A determinacy transfer principle

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In descriptive set theory, we study sets of “real” numbers in terms of their complexity.
Instead of the complete ordered field $\mathbb{R}$, we use the Baire space $\mathcal{N} = \omega^\omega$ with the product of the discrete topologies on ω.

- This is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$
- We refer to elements of $\mathcal{N}$ as “reals”
- Any finite product $\mathcal{X}$ of copies of $\mathcal{N}$ and ω is homeomorphic to $\mathcal{N}$ (or ω)
- We refer to elements of $\mathcal{X}$ also as “reals”

A *pointclass* Γ is a collection of sets of reals, typically corresponding to a degree of complexity.

Example

- The closed sets of reals
- The analytic sets of reals (projections of closed sets)
- The inductive sets of reals
- The sets of reals in $L(\mathbb{R})$, the smallest transitive model of set theory containing the reals and ordinals

Our main example today is the (absolutely) inductive sets

$$\Gamma = \mathrm{IND}\,.$$

- This is the pointclass of sets definable by positive elementary induction over the reals.
- Equivalently, it is the pointclass of sets Σ_1-definable over the least admissible level $L_\kappa(\mathbb{R})$ of $L(\mathbb{R})$.

Notation

For a pointclass Γ we let

$$\check{\Gamma} = \{\neg A : A \in \Gamma\} \qquad \text{(dual pointclass)}$$
$$\Delta = \Gamma \cap \check{\Gamma} \qquad \text{(ambiguous part)}$$

- If Γ is IND then Δ is HYP, the (absolutely) hyperprojective sets.
- We get $\underset{\sim}{\boldsymbol{\Gamma}}$, $\underset{\sim}{\check{\boldsymbol{\Gamma}}}$, and $\underset{\sim}{\boldsymbol{\Delta}}$ by allowing arbitrary real parameters.

Example

If $\Gamma = \text{IND} = \Sigma_1^{L_\kappa(\mathbb{R})}$ then $\underset{\sim}{\boldsymbol{\Delta}}$ consists of all sets of reals *in* the least admissible level $L_\kappa(\mathbb{R})$ of $L(\mathbb{R})$.

Definition

We say a pointclass Γ is inductive-like if it has some nice closure properties, including closure under real quantification (but not negation), and it has the *pre-wellordering property*.

The pre-wellordering property says that every set $A \in \Gamma$ has a Γ-norm $\varphi : A \to \mathrm{Ord}$; roughly, the *approximations* $A_\alpha = \{x \in A : \varphi(x) \leq \alpha\}$ to A are "uniformly $\boldsymbol{\Delta}$."

Example

$\Gamma = \mathrm{IND} = \Sigma_1^{L_\kappa(\mathbb{R})}$ is inductive-like. For the pre-wellordering property let $\varphi(x)$ be the level $\alpha < \kappa$ of the first witness to the Σ_1 fact about x.

Here is a more general way of approximating a set of reals by simpler sets of reals:

Definition (Martin)

For a sequence of sets of reals $\mathcal{S} = (A_\alpha : \alpha < \kappa)$, we say $A \in \overline{\mathcal{S}}$ if for every countable set of reals $\mathcal{I}$, some $A_\alpha \cap \mathcal{I}$ is equal to $A \cap \mathcal{I}$.

Example

If $A \in \Gamma$ has a Γ-norm $A \to \kappa$ then $A \in \overline{\mathcal{S}}$ for some κ-sequence $\mathcal{S}$ of $\underset{\sim}{\boldsymbol{\Delta}}$ sets. (This uses that κ has uncountable cofinality.)

Definition

We say a sequence $(A_\alpha : \alpha < \kappa)$ of sets of reals is uniformly Γ if for every Γ-norm φ on a complete Γ set U,

$$\{(x, y) : y \in U \ \& \ x \in A_{\varphi(y)}\} \in \Gamma.$$

In particular, each A_α is in $\utilde{\Gamma}$.

Remark

The *Axiom of Determinacy* implies *any* sequence $(A_\alpha : \alpha < \kappa)$ of $\utilde{\Gamma}$ sets is uniformly $\utilde{\Gamma}$, by Moschovakis's coding lemma.

