Contributions to Descriptive Inner Model Theory

by

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Abstract

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Descriptive inner model theory is the study of connections between descriptive set theory and inner model theory. Such connections form the basis of the core model induction, which we use to prove relative consistency results relating strong forms of the Axiom of Determinacy with the existence of a strong ideal on $\wp_{\omega_1}(\mathbb{R})$ having a certain property related to homogeneity. The main innovation is a unified approach to the "gap in scales" step of the core model induction.

Contents

Introduction

We will prove the following relative consistency statements.

Main Theorem.

(1) *Assuming* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ + "Θ *is regular," there is a forcing extension where* ZFC *holds,*

  (a) *The nonstationary ideal* $\mathrm{NS}_{\omega_1,\mathbb{R}}$ *is strong and pseudo-homogeneous, and*

  (b) *There is a* $\mathfrak{c}$*-dense pseudo-homogeneous ideal on* $\wp_{\omega_1}(\mathbb{R})$.

(2) *Assuming* ZFC *and the existence of a strong pseudo-homogeneous ideal on* $\wp_{\omega_1}(\mathbb{R})$*, there is an inner model of* $\mathsf{ZF} + \mathsf{AD} + \theta_0 < \Theta$ *containing all the reals and ordinals.*

The theories $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" and $\mathsf{AD} + \theta_0 < \Theta$ are both natural strengthenings of AD, the Axiom of Determinacy. Strength is a property of ideals introduced in [**1**] that is intermediate between precipitousness and pre-saturation. Pseudo-homogeneity is a property of ideals introduced in Chapter 1 that is similar to homogeneity except that it pertains to the theory of the generic ultrapower rather than to that of the generic extension. The ideals in the conclusions of (1a) and (1b) both satisfy the hypothesis of (2), and in turn the model of $\mathsf{AD} + \theta_0 < \Theta$ in the conclusion of (2) is a significant step toward constructing a model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" and thereby acheving equiconsistency.

Throughout the text we assume the following base theory unless otherwise noted:

Assume $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$.

So a theorem is a theorem of $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$, and AD means $\mathsf{AD} + \mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$, for example.

Part (1) of the Main Theorem, the forcing direction, is established in Chapter 1. We define a fairly general class of forcing notions $\mathbb{P}$ that can be used to force the conclusion of (1). This class of forcing notions contains $\mathbb{P} = \mathrm{Col}(\omega_1, \mathbb{R})$ and $\mathbb{P} = \mathbb{P}_{\max}$, showing respectively that we can add CH or $\neg\mathsf{CH}$ to the conclusion. These forcing extensions satisfy $\mathfrak{c}$-DC and we get full AC by a second forcing with $\mathrm{Col}(\Theta, \wp(\mathbb{R}))$ whose role in the argument is negligible.

We show that these forcing extensions have ideals $\mathcal{I}$ with the *ordinal covering property*, which says that for every function $\wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ there are densely many $\mathcal{I}$-positive sets on which the function agrees with a function in the ground model. The ordinal covering property in turn can be used to show that $\mathcal{I}$ is strong, and together with the homogeneity of $\mathbb{P}$, to show that $\mathcal{I}$ is pseudo-homogeneous.

The remaining chapters are all devoted to establishing the inner model direction (2) of the Main Theorem. From a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$ we construct an inner model of $\mathsf{AD} + \theta_0 < \Theta$ via a core model induction. The basic idea of the core model induction is to analyze the extent of determinacy using mice with Woodin cardinals. The mice with Woodin cardinals themselves are obtained by core model theory, namely the K^c constructions of [**40**] and relativized versions that we call $K^{c,F}$ constructions. Our core model induction argument is similar to that used to prove the following related theorem:

Theorem (Ketchersid [**18**]). *If* $\mathsf{ZFC} + \mathsf{CH}$ *holds, the nonstationary ideal* NS_{ω_1} *is* ω_1*-dense below a stationary set, and the corresponding generic elementary embedding* $j \restriction \mathrm{Ord}$

is independent of the generic filter, then there is an inner model of $\mathsf{AD} + \theta_0 < \Theta$ *containing all the reals and ordinals.*

By a theorem whose proof is outlined in Sargsyan's thesis [**30**], the conclusion of Ketchersid's theorem can be strengthened to the existence of an inner model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" containing all the reals and ordinals. This produces an equiconsistency because Woodin has shown in unpublished work that the hypothesis can be forced from $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" by a similar method to the one we use to prove (1) of the Main Theorem.

The primary difference from Ketchersid's theorem is that our ideal is not assumed to be dense or even pre-saturated but merely strong. A generic ultrapower $\mathrm{Ult}(V, H)$ by a strong ideal may fail to contain all the reals of the generic extension $V[H]$. Although strong ideals were introduced thirty years ago in [**1**] it was not known how to derive significant large cardinal strength from them until recently. In [**3**] the existence of a strong ideal is shown to be equiconsistent with the existence of a Woodin cardinal.

Another difference is that we take a new approach to the "gap in scales" case of the core model induction, which first occurs when going beyond hyperprojective determinacy. In Chapter 3, which can be read independently of the rest of the paper, we develop some descriptive-set-theoretic tools for analyzing a gap in scales. Then in Chapters 4 and 5 we present two parallel methods for "sealing" the gap; one using weakly homogeneous trees and the other using directed systems of quasi-iterable pre-mice. These methods are interchangable in our argument except that the latter depends on a conjecture (Conjecture 5.1.2) and so it is not officially part of the proof of the Main Theorem.

Changing the approach to the "gap in scales" case is not necessary to prove the Main Theorem but we have chosen to do so because it results in a substantial simplification of the argument. In particular, it erases the distinctions between weak and strong gaps, and between gaps that end inside a premouse over $\mathbb{R}$ and those that do not.

The Main Theorem is a step toward our original goal, which was to find a theory satisfied by the $\mathbb{P}_{\max}$ extension of a model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" that is equiconsistent with the theory $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular." Originally it was anticipated that this theory would involve forcing axioms and properties of the nonstationary ideal on ω_1 because these are the statements that $\mathbb{P}_{\max}$ was designed to force. For example, under $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" the forcing axiom $\mathsf{MM}(\mathfrak{c})$, which is Martin's Maximum for posets no larger than the continuum, is forced by $\mathbb{P}_{\max}$. In turn $\mathsf{MM}(\mathfrak{c})$ implies that AD holds in $L(\mathbb{R})$ (Steel–Zoble [**38**]) and is expected to be stronger, possibly equiconsistent with $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" itself. However, determining the consistency strength of $\mathsf{MM}(\mathfrak{c})$ appears to be a hard problem because the known methods for getting a model of $\mathsf{AD} + \theta_0 < \Theta$, including those of Chapters 4 and 5, do not seem to apply.

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CHAPTER 1

Forcing strong ideals from determinacy

In this chapter we prove the forcing direction (1) of the Main Theorem. We define the relevant properties of ideals and give examples of such ideals in forcing extensions of models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular." In Sections 1.1 and 1.2 we introduce the elements of determinacy theory required for the analysis of these forcing extensions. In Section 1.3 we introduce some properties of ideals and discuss their relations to one another. In Sections 1.4 and 1.5 we exhibit some ideals with these properties: the nonstationary ideal $\mathrm{NS}_{\omega_1,\mathbb{R}}$ on $\wp_{\omega_1}(\mathbb{R})$ in Section 1.4, and an induced ideal on $\wp_{\omega_1}(\mathbb{R})$ in Section 1.5.

In this chapter, and only in this chapter, we work under the assumption

$$\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta \text{ is regular} + V = L(\wp(\mathbb{R})).$$

Strategies for real games are coded by sets of reals, so $\mathsf{AD}_{\mathbb{R}}$ is absolute to $L(\wp(\mathbb{R}))$ and the consistency of this assumption follows from that of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

1.1. The theory "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular"

The Axiom of Determinacy (AD) is the statement, proposed as an axiom by Mycielski and Steinhaus in [**29**], that every two-player ω-length game of perfect information on the integers is determined. By "determined" we mean that one player or the other has a winning strategy in the game. Such a game is specified by its payoff set, the subset of ω^ω corresponding to wins by the first player. By convention we use the symbol $\mathbb{R}$ to denote the "logician's reals," that is, the Baire space

$$\mathbb{R} = \omega^\omega.$$

So for every set of reals $A \subset \mathbb{R}$ there is a corresponding game G_A where player I wins if the sequence of integers played is in A and player II wins if it is in the complement of A. We say "A is determined" if the game G_A is determined.

Open and closed integer games are determined by the Gale–Stewart Theorem [**5**]. That is, if $A \subset \mathbb{R}$ is an open or closed set then the game G_A is determined. Because the Axiom of Choice implies the existence of undetermined integer games, AD does not describe the universe itself, but rather the realm of "nice" sets of reals. For example, AD implies that every set of reals is Lebesgue measurable and has the Baire Property and the Perfect Set Property. Another consequence of AD is that sets of reals are pre-wellordered by the *Wadge ordering* $\leq_W$ defined by

$$A \leq_W B \iff A = f^{-1}(B) \text{ or } A = f^{-1}(\mathbb{R} \setminus B) \text{ for some continuous function } f.$$

The Wadge hierarchy extends and refines the hierarchy of Borel sets studied in analysis. Given a set A of reals, its rank $|A|_W$ in this prewellordering is a natural way to measure its logical complexity by an ordinal number. We define

$$\wp_\alpha(\mathbb{R}) = \{A \subset \mathbb{R} : |A|_W < \alpha\}.$$

If AD holds then it holds in $L(\mathbb{R})$ because strategies for integer games are coded by reals. Therefore if AD is consistent then so is $\mathsf{AD} + V = L(\mathbb{R})$. So far it seems that all natural theories that imply the consistency of AD (such as large cardinals) actually imply that AD holds in $L(\mathbb{R})$. Moreover AD can be used to develop a nice structure theory for $L(\mathbb{R})$ that avoid the trivialities of $V = L$. For these reasons AD is often regarded as a good candidate for the true theory of $L(\mathbb{R})$. The theory AD has considerable large cardinal strength. For example, it implies that ω_1 is a measurable cardinal as witnessed by the club filter (Solovay, see [**28**]). In fact, it is equiconsistent with the existence of infinitely many Woodin cardinals (see [**31**].)

The axiom $\mathsf{AD}_\mathbb{R}$ states that every two-player ω-length game of perfect information on the *reals* is determined. The consistency strength of $\mathsf{AD}_\mathbb{R}$ is strictly greater than that of AD, and the former cannot hold in $L(\mathbb{R})$. The further strengthening "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" simply says that the ordinal Θ defined by

$$\Theta = \sup\{\alpha \in \mathrm{Ord} : \text{there is a surjection } \mathbb{R} \to \alpha\}$$

is a regular cardinal. (In general under AD it is a strong limit cardinal.) Notice that the regularity of Θ is equivalent to the nonexistence of a cofinal map $\mathbb{R} \to \Theta$: if there were a cofinal map $f : \mathbb{R} \to \Theta$ then Θ would be singular as witnessed by a map from the order type of $\operatorname{ran} f$, and the converse is clear. Methods of forcing over models of "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" are developed in Woodin's book [**45**].

The relative strength of extensions of AD such as $\mathsf{AD}_\mathbb{R}$ and "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" can be analyzed in terms of the *Solovay sequence*, a natural sequence of markers along the Wadge hierarchy, defined as follows.

Definition 1.1.1. The Solovay sequence $\{\theta_\alpha : \alpha < \Omega\}$ is defined by

- $\theta_0 = \sup\{|A|_\mathrm{W} : A \in \mathrm{OD}\}$,
- $\theta_{\alpha+1} = \sup\{|A|_\mathrm{W} : A \in \mathrm{OD}_B\}$ for any $B \subset \mathbb{R}$ with $|B|_\mathrm{W} = \theta_\alpha$ if it exists, and
- $\theta_\lambda = \sup\{\theta_\alpha : \alpha < \lambda\}$ if λ is limit.

The height Θ of the Wadge hierarchy is equal to θ_Ω.

1.2. $\mathrm{Col}(\omega, \mathbb{R})$-generic ultrapowers

One consequence of $\mathsf{AD}_\mathbb{R}$ that is particularly relevant for us is that every subset of $\wp_{\omega_1}(\mathbb{R})$ either contains, or is disjoint from, a club set (Solovay, [**33**]). So the club filter is a measure (the "Solovay measure", denoted here by μ,) that witnesses the $\mathbb{R}$-supercompactness of ω_1.

$$\mu = \{A \subset \wp_{\omega_1}(\mathbb{R}) : A \text{ contains a club set}\}$$

If we wish to force the Axiom of Choice then of course such large cardinal properties of ω_1 must be destroyed. However, in later sections we will show that there are useful "traces" of the $\mathbb{R}$-supercompactness of ω_1 apparent in certain generic extensions satisfying AC.

