

Theorem (Viale)

Let κ be a singular cardinal of countable cofinality. If there is an inner model W such that

1. $(\kappa^+)^W = \kappa^+$, and
2. κ is regular in W,

then there is a coherent covering matrix for κ^+.

Remark

The hypothesis is consistent: it can be obtained from a measurable cardinal κ by Prikry forcing.

Sketch of proof

- $(\kappa^+)^W = \kappa^+$ so for every $\alpha < \kappa^+$ there is a surjection $\pi_\alpha : \kappa \to \alpha$ in W
- If $\alpha < \beta < \kappa^+$ then the sequences
    - $(\pi_\alpha[\xi] : \xi < \kappa)$ and
    - $(\pi_\beta[\xi] \cap \alpha : \xi < \kappa)$

  are cofinally interleaved because π_α and π_β live in a model W where κ is regular
- κ has countable cofinality, say $\kappa = \sup_{i \in \mathbb{N}} \kappa_i$
- Define the covering matrix: $K_\alpha(i) = \pi_\alpha[\kappa_i]$

Combining these two theorems:

Corollary

Let κ be a singular cardinal of countable cofinality that is regular in an inner model W. If PFA holds, then $(\kappa^+)^W < \kappa^+$.

Remark

- This also follows from work of Cummings–Schimmerling and Džamonja–Shelah, using the square principle $\square_\kappa^\omega$ instead of coherent covering matrices.
- The relationship between coherent covering matrices and $\square_\kappa^\omega$ is not clear to me.

Let's consider the regularity or singularity of ω_1 instead of κ.

Definition

The *Axiom of Countable Choice* says that whenever $(S_i : i \in \mathbb{N})$ is a sequence of nonempty sets, there is a sequence $(x_i : i \in \mathbb{N})$ such that $x_i \in S_i$ for all $i \in \mathbb{N}$.

Definition

The *Axiom of Countable Choice for Reals* ($\mathsf{CC}_{\mathbb{R}}$) is the special case where the sets S_i are sets of reals.

Remark

$\mathsf{CC}_{\mathbb{R}}$ implies that ω_1 is regular.

Given a singular strong limit cardinal of countable cofinality, say $\kappa = \sup_{i\in\mathbb{N}} \kappa_i$, we can obtain a model where $\mathsf{CC}_{\mathbb{R}}$ fails:

Definition

Force with the Levy collapse $\mathrm{Col}(\omega, {<}\kappa)$ to get a V-generic filter G and define:

- the *symmetric reals* $\mathbb{R}^*_G = \bigcup_{\xi<\kappa} \mathbb{R}^{V[G\restriction\xi]}$, and
- the *symmetric extension* $V(\mathbb{R}^*_G)$.

In the symmetric extension every ordinal $\xi < \kappa$ is collapsed to be countable but κ itself is not collapsed:

- $\omega_1^{V(\mathbb{R}^*_G)} = \kappa$
- $V(\mathbb{R}^*_G)$ satisfies "ω_1 is singular"
- $\mathsf{CC}_{\mathbb{R}}$ fails in $V(\mathbb{R}^*_G)$.

Remark

- We obtained a coherent covering matrix from an inner model W of V that was big enough to compute κ^+ as a successor, but not so big that κ was singular.
- Now consider an inner model W of $V(\mathbb{R}^*_G)$ that is big enough to compute κ^+ as a successor (in the following sense) but not so big that $\mathsf{CC}_{\mathbb{R}}$ fails.

Definition

θ_0 is the least ordinal α such that there is no ordinal-definable surjection π_α from the reals (here $\mathbb{R}^*_G$) onto α.

Proposition (W.)

Levy collapse a singular strong limit cardinal κ of countable cofinality to get a generic filter G. If there is a definable inner model W of $V(\mathbb{R}^*_G)$ containing $\mathbb{R}^*_G$ and such that

1. $(\theta_0)^W = \kappa^+$, and
2. countable choice for reals holds in W,

then there is a coherent covering matrix for κ^+ in V.

Sketch of proof

- For $\alpha < \kappa^+$ take the least OD^W surjection $\pi_\alpha : \mathbb{R}^*_G \to \alpha$
- If $\kappa = \sup_{i \in \mathbb{N}} \kappa_i$ then $\alpha = \bigcup_{i \in \mathbb{N}} \pi_\alpha[\mathbb{R}^{V[G \restriction \kappa_i]}]$
- Coherence follows from countable choice for reals

- As before we can obtain the hypothesis by Prikry forcing starting from a measurable cardinal κ.
- But now there is a more interesting way to obtain an inner model W of $V(\mathbb{R}^*_G)$ satisfying $\mathsf{CC}_{\mathbb{R}}$:
- If κ is a limit of *Woodin cardinals*, then $L(\mathbb{R}^*_G)$ satisfies AD, the Axiom of Determinacy, which implies $\mathsf{CC}_{\mathbb{R}}$.