Definition

Let Γ be inductive-like. The envelope of Γ, denoted by $\text{Env}(\Gamma)$, consists of sets of reals A such that $A \in \overline{\mathcal{S}}$ for some uniformly Γ sequence $\mathcal{S} = (A_\alpha : \alpha < \kappa)$ such that $(\neg A_\alpha : \alpha < \kappa)$ is *also* uniformly Γ.

In particular, each A_α is in $\underset{\sim}{\boldsymbol{\Delta}}$.

Remark

The Axiom of Determinacy implies $\text{Env}(\underset{\sim}{\boldsymbol{\Gamma}})$ consists of the sets of reals A such that $A \in \overline{\mathcal{S}}$ for some sequence $\mathcal{S}$ of $\underset{\sim}{\boldsymbol{\Delta}}$ sets.

Under AD, our definition of $\text{Env}(\underset{\sim}{\boldsymbol{\Gamma}})$ is equivalent to Martin's original definition where uniformity is not explicitly required.

Remark

Without AD the sequence of $\underset{\sim}{\boldsymbol{\Delta}}$ sets can code too much information:

- Every countable set of reals is in $\underset{\sim}{\boldsymbol{\Delta}}$.
- If AC holds then any set of reals A is in $\overline{\mathcal{S}}$ where $\mathcal{S}$ is a sequence enumerating all countable sets of reals.

The "uniform" definition of $\text{Env}(\underset{\sim}{\boldsymbol{\Gamma}})$ seems to be the right one in the non-AD context.

What is the Axiom of Determinacy?

Definition

The Axiom of Determinacy, AD, states that for every set of reals A, one player or the other has a winning strategy in the game $\mathcal{G}_A$:

$$\begin{array}{c|ccccccc} \text{I} & x(0) & & x(1) & & \ldots & \\ \text{II} & & y(0) & & y(1) & & \ldots \end{array}$$

where Player I wins if the sequence $(x(0), y(0), x(1), y(1), \ldots)$ is in A and Player II wins otherwise.

AD contradicts AC, but large cardinals imply that "nice" sets of reals A are determined—that is, some player has a winning strategy in $\mathcal{G}_A$.

Example

- If there is a measurable cardinal, then the analytic sets are determined. (Martin)
- If there are n many Woodin cardinals below a measurable cardinal, then $\utilde{\boldsymbol{\Sigma}}^1_{n+1}$ sets are determined. (Martin–Steel)
- If there are ω many Woodin cardinals below a measurable cardinal, then every set of reals in $L(\mathbb{R})$ is determined. (Woodin)

Sometimes more large cardinals are *not* required to establish more determinacy. We call this determinacy transfer.

Theorem (Kechris–Woodin)

- If $\underset{\sim}{\mathbf{HYP}}$ sets (*i.e.* sets of reals in $L_\kappa(\mathbb{R})$) are determined, so are $\underset{\sim}{\mathbf{\Sigma}}^*_n$ sets (*i.e.* sets of reals in $L_{\kappa+1}(\mathbb{R})$).
- If all Suslin co-Suslin sets in $L(\mathbb{R})$ are determined, then all sets of reals in $L(\mathbb{R})$ are determined.

A set is Suslin if it is the projection of a tree on $\omega \times \kappa$ for some ordinal κ (generalizing *analytic* sets, where $\kappa = \omega$.)
A set is Suslin if and only if it has a scale, which is a kind of sequence $\vec{\varphi}$ of norms.

Generalizing the Kechris–Woodin argument, we can show

Theorem (W.)

Assume $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$. Let Γ be an inductive-like pointclass. If Δ is determined, then $\mathrm{Env}(\Gamma)$ is determined.

Remark

- We have $\Delta \subsetneq \Gamma \subsetneq \mathrm{Env}(\Gamma)$, so this is a determinacy transfer principle.
- Together with closure properties of the envelope due to Martin, and Steel's construction of scales in $L(\mathbb{R})$, it yields the Kechris–Woodin results.

Proof idea

- Suppose $A \in \mathrm{Env}(\Gamma)$ is not determined.
- By a Skolem hull argument we have many "locally non-determined" games on countable $\mathcal{I} \subset \mathbb{R}$.
- A is uniformly approximated by $\utilde{\boldsymbol{\Delta}}$ sets (in fact Δ in ordinal parameters.)
- Piece together the *least* "locally non-determined" games on various countable sets into a single non-determined game with payoff set in Δ, giving a contradiction.