Under our assumption of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular $+ V = L(\wp(\mathbb{R}))$ we have DC, the Axiom of Dependent Choice. To see this, first notice that it suffices to show DC for relations on $\wp(\mathbb{R})$ because every set is definable from a set of reals and an ordinal. Second, because Θ is regular, for every total relation R on $\wp(\mathbb{R})$ there is an $\alpha < \Theta$ such that the restriction $R \cap (\wp_\alpha(\mathbb{R}) \times \wp_\alpha(\mathbb{R}))$ is total. This restriction is coded by a set of reals, so it remains to notice that DC for sets of reals follows from uniformization for relations on $\mathbb{R}$, which in turn follows from $\mathsf{AD}_{\mathbb{R}}$.

The ultrapower $\mathrm{Ult}(\mathrm{Ord}, \mu)$ of the ordinals by the Solovay measure is wellfounded: If $(\alpha_i : i < \omega)$ is a decreasing sequence of ordinals in the generic ultrapower, we can use DC to choose functions $F_i : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ representing α_i. By the countable completeness of the Solovay measure there is a club of σ such that $(F_i(\sigma) : i < \omega)$ is a decreasing sequence of ordinals in V, a contradiction.

Therefore we can define the μ-ultrapower map

$$j_\mu : \mathrm{Ord} \to \mathrm{Ord}.$$

We can extend j_μ in a natural way to act on any hereditarily wellorderable set. In particular if $S \subset \mathrm{Ord}$ we can define

$$j_\mu : \mathrm{HOD}_S \to \mathrm{Ult}(\mathrm{HOD}_S, \mu).$$

We cannot extend j_μ to an elementary embedding on V because Łòs's Theorem need not hold in the absence of AC. The relevant failure of choice here is that we cannot choose an ω-length enumeration $f(\sigma)$ of each $\sigma \in \wp_{\omega_1}(\mathbb{R})$, or else the choice function would represent an ω-length enumeration $[f]_\mu$ of $\mathbb{R}$ in the ultrapower, which is impossible.

However, if we are given a generic enumeration of $\mathbb{R}$ then can generically extend j_μ to an ultrapower map defined on all of V by the following argument due to Woodin (unpublished.)

Definition 1.2.1. Let $A \subset \mathbb{R}^\omega$.

- For $p \in \mathrm{Col}(\omega, \mathbb{R})$ we say A is *weakly comeager below* p if, for a club set of countable $\sigma \subset \mathbb{R}$, the set $A \cap \sigma^\omega$ is comeager[1] below p in σ^ω
- A is *weakly comeager* if it is weakly comeager below $\emptyset$.

Lemma 1.2.2. *Let $A \subset \mathbb{R}^\omega$ and $p \in \mathrm{Col}(\omega, \mathbb{R})$. Either A is weakly comeager below p, or $\mathbb{R}^\omega \setminus A$ is weakly comeager below some condition $q \leq p$ in* $\mathrm{Col}(\omega, \mathbb{R})$.

Proof. If A is not weakly comeager below p, then because the club filter is an ultrafilter there is a club set of countable $\sigma \subset \mathbb{R}$ such that the set $A \cap \sigma^\omega$ is not comeager below p. For such σ, because the topological space σ^ω is homeomorphic to $\mathbb{R}$ every subset of σ^ω has the Property of Baire by AD, so there is a condition $q \in \mathrm{Col}(\omega, \sigma)$ such that the set

[1]We equip σ^ω with the product of the discrete topologies on σ, so it is homeomorphic to $\mathbb{R}$.

$\neg A \cap \sigma^\omega$ is comeager below q. By the normality of the club filter, there is a single condition $q \in \mathrm{Col}(\omega, \mathbb{R})$ such that, for a club set of σ, the set $\neg A \cap \sigma^\omega$ is comeager below q. $\square$

Given a V-generic filter $h \subset \mathrm{Col}(\omega, \mathbb{R})$ we can define a filter on the subsets of $\mathbb{R}^\omega$ in V by

$$U_h = \{A \subset \mathbb{R}^\omega : A \text{ is weakly comeager below some condition } p \in h).$$

By Lemma 1.2.2 this filter is an ultrafilter on V.

THEOREM 1.2.3. *Given a V-generic filter $h \subset \mathrm{Col}(\omega, \mathbb{R})$, Łòs's Theorem holds for the ultrapower* $\mathrm{Ult}(V, U_h) = \{F : \mathbb{R}^\omega \to V\}^V / U_h$. *In particular, we get an elementary embedding*

$$j_h : V \to \mathrm{Ult}(V, U_h) \subset V[h].$$

PROOF. We lack the Axiom of Choice, so we must verify the existential quantification step in the proof of Łòs's Theorem. That is, given a formula φ in the language of set theory and a function $F : \mathbb{R}^\omega \to V$ such that the set $A = \{f \in \mathbb{R}^\omega : \exists X\, \varphi[F(f), X]\}$ is in U_h, we must show that there is a function $G : \mathbb{R}^\omega \to V$ such that the set $A' = \{f \in \mathbb{R}^\omega : \varphi[F(f), G(f)]\}$ is in U_h. In fact we do not need to shrink the set at all: $A' = A$ works.

Because $V = L(\wp(\mathbb{R}))$, every set $X \in V$ is OD from a set of reals. Because Θ is regular, there is an ordinal $\eta < \Theta$ such that for all $f \in A$ we have $\varphi[F(f), X]$ for some set X that is OD from a set of reals of Wadge rank $< \eta$. By $\mathsf{AD}_\mathbb{R}$ and in particular by uniformization for relations on $\mathbb{R}^\omega \times \wp_\eta(\mathbb{R})$, which is a surjective image of $\mathbb{R}$, there is a choice function $C : \mathbb{R}^\omega \to \wp_\eta(\mathbb{R})$ such that for all $f \in A$ we have $\varphi[F(f), X]$ for some set X that is OD from $C(f)$. Define $G(f)$ to be the least set X that is OD from $C(f)$, in the natural wellordering of such sets, such that $\varphi[F(f), X]$ holds. $\square$

LEMMA 1.2.4. *Given a function $F : \mathbb{R}^\omega \to \mathrm{Ord}$ and a condition $p \in \mathrm{Col}(\omega, \mathbb{R})$, there is a function $F_0 : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ and a condition $q \leq p$ in $\mathrm{Col}(\omega, \mathbb{R})$ such that the set*

$$A = \{f \in \mathbb{R}^\omega : F(f) = F_0(\operatorname{ran} f)\}$$

is weakly comeager below q.

PROOF. For $\sigma \in \wp_{\omega_1}(\mathbb{R})$ the topological space σ^ω is homeomorphic to $\mathbb{R}$, so it is not a wellordered union of meager sets by AD. Define $F_0(\sigma)$ as the least ordinal α such that the set $\{f \in \sigma^\omega : F(f) = \alpha\}$ is nonmeager in σ^ω. This set is comeager below some condition $q \in \mathrm{Col}(\omega, \sigma)$. By the normality of the club filter, there is a single condition $q \in \mathrm{Col}(\omega, \mathbb{R})$ such that for a club set of σ, the set

$$\{f \in \sigma^\omega : F(f) = F_0(\sigma)\}$$

is comeager in σ^ω below q. It remains to observe that for comeager many $f \in \sigma^\omega$ we have $\sigma = \operatorname{ran} f$. $\square$

As an immediate consequence, we see that the U_h-ultrapower is wellfounded and extends the μ-ultrapower:

Theorem 1.2.5. *Let $h \subset \mathrm{Col}(\omega, \mathbb{R})$ by V-generic. There is an isomorphism*

$$\begin{aligned}&\mathrm{Ult}(\mathrm{Ord}, \mu) \to \mathrm{Ult}(\mathrm{Ord}, U_h)\\ &\qquad [F_0]_\mu \mapsto [F_0 \circ \mathrm{ran}]_{U_h}.\end{aligned}$$

In particular, $\mathrm{Ult}(\mathrm{Ord}, U_h)$ *is wellfounded and* $j_h \restriction \mathrm{Ord} = j_\mu \restriction \mathrm{Ord}$.

Proof. First we show that this definition gives a well-defined embedding $\mathrm{Ult}(\mathrm{Ord}, \mu) \to \mathrm{Ult}(\mathrm{Ord}, U_h)$. Let R denote the relation $=$, $\neq$, $\in$, or $\notin$. Suppose $F_0, F_1 : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ are such that $F_0(\sigma)\ R\ F_1(\sigma)$ for a club set of σ. Then for each σ in this club, comeager many $f \in \sigma^\omega$ have range σ, so this club witnesses that $F_0(\mathrm{ran}\, f)\, R\, F_1(\mathrm{ran}\, f)$ for a weakly comeager set of functions $f : \mathbb{R}^\omega \to \mathrm{Ord}$. Now the surjectivity of this embedding follows from Lemma 1.2.4. $\square$

By the same argument, we get $j_h \restriction \mathrm{HOD}_S = j_\mu \restriction \mathrm{HOD}_S$ for any set of ordinals S.

Lemma 1.2.6. *Let $h \subset \mathrm{Col}(\omega, \mathbb{R})$ be V-generic. The generic ultrapower* $\mathrm{Ult}(V, U_h)$ *contains all the reals of the generic extension $V[h]$.*

Proof. Let $\dot{x}$ be a $\mathrm{Col}(\omega, \mathbb{R})$-term for a real. We define the associated continuous function $F : \mathbb{R}^\omega \to \mathbb{R}$ by

$$F(f) = \bigcup \big\{ s \in \omega^{<\omega} : (\exists i < \omega)\, (f \restriction i \Vdash s \subset \dot{x}) \big\},$$

a routine calculation shows that $[F]_{U_h} = \dot{x}_h$. $\square$

We say that a set $A \subset \mathbb{R}$ is $\mathbb{R}$-universally Baire if there are trees S and T such that $A = p[S] = \mathbb{R} \setminus p[T]$ and $V^{\mathrm{Col}(\omega,\mathbb{R})} \models p[S] = \mathbb{R} \setminus p[T]$.

Corollary 1.2.7. *Every set of reals is $\mathbb{R}$-universally Baire.*

Proof. Under $\mathsf{AD}_\mathbb{R}$ every set of reals A is Suslin and co-Suslin. Let S and T be trees with $p[S] = \neg p[T] = A$. Take a generic enumeration $h \subset \mathrm{Col}(\omega, \mathbb{R})$ and let $j_h : V \to \mathrm{Ult}(V, h)$ denote the associated elementary embedding. The trees $S^* = j_\mu(S) = j_h(S)$ and $T^* = j_\mu(T) = j_h(T)$ are in V and project to complements in $\mathrm{Ult}(V, U_h)$. Because $\mathrm{Ult}(V, U_h)$ contains all the reals of $V[h]$, this shows that A is $\mathbb{R}$-universally Baire. $\square$

Corollary 1.2.8. $j_\mu(\omega_1) = \Theta$.

Proof. Take a generic enumeration $h \subset \mathrm{Col}(\omega, \mathbb{R})$. Forcing with $\mathrm{Col}(\omega, \mathbb{R})$ does not collapse Θ because it is regular, so we have $j_\mu(\omega_1) = j_h(\omega_1) = \omega_1^{\mathrm{Ult}(V,U_h)} = \omega_1^{V[h]} = \Theta$. $\square$

1.3. A covering property for ideals in generic extensions

Still working under the assumption "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular $+ V = L(\wp(\mathbb{R}))$," we consider ideals in certain generic extensions

$$V[G][H] \models \mathsf{ZFC}$$

where G is V-generic for a poset $\mathbb{P}$ with the following properties (two examples to keep in mind are $\mathbb{P} = \mathrm{Col}(\omega_1, \mathbb{R})$ and $\mathbb{P} = \mathbb{P}_{\max}$):

- $\mathbb{P}$ is coded by a set of reals,
- $\mathbb{P}$ is σ-closed,
- $\mathbb{P}$ is homogeneous,
- $1 \Vdash_{\mathbb{P}} \mathbb{R}$ is wellorderable, and
- $1 \Vdash_{\mathbb{P}} \mathfrak{c}$-$\mathsf{DC}$, dependent choice for $\mathfrak{c}$-sequences,

and H is $V[G]$-generic for $\mathrm{Col}(\Theta, \wp(\mathbb{R}))^{V[G]}$.

Because Θ is regular we have $\Theta = \mathfrak{c}^+$ in $V[G]$. Because the forcing $\mathrm{Col}(\Theta, \wp(\mathbb{R}))^{V[G]}$ is $\mathfrak{c}$-closed in $V[G]$, it does not add any sets of reals or subsets of $\wp_{\omega_1}(\mathbb{R})$. Notice that the proof of this basic forcing fact requires dependent choice for $\mathfrak{c}$-sequences. So we have $\Theta = \mathfrak{c}^+$ in $V[G][H]$. Finally, notice that $V[G][H]$ satisfies AC because it has a wellordering of $\wp(\mathbb{R})^V$ and $V = L(\wp(\mathbb{R}))$.