More generally:

Definition

Let κ be a limit of Woodin cardinals. The derived model of V at κ by a generic filter G for $\mathrm{Col}(\omega, {<}\kappa)$ is (approximately) the largest inner model of $V(\mathbb{R}^*_G)$ satisfying AD.

$\mathsf{CC}_{\mathbb{R}}$ follows from AD, so it holds in derived models. Therefore we have:

Proposition (W.)

Let κ be a singular limit of Woodin cardinals of countable cofinality and let $D(V, \kappa)$ be a derived model of V at κ. If

$$(\theta_0)^{D(V,\kappa)} = \kappa^+,$$

then there is a coherent covering matrix for κ^+ in V.

Remark

The hypothesis "$(\theta_0)^{D(V,\kappa)} = \kappa^+$" is consistent relative to the hypothesis "κ is a limit of Woodin cardinals."

Because PFA rules out coherent covering matrices, we get:

Theorem (W.)

Let κ be a singular limit of Woodin cardinals of countable cofinality and let $D(V, \kappa)$ be a derived model of V at κ. If PFA holds, then $(\theta_0)^{D(V,\kappa)} < \kappa^+$.

Remark

- If PFA holds, then $(\kappa^+)^{\mathrm{Lp}(A)} < \kappa^+$ for every $A \subset \kappa$ (*Lower part* mouse over A.)
- The relationship between canonical determinacy models $D(V, \kappa)$ and canonical large cardinal models $\mathrm{Lp}(A)$ is still being worked out. For now the proofs remain separate.

This result suggests the following family of conjectures, listed in increasing order of strength: (1) $\Leftarrow$ (2) $\Leftarrow$ (3).

Conjecture

Assume PFA and let κ be *any* limit of Woodin cardinals.

1. $(\theta_0)^{D(V,\kappa)} < \kappa^+$.
2. $D(V,\kappa) \models \theta_0 < \Theta$.
3. $V^{\mathrm{Col}(\omega,\omega_1)} \models (\Sigma^2_1)^{\mathrm{uB}_\kappa} \subset \mathrm{uB}_\kappa$.

Remark

It would be hard to find counterexamples: the conclusions hold after any forcing that collapses a supercompact cardinal to ω_2, and such a forcing is the only known way to get PFA.

Remark

Very good scales (another incompactness principle) can also be obtained from "very good" covering matrices.

Definition

Let κ be a singular cardinal of countable cofinality. A covering matrix $(K_\alpha(i) : \alpha < \kappa^+, i \in \mathbb{N})$ for κ^+ is very good if, for every $\gamma < \kappa^+$ of uncountable cofinality, there is a club $C \subset \gamma$ and an $i \in \mathbb{N}$ such that for all ordinals $\alpha, \beta \in C$,

$$\alpha < \beta \implies \alpha \in K_\beta(i).$$

(We always have $\forall \alpha, \beta < \kappa^+ \, \exists i \in \mathbb{N} \, \alpha < \beta \implies \alpha \in K_\beta(i)$; "very goodness" says we can switch the quantifiers on a club.)

Proposition

Let κ be a singular cardinal of countable cofinality. If there is an inner model W such that

1. $(\kappa^+)^W = \kappa^+$, and
2. κ is regular in W,

then the coherent covering matrix defined above is very good.

- If $\mathrm{cf}(\gamma) > \omega$ then there is a club $C \subset \gamma$ in W of size $< \kappa$.
- C witnesses "very goodness" because κ is regular in W.

Remark

We can use this result to get a very good scale of length κ^+.

- If there is such an inner model W we already know $\square_\kappa^\omega$ holds, which implies that very good scales exist.
- But for inner models of $V(\mathbb{R}^*_G)$ satisfying $\mathsf{CC}_\mathbb{R}$, and in particular derived models, we get a new result:

Theorem (W.)

Let κ be a singular limit of Woodin cardinals of countable cofinality and let $D(V,\kappa)$ be a derived model of V at κ. If $(\theta_0)^{D(V,\kappa)} = \kappa^+$, then there is a very good scale of length κ^+.

Conjecture

This conclusion can be strengthened to $\square_\kappa^\omega$.