Corollary

Let Γ be an inductive-like pointclass. If Δ is determined and $A, B \in \text{Env}(\Gamma)$ then $A = f^{-1}[B]$ or $B = f^{-1}[\neg A]$ for some continuous f (so A and B line up in the Wadge hierarchy.)

Proof.

Wadge's lemma applies. The game

$$\begin{array}{c|cccccc} \text{I} & x(0) & & x(1) & & \dots & \\ \text{II} & & y(0) & & y(1) & & \dots, \end{array}$$

where Player I wins if $x \in A \iff y \in B$, is determined because $\text{Env}(\Gamma)$ is determined and has some basic closure properties. □

We can use Wadge's lemma for sets in the envelope to give a simple proof of the following theorem.

Theorem (Woodin)

If M_1 and M_2 are transitive models of AD^+ containing the reals and ordinals and are divergent (neither $\mathcal{P}(\mathbb{R}) \cap M_1$ nor $\mathcal{P}(\mathbb{R}) \cap M_2$ is contained in the other) then the model

$$M_0 = L(\mathcal{P}(\mathbb{R}) \cap M_1 \cap M_2)$$

satisfies $\mathsf{AD}_{\mathbb{R}}$.

- AD^+ is a natural strengthening of AD that holds in all known models of AD
- $AD_{\mathbb{R}}$ is the Axiom of Determinacy for games on the reals. It has higher consistency strength than AD (and AD^+)

Remark

Under the same hypothesis Grigor Sargsyan has recently shown that the even *stronger* theory $AD_{\mathbb{R}}$ + "Θ is regular" holds in some submodel of the intersection model M_0.

In the remaining few slides we sketch a proof of Woodin's theorem.

- Real games are a red herring: for $\mathrm{AD}_{\mathbb{R}}$ it suffices to show that M_0 satisfies "every set of reals is Suslin."
- If not, it has a largest Suslin cardinal κ and the pointclass $\utilde{\boldsymbol{\Gamma}} = S(\kappa)$ is non-selfdual by Kechris.
- That is, some $\check{\utilde{\boldsymbol{\Gamma}}}$ set (co-Suslin in M_0 set) is not in $\utilde{\boldsymbol{\Gamma}}$ (is not Suslin in M_0.)
- $\utilde{\boldsymbol{\Gamma}}$ is boldface inductive-like. Consider its envelope $\mathrm{Env}(\utilde{\boldsymbol{\Gamma}})$.

There are no "divergent envelopes" so some model goes beyond the envelope:

- The statement "$A \in \mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})$" is absolute, so $\mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_1}, \mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_2} \subseteq \mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{V}$.
- Wadge's lemma applies, so one of $\mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_1}$ and $\mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_2}$ is contained in the other.
- Without loss of generality $\mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_1} \subseteq \mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_2}$.
- M_1 contains a set of reals not in $\mathrm{Env}(\underset{\sim}{\boldsymbol{\Gamma}})^{M_1}$—otherwise it could not diverge from M_2.

Finally, we use a well-known connection between scales and envelopes in the AD context:

- $M_1 \models \mathrm{Env}(\utilde{\boldsymbol{\Gamma}}) \neq \mathcal{P}(\mathbb{R})$ implies that every $\check{\utilde{\boldsymbol{\Gamma}}}$ set has a scale $\vec{\varphi}$ in M_1 whose norms φ_i are all in $\mathrm{Env}(\utilde{\boldsymbol{\Gamma}})^{M_1}$. (Martin)
- M_2 contains a set of reals above every norm φ_i in the Wadge hierarchy, so $\vec{\varphi}$ is in M_2 also.
- Therefore $\vec{\varphi}$ is in the intersection M_0, and our $\check{\utilde{\boldsymbol{\Gamma}}}$ set is Suslin in M_0.
- So $\check{\utilde{\boldsymbol{\Gamma}}} \subseteq \utilde{\boldsymbol{\Gamma}}$, contradicting that $\utilde{\boldsymbol{\Gamma}}$ is non-selfdual and proving Woodin's theorem.