We use the term "ideal" to refer to what is more correctly called a normal, fine, countably complete, and proper ideal:

Definition 1.3.1. An *ideal* on $\wp_{\omega_1}(\mathbb{R})$ is a set $\mathcal{I} \subset \wp_{\omega_1}(\mathbb{R})$ with the properties that

- if $S, T \in \mathcal{I}$ then $S \cup T \in \mathcal{I}$,
- if $S \in \mathcal{I}$ and $T \subset S$ then $T \in \mathcal{I}$,
- $\emptyset \in \mathcal{I}$ and $\mathbb{R} \notin \mathcal{I}$ (properness,)
- if $x \in \mathbb{R}$ then $\{\sigma \in \wp_{\omega_1}(\mathbb{R}) : x \notin \sigma\} \in \mathcal{I}$ (fineness,)
- if $\{S_i : i < \omega\} \subset \mathcal{I}$ then $\bigcup_{i<\omega} S_i \in \mathcal{I}$ (countable completeness,) and
- if $\{S_x : x \in \mathbb{R}\} \subset \mathcal{I}$ then $\{\sigma \in \wp_{\omega_1}(\mathbb{R}) : \exists x \in \sigma\, (\sigma \in S_x)\} \in \mathcal{I}$ (normality.)

We use $\mathcal{I}^+$ to denote $\wp_{\omega_1}(\mathbb{R}) \setminus \mathcal{I}$, the collection of $\mathcal{I}$-positive sets. Under the Axiom of Choice normality can be characterized by Fodor's property: every function $F : S \to \mathbb{R}$ on a $\mathcal{I}$-positive set S is constant on a $\mathcal{I}$-positive set. Every (normal) ideal $\mathcal{I}$ contains the nonstationary ideal.

Given an ideal $\mathcal{I} \in V[G][H]$ and a $V[G][H]$-generic ultrafilter $K \subset \mathcal{I}^+/\mathcal{I}$ we can define a generic elementary embedding

$$j_K : V[G][H] \mapsto \mathrm{Ult}(V[G][H], K) \subset V[G][H][K].$$

Normality implies that $\mathbb{R}^{V[G][H]}$ $(= \mathbb{R}^V)$ is represented in $\mathrm{Ult}(V[G][H], K)$ by the identity function on $\wp_{\omega_1}(\mathbb{R})$. We list some commonly studied properties of ideals in descending order of strength—that is, each property implies the next under ZFC.

Definition 1.3.2 (ZFC). Let $\mathcal{I}$ be an ideal on $\wp_{\omega_1}(\mathbb{R})$.

- $\mathcal{I}$ is $\mathfrak{c}$-*dense ideal* if there is a family of $\mathfrak{c}$ many $\mathcal{I}$-positive sets $(S_\alpha : \alpha < \mathfrak{c})$ such that for each $S \in \mathcal{I}^+$ there is $\alpha < \mathfrak{c}$ with $S_\alpha \setminus S \in \mathcal{I}$.
- $\mathcal{I}$ is *saturated ideal* if there is no family of $\mathfrak{c}^+$ many $\mathcal{I}$-positive sets $(S_\alpha : \alpha < \mathfrak{c}^+)$ that is an antichain: $S_\alpha \cap S_\beta \in \mathcal{I}$ when $\alpha \neq \beta$.

- $\mathcal{I}$ is *pre-saturated ideal* if for every sequence of antichains $(A_i : i < \omega)$ there is a $\mathcal{I}$-positive set S such that for each $i < \omega$ we have $|\{T \in A_i : T \cap S \in \mathcal{I}^+\}| \leq \mathfrak{c}$.
- $\mathcal{I}$ is *strong ideal* if it is precipitous (see below) and $1 \Vdash_{\mathcal{I}^+/\mathcal{I}} j(\omega_1) = \mathfrak{c}^+$.
- $\mathcal{I}$ is *precipitous ideal* if $1 \Vdash_{\mathcal{I}^+/\mathcal{I}}$ (Ult(V, K) is wellfounded).

These implications are all trivial except the one from pre-saturation to strength, which is a consequence of the well-known facts that in the forcing extension by a pre-saturated ideal on $\wp_{\omega_1}(\kappa)$ the cardinal successor κ^+ is not collapsed and the generic ultrapower is closed under countable sequences. We define a property of ideals in $V[G][H]$ relative to the ground model V.

Definition 1.3.3. In $V[G][H]$, an ideal $\mathcal{I}$ has the *ordinal covering property* with respect to V if for every function $F : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ and every $\mathcal{I}$-positive set S there is an $\mathcal{I}$-positive subset $S_0 \subset S$ and a function $F_0 : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ in V such that $F \restriction S_0 = F_0 \restriction S_0$.

Just like with Lemma 1.2.4 on functions $\mathbb{R}^\omega \to \mathrm{Ord}$, we can use this property to show that the generic ultrapower extends j_μ.

Lemma 1.3.4. *In $V[G][H]$, if $\mathcal{I}$ is an ideal on $P_{\omega_1}(\mathbb{R})$ with the ordinal covering property with respect to V and $K \subset \mathcal{I}^+ \setminus \mathcal{I}$ is a $V[G][H]$-generic filter, then*
(a) the corresponding generic embedding $j_K \restriction \mathrm{Ord}$ is equal to $j_\mu \restriction \mathrm{Ord}$, and
(b) $\mathcal{I}$ is strong.

Proof. (a) For any function $F : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ there are densely many $\mathcal{I}$-positive sets $S \subset \wp_{\omega_1}(\mathbb{R})$ such that $F \restriction S = F_0 \restriction S$ for some $F_0 \in V$ so by genericity there is $S \in K$ with this property. We have $K \cap V = \mu$, so this shows that

$$\begin{aligned}\{F : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}\}^{V[G][H]}/K &= \{F : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}\}^V/K \\ &= \{F : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}\}^V/\mu\end{aligned}$$

and also that $j_K \restriction \mathrm{Ord} = j_\mu \restriction \mathrm{Ord}$.

(b) By part (a) and Corollary 1.2.8 applied in V we have $j_K(\omega_1) = j_\mu(\omega_1) = \Theta^V = \mathfrak{c}^+$. ☐

Using the homogeneity of $\mathbb{P}$ we can derive an additional consequence of the ordinal covering property:

Definition 1.3.5 (ZFC). An ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$ is *pseudo-homogeneous* if, whenever $\alpha \in \mathrm{Ord}$, $s \in \mathrm{Ord}^\omega$, $\lambda < \mathfrak{c}^+$, and θ is a formula of set theory, the statement

$$\mathrm{Ult}(V, K) \models \theta[\alpha, j(s), j``(\lambda^\omega)]$$

is independent of the choice of generic filter $K \subset \mathcal{I}^+ \setminus \mathcal{I}$.

One consequence of pseudo-homogeneity is that $j_K \restriction \mathrm{Ord}$ is independent of the generic filter K. Another consequence is that the class HOD of the generic ultrapower is a subclass of V and is independent of the generic filter, because HOD is coded by the theory of the ordinals.

Lemma 1.3.6. *In $V[G][H]$, if $\mathcal{I}$ is an ideal on $\wp_{\omega_1}(\mathbb{R})$ with the ordinal covering property relative to V then $\mathcal{I}$ is pseudo-homogeneous.*

Proof. Let $\alpha \in \mathrm{Ord}$, $s \in \mathrm{Ord}^\omega$, $\lambda < \mathfrak{c}^+$, and let θ be a formula of set theory. Take a function $F_0 : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ in V representing α in $\mathrm{Ult}(V, \mu)$. By the ordinal covering property F_0 also represents α in $\mathrm{Ult}(V[G][H], K)$. In both ultrapowers $j(s)$ is represented by the constant function F_1 given by $F_1(\sigma) = s$, which is in V because the forcing $\mathbb{P}$ is countably closed. Let $\pi : \mathbb{R} \to \lambda^\omega$ be a surjection in V. In both ultrapowers $\mathbb{R}^V$ is represented by the identity function on $\wp_{\omega_1}(\mathbb{R})$, so $j``(\lambda^\omega)$ is represented by the function $F_2 \in V$ given by $F_2(\sigma) = \pi``\sigma$.

So we have $\mathrm{Ult}(V[G][H], K) \models \theta[\alpha, j(s), j``(\lambda^\omega)]$ if and only if the set

$$S = \{\sigma : V[G][H] \models \theta[F_0(\sigma), F_1(\sigma), F_2(\sigma)]\}$$

is in K. We have $S \in V$ by the homogeneity of $\mathbb{P}$, so it is in K if and only if it is in μ—that is, if and only if it contains a club. $\square$

In the remaining two sections of this chapter we will prove (1) of the Main Theorem using Lemmas 1.3.4 and 1.3.6. In Section 1.4 we will show that the nonstationary ideal has the ordinal covering property in $V[G][H]$, establishing (1a), and in Section 1.5 we will construct a $\mathfrak{c}$-dense ideal in $V[G][H]$ with the ordinal covering property, establishing (1b).

1.4. The covering property for $\mathrm{NS}_{\omega_1,\mathbb{R}}$

We continue to work in $V[G][H]$ under the assumptions stated at the beginning of Section 1.3. The material in this section is adapted from the proofs of Lemmas 9.143 and 9.144 in [**45**], which are related results about J_{NS}, the nonstationary ideal on ω_2 restricted to ordinals of cofinality ω, in the $\mathbb{P}_{\max}$ extension. We instead consider $\mathrm{NS}_{\omega_1,\mathbb{R}}$, the nonstationary ideal on $\wp_{\omega_1}(\mathbb{R})$, and generalize to the class of forcings defined in Section 1.3.

To establish the ordinal covering property for $\mathrm{NS}_{\omega_1,\mathbb{R}}$ in $V[G][H]$—or equivalently in $V[G]$—we will need the following lemma, which gives a criterion for a subset of $\wp_{\omega_1}(\mathbb{R})$ to be stationary.

Lemma 1.4.1. *Let $\dot{S}$ be a $\mathbb{P}$-name for a subset of $\wp_{\omega_1}(\mathbb{R})$. The following are equivalent:*

(1) $p \Vdash \dot{S}$ contains a club

(2) For a club of σ, we have

(*) $\quad (\forall^* g \subset \mathbb{P} \restriction \sigma \text{ containing } p)(\forall q \leq g)\,(q \Vdash \sigma \in \dot{S})$

where $\forall^*$ denotes the category quantifier for filters and "$q \leq g$" is shorthand for "$q \leq r$ for all $r \in g$."

Proof. Assume (1) holds, and let $\dot{f}$ be a $\mathbb{P}$-name for a function $\mathbb{R}^{<\omega} \to \mathbb{R}$ such that p forces $\dot{S}$ to contain the club generated by $\dot{f}$. Assume without loss of generality that the conditions of $\mathbb{P}$ are reals. To establish (2) it is enough to observe that there is a club of σ

such that for every $t \in \sigma^{<\omega}$, the set

$$D_t = \{q \in \mathbb{P} \cap \sigma : (\exists x \in \sigma)\,(q \Vdash \dot{f}(t) = x)\}$$

is dense below p in $\mathbb{P} \cap \sigma$.

Now assume (2). Take $N = L_\alpha(P_\beta(\mathbb{R}))$ satisfying "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" and a reasonable fragment of ZF, containing the set of reals

$$A = \{(q, x) : x \text{ codes } \sigma \in \wp_{\omega_1}(\mathbb{R}) \text{ and } q \Vdash \sigma \in \dot{S}\},$$

and admitting a surjection $F : \mathbb{R} \to N$. Under $\mathsf{AD}_\mathbb{R}$, every set of reals is $\mathbb{R}$-universally Baire, including the sets of reals coding the first-order theory of the structure $(V_{\omega+1}, \in, A)$. Therefore a club C of $\sigma \in \wp_{\omega_1}(\mathbb{R})$ have the following properties:

- property (*) holds,
- defining $X_\sigma = F``\sigma$ we have $X_\sigma \prec N$ and $X_\sigma \cap \mathbb{R} = \sigma$, and
- defining $\pi_\sigma : X_\sigma \cong N_\sigma$ as the transitive collapse of X_σ, we have

  $$(V_{\omega+1} \cap N_\sigma[h], \in, A \cap N_\sigma[h]) \prec (V_{\omega+1}, \in, A)$$

  for any N_σ-generic filter $h \subset \mathrm{Col}(\omega, \sigma)$.

All $\sigma \in C$ have the following property:

$$(**) \qquad N_\sigma \models p \Vdash^g_{\mathbb{P}\restriction\sigma} \Big(1 \Vdash^h_{\mathrm{Col}(\omega,\sigma)} (\forall q \leq g)\,\big((q, \sigma_h) \in \pi_\sigma(A)_{g\times h})\big)\Big),$$

where by σ_h we mean the real generically coding σ relative to h, and by $\pi_\sigma(A)_{g\times h}$ we mean the unique extension of $\pi_\sigma(A)$ to a set of reals in $N_\sigma[g][h]$—which can be rearranged as a generic extension of N_σ by $\mathrm{Col}(\omega, \sigma)$—given by universal Baire-ness.

Now given an V-generic filter $G \subset \mathbb{P}$ containing p, there is a club of σ such that $\sigma \in C$ and $X_\sigma[G] \prec N[G]$ and $X_\sigma[G] \cap V = X_\sigma$. Given any σ in this club, take $g = G \cap \sigma$. The set of conditions $q \leq p$ that are either below every element of g or incompatible with some element of g is dense below p. Therefore we can take $q \leq g$ in G. Any lower bound $q \leq g$ forces $\sigma \in \dot{S}$ by (**). So this club witnesses (1). □

Theorem 1.4.2. *In $V[G][H]$, the nonstationary ideal $\mathrm{NS}_{\omega_1,\mathbb{R}}$ has the ordinal covering property with respect to V.*

Proof. Suppose that the condition p_0 forces that $\dot{F}$ is an ordinal-valued function defined on a stationary set $\dot{S} \subset \wp_{\omega_1}(\mathbb{R})$. By Lemma 1.4.1 an equivalent condition for stationarity is that for all $p \leq p_0$, the formula (*) fails for a term for the complement of $\dot{S}$, that is, for stationary many (in fact club many by $\mathrm{AD}_\mathbb{R}$) countable $\sigma \subset \mathbb{R}$ we have

$$(\exists^* g \subset \mathbb{P} \restriction \sigma \text{ containing } p)(\exists q \leq g)\,(q \Vdash \sigma \in \dot{S}).$$

Under AD a wellordered union of meager sets is meager, so we can let $F_0(\sigma)$ be the least ordinal α such that

$$(\exists^* g \subset \mathbb{P} \restriction \sigma \text{ containing } p)(\exists q \leq g)\,(q \Vdash \dot{F}(\sigma) = \alpha).$$

Using our equivalent condition for stationarity again, we see that p_0 forces the set of $\sigma \in \dot{S}$ such that $\dot{F}(\sigma) = F_0(\sigma)$ to be stationary. $\square$

Together with Lemmas 1.3.4 and 1.3.6 this completes the proof of (1a) of the Main Theorem.

1.5. A $\mathfrak{c}$-dense ideal with the covering property

We continue to work in $V[G][H]$ under the assumptions stated at the beginning of Section 1.3. In this section we slightly generalize a theorem of Woodin, originally proved for $\mathbb{P} = \mathrm{Col}(\omega_1, \mathbb{R})$. It easily generalizes to the class of forcing extensions $\mathbb{P}$ we are considering.

Lemma 1.5.1. *If $h \subset \mathrm{Col}(\omega, \mathbb{R})$ is $V[H]$-generic and $G \in V[h]$, then letting $j_h : V \to \mathrm{Ult}(V, U_h) \subset V[h]$ denote the corresponding elementary embedding, in $V[h][H]$ there is an $\mathrm{Ult}(V, U_h)$-generic filter $G' \subset j_h(\mathbb{P})$ extending j_h“G.*

Proof. The poset $j_h(\mathbb{P})$ is countably closed in $\mathrm{Ult}(V, U_h)$ by elementarity and is coded by a set of reals there. We have $\mathbb{R} \cap V[h] = \mathbb{R} \cap \mathrm{Ult}(V, U_h)$ by Lemma 1.2.6, so $j_h(\mathbb{P})$ is countably closed in $V[h]$. Therefore j_h“G has a lower bound p in $j_h(\mathbb{P})$ because it is countable in $V[h]$.

A subset of $j_h(\mathbb{P})$ in $\mathrm{Ult}(V, U_h)$ is represented by a function $\mathbb{R}^\omega \to \wp(\mathbb{P})$ in V, which in turn is coded by a set of reals in V. Therefore in $V[h]$ there is a surjection $\wp(\mathbb{R})^V \to \wp(j_h(\mathbb{P}))^{\mathrm{Ult}(V,U_h)}$. In $V[G][H]$, by the definition of H, there is a surjection $\omega_1^{V[h]} = \Theta^V \to \wp(\mathbb{R})^V$ whose proper initial segments are in $V[G] \subset V[h]$. So composing these surjections we get a surjection $\omega_1^{V[h]} \to \wp(j_h(\mathbb{P}))^{\mathrm{Ult}(V,U_h)}$ whose proper initial segments are in $V[h]$. We can use this surjection to recursively define a decreasing ω_1-sequence $(p_\alpha : \alpha < \omega_1)$ of conditions in $j_h(\mathbb{P})$ below p whose proper initial segments are in $V[h]$ and which generates the desired filter G'. (The reason that we want the proper initial segments to be in $V[h]$ is that this is is the model in which $j_h(\mathbb{P})$ is countably closed.) $\square$

We can use such a generic G' to extend j_h to an elementary embedding j_h^* on $V[G]$ and then define the *induced* ideal from j_h^* (see [**4**]). This induced ideal will automatically have the ordinal covering property.

Theorem 1.5.2. *In $V[G][H]$ there is a $\mathfrak{c}$-dense ideal on $\wp_{\omega_1}(\mathbb{R})$ with the ordinal covering property relative to V.*

Proof. The size of the poset $\mathbb{P}$ is the continuum, so $\mathbb{P} \times \mathrm{Col}(\omega, \mathbb{R})$ is forcing-equivalent to $\mathrm{Col}(\omega, \mathbb{R})$. This means that in $V[G][H]$ there is a $\mathrm{Col}(\omega, \mathbb{R})$-term h such that

$$\emptyset \Vdash_{\mathrm{Col}(\omega,\mathbb{R})} h \subset \mathrm{Col}(\omega, \mathbb{R}) \text{ is a } V[H]\text{-generic filter and } G \in V[h].$$

Therefore by Lemma 1.5.1, forcing with $\mathrm{Col}(\omega, \mathbb{R})$ over $V[G][H]$ adds an $\mathrm{Ult}(V, U_h)$-generic filter $G' \subset j_h(\mathbb{P})$ extending j“G . We can extend j_h to an elementary embedding $j_h^* : V[G] \to \mathrm{Ult}(V, U_h)[G']$ by defining $j_h^*(\tau_G) = j_h(\tau)_{G'}$. Because $V[G]$ contains all the subsets of $\wp_{\omega_1}(\mathbb{R})$

in $V[G][H]$, we can define an ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$ in $V[G][H]$ by

$$S \in \mathcal{I} \iff \emptyset \Vdash_{\mathrm{Col}(\omega,\mathbb{R})} \check{\mathbb{R}} \notin j_h^*(\check{S}).$$

The quotient $\wp(\wp_{\omega_1}(\mathbb{R}))/\mathcal{I}$ is isomorphic to the subalgebra $\mathcal{B} = \{||\check{\mathbb{R}} \in j_h^*(\check{S})|| : S \subset \wp_{\omega_1}(\mathbb{R})\}$ of the regular-open algebra $\mathrm{RO}(\mathrm{Col}(\omega,\mathbb{R}))$.

Clearly $\mathcal{I}$ is a fine ideal. If $(S_x : x \in \mathbb{R})$ is a family of subsets of $\wp_{\omega_1}(\mathbb{R})$ and S is its diagonal union, then

$$||\check{\mathbb{R}} \in j_h^*(S)|| = ||\exists x \in \check{\mathbb{R}}\, (\check{\mathbb{R}} \in j_h^*(S_x))|| = \sup_{x\in\mathbb{R}} ||\check{\mathbb{R}} \in j_h^*(S_x)||.$$

Therefore $\mathcal{I}$ is normal and $\mathcal{B}$ is an $\mathfrak{c}$-complete subalgebra of $\mathrm{RO}(\mathrm{Col}(\omega,\mathbb{R}))$. Because the Boolean algebra $\mathrm{RO}(\mathrm{Col}(\omega,\mathbb{R}))$ has size $\mathfrak{c}^+$ and has the $\mathfrak{c}^+$-chain condition, $\mathcal{B}$ is a complete subalgebra. In general, if κ is a cardinal then every complete subalgebra of a complete κ-dense Boolean algebra is κ-dense, so $\mathcal{B}$ is $\mathfrak{c}$-dense.

We will show that $\mathcal{I}$ has the ordinal covering property relative to V. In $V[G][H]$, suppose that $F : S \to \mathrm{Ord}$ where $S \in \mathcal{I}^+$. Note that $F \in V[G]$ because H does not add any functions from $\wp_{\omega_1}(\mathbb{R})$. Take $p \in \mathrm{Col}(\omega,\mathbb{R})$ forcing $\check{\mathbb{R}} \in j^*(S)$. Take $q \leq p$ forcing $j_h^*(F)(\check{\mathbb{R}}) = \alpha$ for some ordinal α. Take a function $F_0 : \wp_{\omega_1}(\mathbb{R}) \to \mathrm{Ord}$ in V such that $[F_0]_\mu = \alpha$. The empty condition forces $[F_0]_\mu = j_h(F_0)(\check{\mathbb{R}}) = j_h^*(F_0)(\check{\mathbb{R}})$, so q forces $j_h^*(F)(\check{\mathbb{R}}) = j_h^*(F_0)(\check{\mathbb{R}})$. Therefore the set $\{\sigma \in S : F(\sigma) = F_0(\sigma)\}$ is $\mathcal{I}$-positive. $\square$

Together with Lemmas 1.3.4 and 1.3.6 this completes the proof of (1b) of the Main Theorem.

CHAPTER 2

The core model induction

The core model theory in a core model induction can be summarized by the K^F existence dichotomy, which is a straightforward extension of the K existence dichotomy to a kind of relativized mice. We call the new mice *hybrid mice*. They are relativised in two ways: by starting with some fixed transitive set x as a first level, and by closing under some fixed function F. We think of F as the "small next step" function of a hybrid premouse. In ordinary premice, the small next steps consist in moving from $\mathcal{P}$ to $\mathrm{rud}(\mathcal{P})$, so ordinary premice will be F-premice where $F(\mathcal{P}) = \mathrm{rud}(\mathcal{P})$. The "large next steps" in an F-premouse, as in an ordinary premouse, come from adding extenders.

We can start with any transitive set x. All the critical points of extenders on the sequence of a mouse $\mathcal{M}$ over x are above $\mathrm{rank}(x)$, so all iterations of $\mathcal{M}$ fix x. We put $x \cup \{x\}$ into all hulls we take, because we are only looking for fine structure "above x." Therefore the iterates and hulls of mice over x are themselves mice over x. Everything works just as in the case $x = \emptyset$, with no additional difficulty except for some small points in the case x is not equipped with a well-ordering.[1]

Not all functions F will do. For example, we want the appropriately elementary hulls of an F-mouse over x to also be F-mice over x. Because F will be defined throughout the universe of our mouse, we cannot simply work above it as we did with x. We need a condensation property of F that says roughly that if $F(a) = b$, and $(\bar{a}, \bar{b})$ is the image of (a, b) in a sufficiently elementary collapse, then $F(\bar{a}) = \bar{b}$. Condensation for F also enables us to construct iteration strategies for F-mice whose associated iteration maps move F to itself, so that the iterates remain F-mice.[2] The restriction that F condenses to itself is actually quite severe, but there are important examples that will play a crucial role in our core model induction.

All unattributed results in Sections 2.1, 2.2, and 2.3 are due to Steel. Some of the language and structure of these sections (and in this introduction to the chapter) was taken with permission from a draft of the book [**32**], subject to minor adaptations.

[1]In particular, we must require that if $\mathcal{M}$ is an active premouse then the measures forming its top extender are complete with respect to intersections of sequences of size $a \times \gamma$ in $\mathcal{M}$ for all γ less than the critical point. This ensures that the relevant version of Łoś's theorem holds for ultrapowers of $\mathcal{M}$ and that if $g \subset \mathrm{Col}(\omega, a)$ is a $\mathcal{M}$-generic filter then $\mathcal{M}[g]$ can be reorganized as a premouse over the real $a_g = \{(m, n) : (\bigcup g)(m) \in (\bigcup g)(n)\}$.
[2]To do this, we apply condensation to a map realizing the iterate in a level of some K^c construction relative to F.

2.1. Model operators

Definition 2.1.1. Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}$, $\dot{B}$, and $\dot{S}$, and constant symbols $\dot{l}$ and $\dot{a}$. Let a be a given transitive set. A *model with parameter* a is an $\mathcal{L}_0$-structure of the form

$$\mathcal{M} = (M; \in, E, B, \mathcal{S}, l, a)$$

such that M is a transitive rud-closed set containing a, the structure $\mathcal{M}$ is amenable, $\dot{a}^{\mathcal{M}} = a$, and $\mathcal{S}$ is a (possibly empty) sequence of models with parameter a such that, letting S_ξ be the universe of $\mathcal{S}_\xi$,

- $\dot{S}^{\mathcal{S}_\xi} = \mathcal{S} \restriction \xi$ for all $\xi \in \operatorname{dom} \mathcal{S}$, and $\dot{S}^{\mathcal{S}_\xi} \in S_\xi$ if ξ is a successor ordinal,
- $S_\xi = \bigcup_{\alpha < \xi} S_\alpha$ for all limit $\xi \in \operatorname{dom}(\mathcal{S})$,
- if $\operatorname{dom}(\mathcal{S})$ is a limit ordinal, then $M = \bigcup_{\alpha \in \operatorname{dom}(\mathcal{S})} S_\alpha$ and $l = 0$, and
- if $\operatorname{dom}(\mathcal{S})$ is a successor ordinal, then $\operatorname{dom}(\mathcal{S}) = l$.

Here we are thinking of $\mathcal{M}$ as a potential level of one of our hybrid premice. The set a is some parameter fixed in advance and put in at the bottom of all mice of the type being considered. The predicates E and B capture the new information added at level $\mathcal{M}$—in a typical example E codes an extender over M added in a relativized K^c construction, and B codes some other kind of information. The predicate S is the sequence of previous levels and $l - 1$ is the index of the immediately preceding level, if there is one. The previous levels must also be models with parameter a, so "model with parameter a" is being defined by $\in$-recursion.

Definition 2.1.2. Let $\mathcal{M}$ be a model with parameter a. Then $|\mathcal{M}|$ denotes the universe of $\mathcal{M}$. We define the *length of* $\mathcal{M}$ by $l(\mathcal{M}) = \operatorname{dom}(\dot{S}^{\mathcal{M}})$ and set $\mathcal{M}|\xi = \dot{S}^{\mathcal{M}}_\xi$ for all $\xi < l(\mathcal{M})$. The model $\mathcal{M}|0$ is called the *base model* of $\mathcal{M}$. We set $\mathcal{M}|l(\mathcal{M}) = \mathcal{M}$. If $l(\mathcal{M})$ is a successor ordinal then we set $\mathcal{M}^- = \mathcal{M}|(l(\mathcal{M}) - 1)$.

Definition 2.1.3. Let $\mathcal{M}$ be a model with parameter a. The *coarse projectum of* $\mathcal{M}$, denoted by $\rho(\mathcal{M})$, is the least ordinal $\rho \le l(\mathcal{M})$ such that $A \cap |\mathcal{M}|\rho| \notin M$ for some set $A \subseteq M$ that is definable from parameters over the structure $\mathcal{M}$.

We remark that the coarse projectum of $\mathcal{M}$ is equal to its ω^{th} projectum if $\mathcal{M}$ is ω-sound, which will always be the case in our applications, but in general it could be lower. (The notions of ω^{th} projectum and ω-soundness will be defined later.)

Definition 2.1.4. Let ν be an uncountable cardinal, and let $a \in H_\nu$. A *model operator with parameter* a *on* H_ν is a function F that maps every model $\mathcal{M} \in H_\nu$ with parameter a to a model $F(\mathcal{M}) \in H_\nu$ with parameter a such that

- Whenever $x \in |F(\mathcal{M})|$ and $y \in |\mathcal{M}|\rho(\mathcal{M})|$, we have $x \cap y \in |\mathcal{M}|$,
- $F(\mathcal{M}) = \operatorname{Hull}^{F(\mathcal{M})}_{\Sigma_1}(|\mathcal{M}|)$,

- $\dot{E}^{F(\mathcal{M})} = \emptyset$,[3] and
- $\dot{S}^{F(\mathcal{M})} = \dot{S}^{\mathcal{M}\frown}\mathcal{M}$.

The first examples of model operators F that we will encounter, beyond the rudimentary closure operator

$$F(\mathcal{M}) = (\mathrm{rud}(\mathcal{M}), \in, \emptyset, \emptyset, \dot{S}^{\mathcal{M}\frown}\mathcal{M}, l(\mathcal{M}) + 1, a),$$

are derived from mouse operators J such as the $M_n^\sharp$ operator mapping $\mathcal{M}$ to $M_n^\sharp(\mathcal{M})$. To define the $M_n^\sharp$ operator we first introduce the following standard notation.

Definition 2.1.5 (Lp). Let $\mathcal{M}$ be a transitive set (*e.g.* a model over a.) The *lower part mouse* $\mathrm{Lp}(\mathcal{M})$ is defined as the union of ω-sound premice $\mathcal{P}$ over $\mathcal{M}$, projecting to $\mathcal{M}$, and with the property that whenever $\pi : \bar{\mathcal{P}} \to \mathcal{P}$ is an elementary embedding with $\bar{\mathcal{P}}$ countable and $\pi(\bar{\mathcal{M}}) = \mathcal{M}$ we have that $\bar{\mathcal{P}}$ is an $(\omega_1 + 1)$-iterable premouse over $\bar{\mathcal{M}}$.

This is enough iterability for comparison, so $\mathrm{Lp}(\mathcal{M})$ can be reorganized as a premouse over $\mathcal{M}$, and we will tend to confuse it with its reorganization as such. We remark that the extenders on the sequence of a premouse $\mathcal{P}$ over $\mathcal{M}$ are all above $\mathcal{M}$, and in the definition of ω-soundness the set $\mathcal{M}$ is included in all hulls that we take.

Definition 2.1.6. Let $\mathcal{M}$ be a transitive set (*e.g.* a model over a.)

- $\mathcal{M}^\sharp$ is the least level of $\mathrm{Lp}(\mathcal{M})$ that is active, if it exists.
- $M_n^\sharp(\mathcal{M})$ is the least level of $\mathrm{Lp}(\mathcal{M})$ that is active and has n cardinals that are witnessed to be Woodin by extenders on the extender sequence of $\mathrm{Lp}(\mathcal{M})$, if it exists.

Next we define a notion of mouse operator that generalizes the essential features of the sharp operator and the $M_n^\sharp$ operators. Some sources use the term "first-order mouse operator" instead. We have no use for any other kind of mouse operator, so we omit "first-order" from the name.

Definition 2.1.7. A mouse operator with parameter a on H_ν is a function J that assigns to every transitive set $\mathcal{M} \in H_\nu$ containing the set a a structure $J(\mathcal{M})$ that is the least level of $\mathrm{Lp}(\mathcal{M})$ satisfying $\varphi[\mathcal{M}, a]$ for some fixed rQ formula $\varphi(v_1, v_2)$ in the language of premice.

For the definition of an rQ formula, see [**24**, Def. 2.3.9]. The relevant properties are that "I am an active premouse" and "I am a passive premouse" can be expressed by rQ formulas, and rQ formulas are preserved downward under Σ_1-elementary embeddings and upward under 0-embeddings and Σ_2-elementary embeddings.

Given a mouse operator, we can define a corresponding model operator.

Definition 2.1.8 (F_J). Let J be a mouse operator with parameter a on H_ν. We code J into a model operator F_J with parameter a on H_ν as follows. Let $\mathcal{M}$ be a model with parameter a in H_ν.

[3] The E-predicate is reserved for extenders that may be added at limit stages of a $K^{c,F}$ construction as defined in the next section.

(1) If every set in $J(\mathcal{M})$ is amenable to $\mathcal{M}|\rho(\mathcal{M})$, then $F_J(\mathcal{M})$ is simply $J(\mathcal{M})$ with the appropriate predicates added. Namely, we let $(M, \in, B)$ be the amenable code of $\mathcal{M}$ (which we usually identify with $\mathcal{M}$) and we let

$$F_J(\mathcal{M}) = (M; \in, \emptyset, B, \dot{S}^{\mathcal{M}\frown}\mathcal{M}, l(\mathcal{M}) + 1, a).$$

(2) If some set in $J(\mathcal{M})$ is not amenable to $\mathcal{M}|\rho(\mathcal{M})$, let (ξ, n) be lexicographically least such that some $\underset{\sim}{\mathbf{\Sigma}}_{n+1}^{J(\mathcal{M})|\xi}$ set is not amenable to $\mathcal{M}|\rho(\mathcal{M})$. Let $(N, \in, B)$ be the n^{th} master code of the premouse $J(\mathcal{M})|\xi$ and define

$$F_J(\mathcal{M}) = (N; \in, \emptyset, B, \dot{S}^{\mathcal{M}\frown}\mathcal{M}, l(\mathcal{M}) + 1, a).$$

Notice that even if $\mathcal{P}$ is an active premouse, its top extender is coded in the $\dot{B}$ predicate of $F_J(\mathcal{M})$ and not the $\dot{E}$ predicate.

Ordinary mouse operators J and their associated model operators F_J will be enough to prove hyperprojective determinacy. Going beyond this, the "gap in scales" step of the core model induction requires consideration of term-relation hybrid mouse operators. A term-relation hybrid mouse is like an ordinary mouse except that at certain limit stages we add a "term relation" for a self-justifying system that seals the gap in scales. If $\vec{A}$ is a self-justifying system then a hybrid mouse with term relations for $\vec{A}$ is called a $\vec{A}$-mouse. The notion of $\vec{A}$-mouse will be defined more precisely in Chapter 4.

We will need to define some notions of elementarity for maps between models with parameter a.

Definition 2.1.9. Let $\pi : \mathcal{M} \to \mathcal{N}$ be an elementary embedding between models $\mathcal{M}$ and $\mathcal{N}$ with parameter a.

- π is a *0-embedding* if it is Σ_0-elementary and its range is $\in$-cofinal in $\mathcal{N}$.
- π is a *weak 0-embedding* if there is a set X such that $|\mathcal{M}| = \bigcup X$ and π is Σ_1-elementary on tuples from X.

These notions are important because in general, Σ_0 ultrapower maps of premice are 0-embeddings, and lifting (or resurrection) maps from one ultrapower to another copied version of it are weak 0-embeddings. More generally, Σ_n ultrapower maps of premice are n-embeddings, which can be considered as 0-embeddings of the corresponding n^{th} master code structures. For more information on these notions, see [**24**].

We will only be interested in model operators which condense to themselves in the following sense.

Definition 2.1.10. Let F be a model operator with parameter a on H_ν. We say that F *condenses well* if it satisfies both of the following conditions:

(1) Let $\mathcal{M}_0, \mathcal{N}_0 \in H_\nu$ be models with parameter a. In $V^{\text{Col}(\omega, \mathcal{M}_0)}$ whenever we have a model $\mathcal{M}$ with parameter a such that $\mathcal{M}^- = \mathcal{M}_0$, and an embedding

$$\mathcal{M} \xrightarrow{\pi} F(\mathcal{N}_0)$$

fixing a and its elements that is either a 0-embedding or is Σ_2-elementary, we have $\mathcal{M} = F(\mathcal{M}_0)$.

(2) Let $\mathcal{P}_0, \mathcal{M}_0, \mathcal{N}_0 \in H_\nu$ be models with parameter a. In $V^{\mathrm{Col}(\omega,\mathcal{M}_0)}$ whenever we have a model $\mathcal{M}$ with parameter a such that $\mathcal{M}^- = \mathcal{M}_0$, and embeddings

$$F(\mathcal{P}_0) \xrightarrow{\sigma} \mathcal{M} \xrightarrow{\pi} F(\mathcal{N}_0)$$

fixing a and its elements, where σ is a 0-embedding or is Σ_2-elementary, π is a weak 0-embedding, and $\pi \circ \sigma \in V$, we have $\mathcal{M} = F(\mathcal{M}_0)$.

The next lemma on uniqueness of extensions shows that a model operator with parameter a that condenses well is determined by its action on $H_{|a|^+}$.

Lemma 2.1.11. *Let ν be an uncountable cardinal and let $a \in H_\nu$. Let F be a model operator with parameter a on $H_{|a|^+}$ that condenses well. Then F has at most one extension to H_ν. (That is, to a model operator with parameter a on H_ν that condenses well.)*

Proof. Let F' and F'' be extensions of F to H_ν and suppose toward a contradiction that there is a model $\mathcal{M} \in H_\nu$ with $F'(\mathcal{M}) \neq F''(\mathcal{M})$. Let $\pi : H \to H_\nu$ be an elementary embedding whose range contains $\mathcal{M}$, $F'(\mathcal{M})$, and $F''(\mathcal{M})$, say $\pi(\bar{\mathcal{M}}) = \mathcal{M}$. Because F' and F'' condense well we have $\pi(F'(\bar{\mathcal{M}})) = F'(\mathcal{M})$ and $\pi(F''(\bar{\mathcal{M}})) = F''(\mathcal{M})$. But $F'(\bar{\mathcal{M}})$ and $F''(\bar{\mathcal{M}})$ are both equal to $F(\bar{\mathcal{M}})$ and so are equal to each other, a contradiction. $\square$

While one can certainly construct pathological model operators using the Axiom of Choice, model operators that condense well are rare and interesting objects. The model operators derived from mouse operators condense well, as do the model operators derived from the term relation hybrid mouse operators that we will define in Chapter 4.

Lemma 2.1.12. *If J is a mouse operator with parameter a on H_ν, then F_J is a model operator with parameter a on H_ν and it condenses well.*

Proof. Let $\mathcal{N} \in H_\nu$ be a model with parameter a. In the first case, we assume that every set in $J(\mathcal{N})$ is amenable to $\mathcal{N}|\rho(\mathcal{N})$, so we are in case (1) of the definition of F_J (2.1.8) and $F_J(\mathcal{N})$ is simply $J(\mathcal{N})$ with the appropriate predicates added.[4] Being a premouse is expressible by an rQ sentence, and so is the defining property of $J(\mathcal{N})$. Therefore, both clauses of the definition of "condenses well" (2.1.10) follow from the fact that rQ formulas are preserved downward under Σ_1-elementary embeddings and upward under 0-embeddings and Σ_2-elementary embeddings, and iterability is preserved downward under weak 0-embeddings.

In the second case, there is a proper initial segment $\mathcal{N}|\xi$ of $\mathcal{N}$ and a least n such that some $\underset{\sim}{\Sigma}{}_{n+1}^{\mathcal{N}|\xi}$ set A is not amenable to $\mathcal{N}|\rho(\mathcal{N})$. Then we are in case (2) of the definition of F_J, and $F_J(\mathcal{N})$ is the n^{th} master code of $\mathcal{N}|\xi$ with the appropriate predicates added. The proof of condensation in this case is similar, using the fact that a Σ_1-elementary embedding of n^{th} master codes induces a Σ_{n+1}-elementary embedding of the underlying premice. $\square$

[4]Recall that our premice are coded in an amenable way; otherwise we would have to say $\mathfrak{C}_0(J(\mathcal{N}))$ here.

2.2. F-mice

Let F be a model operator with parameter a on H_ν that condenses well. We will define the notion of F-premouse—roughly speaking, a model with parameter a whose successor levels come from applying F to the previous level, and whose limit levels come from taking unions and then perhaps adding an extender. Such an extender must *cohere* with the model to which we add it, in the following sense.

DEFINITION 2.2.1 (Coherence). Let $\mathcal{M}$ be a model with parameter a, and let E be a pre-extender over $\mathcal{M}$. Let $\kappa = \mathrm{crit}(E)$, $\nu = \nu(E)$, and $\lambda = \mathrm{lh}(E)$. We say that E *coheres with* $\mathcal{M}$ if

- $\mathcal{M} \models$ "$\wp(\kappa)$ exists,"
- The extender E is the trivial completion[5] of $E \upharpoonright \nu$ and is not of type Z,
- $i_E(\dot{S}^{\mathcal{M}}) \upharpoonright (\lambda + 1) = \dot{S}^{\mathcal{M}} {}^\frown \mathcal{M}$ where $\lambda = l(\mathcal{M})$ and $i_E \colon \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, E)$ is the canonical embedding, and
- (Closure under initial segment) Let $\eta < \nu$, let G be the trivial completion of $E \upharpoonright \eta$, and suppose G is not of type Z. Let $\bar{\lambda} = \mathrm{lh}(G)$ and $\bar{\nu} = \nu(G)$. Then either
  - G is the extender coded by $\dot{E}^{\mathcal{M}|\bar{\lambda}}$, or
  - $\dot{E}^{\mathcal{M}|\bar{\nu}}$ codes an extender H, and letting $\mathcal{N} = \mathrm{Ult}(\mathcal{M}|\xi, H)$, we have that G is the extender coded by $\dot{E}^{\mathcal{N}|\bar{\lambda}}$.

DEFINITION 2.2.2. Let F be a model operator with parameter a on H_ν that condenses well. A *potential F-premouse* is a model $\mathcal{M}$ with parameter a such that

- $\mathcal{M}|(\eta + 1) = F(\mathcal{M}|\eta)$ for all ordinals $\eta < l(\mathcal{M})$, and
- if $\lambda \leq l(\mathcal{M})$ is a limit ordinal, then $\dot{B}^{\mathcal{M}|\lambda} = \emptyset$, and either $\dot{E}^{\mathcal{M}|\lambda} = \emptyset$, or $\dot{E}^{\mathcal{M}|\lambda}$ codes an extender that coheres with $\mathcal{M}|\lambda$.

If F is the rud-closure operator, then a potential F-premouse $\mathcal{M}$ is basically an ordinary potential premouse as in [**24**, Definition 1.0.5], but over the model $\mathcal{M}|0$.

More generally, if J is a mouse operator and $F = F_J$, then potential F-premice are just ordinary potential premice which have been re-stratified by collapsing certain intervals in their hierarchy. So we get nothing new in this case beyond a point of view that is sometimes useful. Later we shall consider information other than extenders which can be fed into canonical inner models. Most importantly, we will look at extender models that are also being told term relations for a self-justifying system. In this case, the associated model operators will give us truly new F-premice.

It is easy to see that there is a fixed Σ_1 formula φ of $\mathcal{L}_0$ such that whenever $\mathcal{M}$ is a potential F-premouse, φ defines a surjection $h : |\mathcal{M}|0| \times l(\mathcal{M})^{<\omega} \to |\mathcal{M}|$ over $\mathcal{M}$. We can also arrange these surjections to fit together in the sense that for $\eta < l(\mathcal{M})$ we have

[5] We define the trivial completion of E to be the equivalent extender of length $\theta(\nu \times a)^{\mathrm{Ult}(\mathcal{M},E)}$ where for a set X the notation $\theta(X)$ indicates the least ordinal that is not a surjective image of X. So if the base model $\mathcal{M}|0$ comes with a well-ordering of a, then the trivial completion of E has length $(\nu^+)^{\mathrm{Ult}(\mathcal{M},E)}$ as in the usual Mitchell–Steel indexing of extenders.

$\varphi^{\mathcal{M}|\eta} \subseteq \varphi^{\mathcal{M}}$. In practice, $\mathcal{M}|0$ very often has a Σ_1-definable wellorder, and then we easily get from φ uniformly $\Sigma_1^{\mathcal{M}|\eta}$ wellorders of $\mathcal{M}|\eta$ that end-extend one another. The main exceptions to this rule are premice over the reals, that is, $\mathcal{M}$ such that $|\mathcal{M}|0|$ is the rud-closure of $V_{\omega+1}$.

In order to see that there is a reasonable fine structure theory for potential F-premice, we need the following straightforward consequence of Definition 2.1.10.

Lemma 2.2.3 (Condensation Lemma). *Let F be a model operator with parameter a on H_ν that condenses well.*

(1) Let $\mathcal{M}, \mathcal{N} \in H_\nu$ be models with parameter a. If $\mathcal{N}$ is a potential F-premouse and in $V^{\mathrm{Col}(\omega,\mathcal{M})}$ there is an embedding

$$\mathcal{M} \xrightarrow{\pi} \mathcal{N}$$

fixing a and its elements that is a 0-embedding or is Σ_2-elementary, then $\mathcal{M}$ is also a potential F-premouse.

(2) Let $\mathcal{P}, \mathcal{M}, \mathcal{N} \in H_\nu$ be models with parameter a. If $\mathcal{P}$ and $\mathcal{N}$ are potential F-premice and in $V^{\mathrm{Col}(\omega,\mathcal{M})}$ there are embeddings

$$\mathcal{P} \xrightarrow{\sigma} \mathcal{M} \xrightarrow{\pi} \mathcal{N}$$

fixing a and its elements such that σ is a 0-embedding or is Σ_2-elementary, π is a weak 0-embedding, and $\pi \circ \sigma \in V$, then $\mathcal{M}$ is also a potential F-premouse.

A trivial consequence of the Condensation Lemma 2.2.3 is that if $\mathcal{M}$ is a potential F-premouse then we have

$$\mathcal{M} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(|\mathcal{M}|0| \cup \mathrm{Ord}^{\mathcal{M}}).$$

That is, $\mathcal{M}$ is the Σ_1-hull generated inside $\mathcal{M}$ from the ordinals in $\mathcal{M}$ together with elements of the base model $\mathcal{M}|0$.

Next we define some fine structural notions for potential F-premice that are parallel to those for ordinary potential premice. For the reader's convenience we have borrowed some of the exposition of fine structure from [**43**] here, making only the obvious changes required by the relativization to F.

We note that because we always code extenders by amenable predicates and our premice are squashed when appropriate (in the active type III case—see [**24**],) an F-premouse $\mathcal{M}$ is literally equal to its Σ_0 code $\mathfrak{C}_0(\mathcal{M})$. Accordingly, we will not use the notation $\mathfrak{C}_0$ in what follows.

Definition 2.2.4. Let F be a model operator that condenses well and let $\mathcal{M}$ be a potential F-premouse.

The Σ_1-*projectum of* $\mathcal{M}$, denoted by $\rho_1(\mathcal{M})$, is the least ordinal $\alpha < l(\mathcal{M})$ such that there is a subset of $\mathcal{M}|\alpha$ that is Σ_1-definable over $\mathcal{M}$ with parameters but is not an element of $\mathcal{M}$. Note that the parameter in the definition can be taken to be a finite sequence of ordinals and elements of $|\mathcal{M}|0|$.

The *first standard parameter of* $\mathcal{M}$, denoted by $p_1(\mathcal{M})$, is the lexicographically least finite decreasing sequence of ordinals $p \in l(\mathcal{M})^{<\omega}$ such that there is a subset of $\mathcal{M}$ that is Σ_1-definable over $\mathcal{M}$ from p and parameters in the base model $|\mathcal{M}|0|$ but whose intersection with $\mathcal{M}|\rho_1(\mathcal{M})$ is not an element of $\mathcal{M}$.

The *first core of* $\mathcal{M}$, denoted by $\mathfrak{C}_1(\mathcal{M})$, is the transitive collapse of the substructure given by all elements of $\mathcal{M}$ that are Σ_1-definable over $\mathcal{M}$ from parameters in $\mathcal{M}|\rho_1(\mathcal{M}) \cup \{p_1(\mathcal{M})\}$.

One can check that $\mathfrak{C}_1(\mathcal{M})$ is itself a potential F-premouse. For the potential F-premice we construct in practice, the first standard parameter will have the following nice properties.

Definition 2.2.5. Let F be a model operator that condenses well and let $\mathcal{M}$ be a potential F-premouse.

We say that $p_1(\mathcal{M})$ is *1-universal* if every subset of $\mathcal{M}|\rho_1(\mathcal{M})$ that is in $\mathcal{M}$ is also in $\mathfrak{C}_1(\mathcal{M})$.

We say that $p_1(\mathcal{M})$ is *1-solid* if, letting $p_1(\mathcal{M}) = (\alpha_0, \dots, \alpha_n)$, for every $i < n$ and for every subset $A \subset \mathcal{M}$ that is Σ_1-definable from $(\alpha_0, \dots, \alpha_i)$ and parameters in the base model $|\mathcal{M}|0|$, we have $A \cap \mathcal{M}|\alpha_{i+1} \in \mathcal{M}$.

If $p_1(\mathcal{M})$ is 1-universal then $\rho_1(\mathfrak{C}_1(\mathcal{M})) = \rho_1(\mathcal{M})$ and $p_1(\mathfrak{C}_1(\mathcal{M}))$ is the image of $p_1(\mathcal{M})$ under the transitive collapse map.

Definition 2.2.6. Let F be a model operator that condenses well and let $\mathcal{M}$ be a potential F-premouse.

- $\mathcal{M}$ is *1-solid* if $p_1(\mathcal{M})$ is 1-universal and 1-solid.
- $\mathcal{M}$ is *1-sound* if $p_1(\mathcal{M})$ is 1-universal and 1-solid, and $\mathcal{M} = \mathfrak{C}_1(\mathcal{M})$.

Continuing in this manner, for every integer n such that $\mathcal{M}$ is n-solid we can define the notions $\mathfrak{C}_{n+1}(\mathcal{M})$, $\rho_{n+1}(\mathcal{M})$, $p_{n+1}(\mathcal{M})$, $(n+1)$-universality, and $(n+1)$-solidity from $\mathfrak{C}_n(\mathcal{M})$[6] just like we defined $\mathfrak{C}_1(\mathcal{M})$ from $\mathcal{M}$ except that we replace Σ_1 formulas with Σ_{n+1} formulas. Finally, we say:

- $\mathcal{M}$ is *ω-solid* if it is n-solid for every integer n.
- $\mathcal{M}$ is *ω-sound* if it is n-sound for every integer n.

If $\mathcal{M}$ is ω-solid then we have $\rho_1(\mathcal{M}) \geq \rho_2(\mathcal{M}) \geq \cdots$ and we let $\rho_\omega(\mathcal{M})$ and $\mathfrak{C}_\omega(\mathcal{M})$ be the eventual values of $\rho_n(\mathcal{M})$ and $\mathfrak{C}_n(\mathcal{M})$ respectively.

Let $n \leq \omega$ and let $\mathcal{M}$ be an n-solid F-premouse, so that the n^{th} core $\mathfrak{C}_n(\mathcal{M})$ is defined. Using the Condensation Lemma 2.2.3 one can show that $\mathfrak{C}_n(\mathcal{M})$ is an n-sound F-premouse with the same base model as $\mathcal{M}$.

Definition 2.2.7. If $\mathcal{M}$ is n-sound, a *(weak) n-embedding* of $\mathcal{M}$ is a (weak) 0-embedding of the corresponding n^{th} master code structure.

If $\mathcal{M}$ is n-sound, this is equivalent to the usual definition (see [**24**].) We have no use for the notion of (weak) n-embedding of $\mathcal{M}$ when $\mathcal{M}$ is not n-sound.

[6] Not from $\mathcal{M}$!

Definition 2.2.8. Let F be a model operator with parameter a on H_ν that condenses well. An *F-premouse* is a potential F-premouse $\mathcal{M}$ whose proper initial segments $\mathcal{M}|\xi$, where $\xi < l(\mathcal{M})$, are all ω-sound. A *hybrid premouse* is an F-premouse for some model operator F.

The most basic examples of F-premice are the levels of L^F, which is the least transitive model closed under F. More precisely:

Definition 2.2.9. Let ν be an uncountable cardinal. Suppose that F is a model operator with parameter a on H_ν that condenses well. Let $\mathcal{P}$ be a model with parameter a. By $L^F_\nu(\mathcal{P})$ we denote the unique F-premouse $\mathcal{M}$ with base model $\mathcal{P}$ and length $l(\mathcal{M}) = \nu$ with no extender on its E-sequence, that is, $\dot{E}^{\mathcal{M}\restriction\xi} = \emptyset$ for all $\xi \leq l(\mathcal{M})$. We also write $L^F(\mathcal{P})$ for $L^F_\nu(\mathcal{P})$ if $\nu = \infty$.

Regarding ordinary premice, we refer the reader to [**43**] for the notions of a k-maximal iteration tree, putative iteration tree, (k, λ, θ)-iteration strategy, and the degree $\deg_\mathcal{T}$ of a model or branch of an iteration tree. For F-premice, we make the following modifications.

Definition 2.2.10. Let ν be an uncountable cardinal. Suppose that F is a model operator with parameter a on H_ν that condenses well. Let $\mathcal{M}$ be an F-premouse.

- An *iteration tree* on $\mathcal{M}$ is like an iteration tree on an ordinary premouse except that the requirement that all the models are well-founded is strengthened to say that all the models are F-premice.
- $\mathcal{M}$ is (k, λ, θ)-*iterable* if there is a (k, λ, θ)-iteration strategy Σ for $\mathcal{M}$ such that if $\mathcal{M}^*$ is an iterate of $\mathcal{M}^*$ according to Σ, then $\mathcal{M}^*$ is an F-premouse.

Note that iteration trees are only allowed to use extenders on the E-sequences of its models, where the *E-sequence* of a hybrid premouse $\mathcal{M}$ is the sequence of extenders $(\dot{E}^{\mathcal{M}|\alpha} : \alpha \leq l(\mathcal{M}))$.

Definition 2.2.11 (Lp^F). Let ν be an uncountable cardinal, let F be a model operator with parameter a on H_ν that condenses well, and let $\mathcal{P}$ be a model with parameter a. The *lower part F-mouse* over $\mathcal{P}$, called $\mathrm{Lp}^F(\mathcal{P})$, is the union of all ω-sound hybrid premice $\mathcal{M}$ with base model $\mathcal{P}$, projecting to $\mathcal{P}$, and such that whenever $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ is an elementary embedding with $\bar{\mathcal{M}} \in H_{|a|^+}$, $\pi(\bar{\mathcal{P}}) = \mathcal{P}$, and $\pi \restriction (a \cup \{a\}) = \mathrm{id}$, we have that $\bar{\mathcal{M}}$ is an $(|a|^+ + 1)$-iterable F-premouse.

This is enough iterability for comparison, so $\mathrm{Lp}^F(\mathcal{P})$ may be reorganized as a hybrid premouse with base model $\mathcal{P}$ and we will tend to confuse it with its reorganization as such. Note that $\mathrm{Lp}^F(\mathcal{P})$ may not literally be an F-premouse—we are not requiring $\mathcal{P}$ to be in H_ν, so F might not be defined on it.

Next we define some important examples of F-premice.

Definition 2.2.12. Let ν be an uncountable cardinal, let F be a model operator with parameter a on H_ν that condenses well, and let $\mathcal{P}$ be a model with parameter a.

- $F^\sharp(\mathcal{P})$ is the least F-premouse $\mathcal{M} \lhd \mathrm{Lp}^F(\mathcal{P})$, if it exists, that is active.[7]
- $M_n^{F,\sharp}(\mathcal{P})$ is the least F-premouse $\mathcal{M} \lhd \mathrm{Lp}^F(\mathcal{P})$, if it exists, that is active and has n many Woodin cardinals that are witnessed to be Woodin by extenders on its E-sequence.

Here we write $\mathcal{M} \lhd \mathcal{N}$ if $\mathcal{M} = \mathcal{N}|\xi$ for some $\xi < l(\mathcal{N})$. Note that $M_0^{F,\sharp}(\mathcal{P})$ is just another name for $F^\sharp(\mathcal{P})$. Also note that the Woodin cardinals of $M_n^{F,\sharp}(\mathcal{P})$ must be below the critical point of the top extender.

We can define the notion of an *F-mouse operator* in much the same way as a mouse operator. Some examples are given by the maps $\mathcal{P} \mapsto F^\sharp(\mathcal{P})$ and $\mathcal{P} \mapsto M_n^{F,\sharp}(\mathcal{P})$. We can derive a model operator F_J from an F-mouse operator J just like we would derive a model operator from an ordinary mouse operator.

In the next section we will show how to construct F-mice (that is, iterable F-premice) by carefully putting extenders on the E-sequence.

2.3. The $K^{c,F}$ construction and the K^F existence dichotomy

In this section we work in ZFC, although the author does not know whether it is necessary to assume the Axiom of Choice. Let z be a real and let F be a model operator with parameter z on H_ν that condenses well. (In our application we only need to consider model operators whose parameters are reals.) Assume that the F-mouse operator $F^\sharp$ is total on H_ν, that is, for every model $\mathcal{M} \in H_\nu$ with parameter z, the F-mouse $F^\sharp(\mathcal{M})$ exists.

Let $\mathcal{P}$ be a countable model with parameter z. The Condensation Lemma 2.2.3 for F-premice will be the key tool for showing that the $K^{c,F}(\mathcal{P})$ construction succeeds unless $M_1^{F,\sharp}(\mathcal{P})$ exists. We are now going to inductively define F-premice $\mathcal{N}_\xi$ over $\mathcal{P}$ in much the same way as in the definition of K^c in [**43**, Definition 6.3], except that we start with $\mathcal{P}$ rather than with $(V_\omega; \in)$, and we use F rather than the rudimentary closure operator at successor stages of the construction. More precisely:

Definition 2.3.1. Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. A $K^{c,F}(\mathcal{P})$ *construction* is a sequence $(\mathcal{N}_\xi \ : \xi \leq \theta)$, where $\theta \leq \nu$, of F-premice in H_ν with base model $\mathcal{P}$ such that the following conditions hold true.

(1) $\mathcal{N}_0 = \mathcal{P}$.
(2) If $\xi + 1 < \theta$, then $\mathcal{N}_\xi$ is an ω-solid F-premouse with base model $\mathcal{P}$, and either
  (a) $\mathcal{N}_{\xi+1} = F(\mathcal{M})$ where $\mathcal{M} = \mathfrak{C}_\omega(\mathcal{N}_\xi)$, or
  (b) $\mathcal{N}_\xi$ is passive of limit length, that is, it has the form $(|\mathcal{N}_\xi|; \in, \emptyset, \emptyset, \mathcal{S}, 0, z)$, there is an extender E that coheres with $\mathcal{N}_\xi$ and is countably certified in the sense of [**43**, Definition 6.2], and

$$\mathcal{N}_{\xi+1} = (|\mathcal{N}_\xi|; \in, E^*, \emptyset, \mathcal{S}, 0, z)$$

[7]A premouse or F-premouse is *active* if $\dot{E}^{\mathcal{M}} \neq \emptyset$.

where E^* is the amenable code of the extender E, and in the case that E has type III, we replace $\mathcal{N}_{\xi+1}$ with the squash of this structure.

(3) If $\lambda < \theta$ be a limit ordinal, then for $\xi < \lambda$ we let $\mathcal{N}_\xi = (|\mathcal{N}_\xi|; \in, E_\xi, B_\xi, \mathcal{S}_\xi, l_\xi, z)$ and let $\mathcal{S}$ be the unique maximal sequence of models such that for all $\beta < \mathrm{lh}(\mathcal{S})$ the initial segment $\mathcal{S}_\xi \restriction \beta$ is eventually equal to $\mathcal{S} \restriction \beta$ as $\xi \to \lambda$. Then we define

$$\mathcal{N}_\lambda = (\bigcup \mathcal{S}; \in, \emptyset, \emptyset, \mathcal{S}, 0, z).$$

Note that we need iterability to prove the solidity used to define the cores, so the construction might break down. The simplest instance of a $K^{c,F}(\mathcal{P})$ construction is the one that produces $L^F_\theta(\mathcal{P})$, that is, the one in which case (2b) never occurs, but there may be more complicated ones. For example, if the F-mouse operator $F^\sharp$ is total on H_ν then extenders may be added at cofinally many points below ν in the construction. As in [**40**] one can show that if $\lambda \leq \theta$ is a regular cardinal, then $\mathrm{Ord}^{\mathcal{N}_\lambda} = l(\mathcal{N}_\lambda) = \lambda$.

Definition 2.3.2 (Realizable branch). Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Let $(\mathcal{N}_\xi : \xi \leq \theta)$ be a $K^{c,F}(\mathcal{P})$ construction where $\theta \leq \nu$. Let $\pi : \bar{\mathcal{N}} \to \mathfrak{C}_k(\mathcal{N}_\xi)$ be a weak k-embedding where $\bar{\mathcal{N}}$ is countable, and let $\mathcal{T}$ be a countable k-maximal iteration tree on $\bar{\mathcal{N}}$. A *realizable branch* of $\mathcal{T}$ is a branch b of $\mathcal{T}$ such that letting $l = \deg^{\mathcal{T}}(b)$, either

- b does not drop and there is a weak l-embedding $\sigma : \mathcal{M}^{\mathcal{T}}_b \to \mathfrak{C}_l(\mathcal{N}_\xi)$ such that $\sigma \circ i^{\mathcal{T}}_b = \pi$, or
- b drops and there is a weak l-embedding $\sigma : \mathcal{M}^{\mathcal{T}}_b \to \mathfrak{C}_l(\mathcal{N}_{\bar{\xi}})$ for some $\bar{\xi} < \xi$.

Lemma 2.3.3. *Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Let $(\mathcal{N}_\xi : \xi \leq \theta)$ be a $K^{c,F}(\mathcal{P})$ construction where $\theta \leq \nu$. Let $\pi : \bar{\mathcal{N}} \to \mathfrak{C}_k(\mathcal{N}_\xi)$ be a weak k-embedding where $\bar{\mathcal{N}}$ is countable and let $\mathcal{T}$ be a countable k-maximal iteration tree on $\bar{\mathcal{N}}$. If b is a realizable branch of $\mathcal{T}$ then the branch model $M^{\mathcal{T}}_b$ is an F-premouse.*

Proof. Let $l = \deg_{\mathcal{T}}(b)$. Apply clause 2 of the Condensation Lemma to the composition of maps

$$\mathcal{M}^{\mathcal{T}}_\eta \xrightarrow{i^{\mathcal{T}}_{\eta,b}} \mathcal{M}^{\mathcal{T}}_b \xrightarrow{\sigma} \mathfrak{C}_l(\mathcal{N}_{\bar{\xi}})$$

where η is large enough such that the branch embedding $i^{\mathcal{T}}_b$ exists, that is, there is no further dropping along b, and the map σ and ordinal $\bar{\xi}$ are as in the definition of a realizable branch. (If the branch b does not drop then we can take $\eta = 0$ and $\bar{\xi} = \xi$.) Note that the model $\mathcal{M}^{\mathcal{T}}_\eta$ is an F-premouse by the definition of an iteration tree on an F-premouse. $\square$

Theorem 2.3.4 (Branch existence). *Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Let $(\mathcal{N}_\xi : \xi \leq \theta)$ be a $K^{c,F}(\mathcal{P})$ construction where $\theta \leq \nu$. Let $\pi : \bar{\mathcal{N}} \to \mathfrak{C}_k(\mathcal{N}_\xi)$ be a weak k-embedding where $\bar{\mathcal{N}}$ is countable and let $\mathcal{T}$ be a countable k-maximal iteration tree on $\bar{\mathcal{N}}$. Then*

- *If $\mathcal{T}$ has a last model $\mathcal{M}^{\mathcal{T}}_\eta$, then the branch $[0,\eta]_{\mathcal{T}}$ is realizable.*
- *If $\mathcal{T}$ has limit length then it has a maximal realizable branch (not necessarily cofinal.)*

This theorem is shown by the method of [**40**, §9]. We will combine it with a theorem on the uniqueness of branches to show that all models from a $K^{c,F}(\mathcal{P})$ construction are "sufficiently iterable" under favorable circumstances.

Definition 2.3.5. An F-premouse is *F-small* if for any ordinal $\xi \le l(\mathcal{M})$ such that the level $\mathcal{M}|\xi$ of $\mathcal{M}$ is active, letting $\kappa = \operatorname{crit}(\dot{E}^{\mathcal{M}|\xi})$ we have

$$\mathcal{M}|\kappa \models \text{"no cardinal is witnessed to be Woodin by extenders on the } E\text{-sequence."}$$

So $M_1^{F,\sharp}(\mathcal{P})$ is the least level of $\operatorname{Lp}^F(\mathcal{P})$ that is not F-small, if it exists. Note that if a model $\mathcal{N}_\xi$ of a $K^{c,F}(\mathcal{P})$ construction is F-small then so is every preceding model $\mathcal{N}_{\bar\xi}$ where $\bar\xi < \xi$. If a $K^{c,F}(\mathcal{P})$ construction produces a model that is not F-small, and $\mathcal{N}_\xi$ is the first such model produced, then $\mathfrak{C}_\omega(\mathcal{N}_\xi) = M_1^{F,\sharp}(\mathcal{P})$.

Definition 2.3.6. Let $\mathcal{N}$ be an F-premouse and let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$.

- $\delta(\mathcal{T}) = \sup\{\operatorname{lh}(E^{\mathcal{T}}_\alpha) : \alpha < \operatorname{lh}\mathcal{T}\}$
- $M(\mathcal{T}) = \bigcup\{\mathcal{M}^{\mathcal{T}}_\alpha | \operatorname{lh}(E^{\mathcal{T}}_\alpha) : \alpha < \operatorname{lh}\mathcal{T}\}$, organized as a passive F-premouse.

Notice that $M(\mathcal{T}) \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ for any cofinal branch b of $\mathcal{T}$. The following definition is the standard one (see [**43**]) adapted to F-mice in a straightforward way.

Definition 2.3.7 (Q-structure). Let $\mathcal{N}$ be an F-premouse and let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{N}$. Let b be a cofinal branch of $\mathcal{T}$. The *Q-structure* for b, if it exists, is the least level $Q = M^{\mathcal{T}}_b|\xi$ of the branch model $M^{\mathcal{T}}_b$ such that either

(1) $\xi < l(M^{\mathcal{T}}_b)$ and $\delta(\mathcal{T})$ is not Woodin in $M^{\mathcal{T}}_b|(\xi+1)$, or
(2) $\xi = l(M^{\mathcal{T}}_b)$, the branch b drops, and $\rho_k(M^{\mathcal{T}}_b|\xi) < \delta(\mathcal{T})$.

Note that if b drops in either model or degree then the Q-structure exists as in (2).

Definition 2.3.8 ($F^\sharp$-guided branch). Let ν be an uncountable cardinal and let F be a model operator with real parameter z on H_ν that condenses well. Let $\mathcal{N} \in H_\nu$ be an F-premouse and let $\mathcal{T} \in H_\nu$ be an iteration tree on $\mathcal{N}$. We say that a cofinal branch b of $\mathcal{T}$ is *$F^\sharp$-guided* if b has a Q-structure $\mathcal{Q}$ such that either $Q \trianglelefteq L^F(M(\mathcal{T}))$, or $F^\sharp(M(\mathcal{T}))$ exists and $Q = F^\sharp(M(\mathcal{T}))$.

An important observation is that an iteration tree $\mathcal{T}$ can have at most one $F^\sharp$-guided branch. If $\mathcal{T}$ has two $F^\sharp$-guided branches b_0 and b_1, then their Q-structures $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are both initial segments of $L^F(M(\mathcal{T}))$, or of $F^\sharp(M(\mathcal{T}))$ if it exists, so by minimality we must have $\mathcal{Q}_0 = \mathcal{Q}_1$. Then by standard arguments as in [**43**] we have $b_0 = b_1$. This argument shows that an $F^\sharp$-guided branch, if it exists, is unique even in generic extensions of V.

Theorem 2.3.9 (Branch uniqueness). *Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Let $\mathcal{N}_\xi$ be a model of a $K^{c,F}(\mathcal{P})$ construction.*

Suppose that either $\mathcal{N}_\xi$ is either an F-small model with no Woodin cardinal, or $\mathfrak{C}_\omega(\mathcal{N}_\xi) = M_1^{F,\sharp}(\mathcal{P})$. Let $k \leq \omega$ and let $\pi : \bar{\mathcal{N}} \to \mathfrak{C}_k(\mathcal{N}_\xi)$ be a weak k-embedding where $\bar{\mathcal{N}}$ is countable. Let $\mathcal{T}$ be a k-maximal iteration tree on $\bar{\mathcal{N}}$ of countable limit length. Then any cofinal realizable branch b of $\mathcal{T}$ is $F^\sharp$-guided. Therefore $\mathcal{T}$ has at most one cofinal realizable branch.

Proof. By Lemma 2.3.3 the branch model $M_b^{\mathcal{T}}$ is an F-premouse. Note that b must have a Q-structure $\mathcal{Q}$. If b drops this is true on general grounds and if it doesn't drop then this follows from the assumption of no Woodin cardinals. If $\mathcal{N}_\xi$ is F-small, then so is $M_b^{\mathcal{T}}$ and we must have $\mathcal{Q} \trianglelefteq L^F(M(\mathcal{T}))$. If $\mathfrak{C}_\omega(\mathcal{N}_\xi) = M_1^{F,\sharp}(\mathcal{P})$ then either $\mathcal{Q} \trianglelefteq L^F(M(\mathcal{T}))$, or $F^\sharp(M(\mathcal{T}))$ exists and $\mathcal{Q} = F^\sharp(M(\mathcal{T}))$. In any case b is $F^\sharp$-guided, and $F^\sharp$-guided branches are unique. □

Definition 2.3.10. Let F be a model operator with real parameter z on H_ν that condenses well and let $\mathcal{N} \in H_\nu$ be an F-premouse.

- An iteration tree $\mathcal{T} \in H_\nu$ on $\mathcal{N}$ is *$F^\sharp$-guided* if for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, the branch $[0, \lambda)_{\mathcal{T}}$ of $\mathcal{T} \restriction \lambda$ is $F^\sharp$-guided.
- A partial (ω, ν, ν)-iteration strategy Σ for $\mathcal{N}$ is *$F^\sharp$-guided* if whenever $\mathcal{T}$ is $F^\sharp$-guided and the branch $b = \Sigma(\mathcal{T})$ is defined, b is $F^\sharp$-guided.

We will refer to the maximal partial iteration strategy that is $F^\sharp$-guided as "the" $F^\sharp$-guided partial iteration strategy, and say that $\mathcal{N}$ is iterable by the $F^\sharp$-guided strategy if this partial iteration strategy is total—that is, if every $F^\sharp$-guided tree on $\mathcal{N}$ has an $F^\sharp$-guided branch.

As a consequence of the branch existence and uniqueness theorems 2.3.4 and 2.3.9 we derive the following iterability result.

Corollary 2.3.11. *Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Let $k \leq \omega$ and let $\mathcal{N}_\xi$ be a model of a $K^{c,F}(\mathcal{P})$ construction.*

Suppose that either $\mathcal{N}_\xi$ has no Woodin cardinal and is F-small, or $\mathfrak{C}_\omega(\mathcal{N}_\xi) = M_1^{F,\sharp}(\mathcal{P})$. Let $\pi : \bar{\mathcal{N}} \to \mathfrak{C}_k(\mathcal{N}_\xi)$ be a weak k-embedding where $\bar{\mathcal{N}}$ is countable. Then $\bar{\mathcal{N}}$ is (k, ω_1, ω_1)-iterable by the $F^\sharp$-guided strategy.

Proof. Let Σ be the partial iteration strategy for $\bar{\mathcal{N}}$ defined by choosing realizable cofinal branches if they exist. We know that realizable cofinal branches are $F^\sharp$-guided and therefore unique. Therefore it suffices to show that Σ is total. Let $\mathcal{T}$ be a putative iteration tree on $\bar{\mathcal{N}}$ as in the definition of (k, ω_1, ω_1)-iterability with the property that the branch $[0, \lambda)_{\mathcal{T}}$ is realizable for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$. We want to show that $\mathcal{T}$ itself has a cofinal realizable branch.

If $\mathcal{T}$ has successor length $\eta + 1$ on $\bar{\mathcal{N}}$ then by the branch existence theorem 2.3.4 the main branch $[0, \eta]_{\mathcal{T}}$ is realizable. Therefore by Lemma 2.3.3 the last model $\mathcal{M}_\eta^{\mathcal{T}}$ is an F-premouse—and in particular is wellfounded—so the putative iteration tree $\mathcal{T}$ is in fact an iteration tree.

If $\mathcal{T}$ has limit length then by the branch existence theorem 2.3.4 it has a maximal realizable branch b. It remains to see that this branch is cofinal. If not, then letting $\lambda = \sup b$ the branch $[0, \lambda)_{\mathcal{T}}$ was played according to Σ, so it is also realizable and applying the branch uniqueness theorem 2.3.9 to the tree $\mathcal{T} \restriction \lambda$ shows that $b = [0, \lambda)_{\mathcal{T}}$. But then we can extend b to the branch $[0, \lambda]_{\mathcal{T}}$, contradicting its maximality. □

Definition 2.3.12. Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Let $\theta \leq \nu$ be an ordinal. The sequence $(\mathcal{N}_\xi : \xi \leq \theta)$ is called a *maximal $K^{c,F}(\mathcal{P})$ construction of height θ* if for all limit ordinals $\xi \leq \theta$ such that the core $\mathcal{M} = \mathfrak{C}_\omega(\mathcal{N}_\xi)$ is passive, say $\mathcal{M} = (|\mathcal{M}|; \in, \emptyset, \emptyset, \mathcal{S}, 0, z)$, and there is an extender E that coheres with $\mathcal{M}$ and is countably certified, we have $\mathcal{N}_{\xi+1} = (|\mathcal{M}|; \in, E, \emptyset, \mathcal{S}, 0, z)$ for some such extender E. That is, extenders must be added in the construction whenever possible.

Using the iterability provided by Corollary 2.3.11 we may inductively prove that if every model of the $K^{c,F}(\mathcal{P})$ construction is F-small—for example, if $M_1^{F,\sharp}(\mathcal{P})$ does not exist—then every standard parameter of every $\mathfrak{C}_\omega(\mathcal{N}_\xi)$ is solid and universal and there is exactly one maximal $K^{c,F}(\mathcal{P})$ construction over $\mathcal{P}$ of any given height $\theta \leq \nu$ (cf. [**43**, §6.3]).

Definition 2.3.13. Let ν be an uncountable cardinal and let F be a model operator with real parameter z on H_ν that condenses well. Let $\mathcal{P}$ be a countable model over z. Moreover assume that ν is regular. We shall then write $K^{c,F}(\mathcal{P})|\nu$ for the last model $\mathcal{N}_\nu$ of the maximal $K^{c,F}(\mathcal{P})$ construction $(\mathcal{N}_\xi : \xi \leq \nu)$ of height ν, provided that the construction exists and is unique. (For example, if the $F^\sharp$ operator is total and $M_1^{F,\sharp}(\mathcal{P})$ does not exist, in which case the requisite iterability is provided by Corollary 2.3.11.)

If the $F^\sharp$ operator is total on H_ν but $M_1^{F,\sharp}(\mathcal{P})$ does not exist, then Corollary 2.3.11 implies that every model of $K^{c,F}(\mathcal{P})$ is countably iterable by the $F^\sharp$-guided strategy. We can use the following general lemma to give an $F^\sharp$-guided iteration strategy for $K^{c,F}(\mathcal{P})$ itself.

Lemma 2.3.14 (Q-structure reflection). *Let ν be an uncountable cardinal and let F be a model operator with real parameter z on H_ν that condenses well. Assume that the $F^\sharp$ operator is total on H_ν. Let $k \leq \omega$ and let $\mathcal{N}$ be a k-sound F-premouse such that any countable k-sound F-premouse $\bar{\mathcal{N}}$ admitting a weak k-embedding $\bar{\mathcal{N}} \to \mathcal{N}$ is (k, ω_1, ω_1)-iterable via the $F^\sharp$-guided iteration strategy. Then $\mathcal{N}$ itself is (k, ν, ν)-iterable via the $F^\sharp$-guided iteration strategy.*

Proof. Let $\mathcal{T}$ be a putative iteration tree on $\mathcal{N}$ as in the definition of (k, ν, ν)-iterability, and such that $\mathcal{T}$ is $F^\sharp$-guided. Take an elementary embedding $\pi : H \to V_{\nu+2}$ such that H is a countable transitive set and $\mathcal{N}, \mathcal{T}, F \in \operatorname{ran}(\pi)$. Write $\bar{\mathcal{N}} = \pi^{-1}(\mathcal{N})$ and $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$. By the condensation lemma we have that $\pi^{-1}(F) = F \restriction H$, and that the tree $\bar{\mathcal{T}}$ is $F^\sharp$-guided.

If the trees $\mathcal{T}$ and $\bar{\mathcal{T}}$ have successor length then because the main branch of $\bar{\mathcal{T}}$ is $F^\sharp$-guided, its last model is an F-premouse by our iterability hypothesis. By the elementarity of π the last model of $\mathcal{T}$ is also an F-premouse and we are done.

If the trees $\mathcal{T}$ and $\bar{\mathcal{T}}$ have limit length, then by our iterability hypothesis $\bar{\mathcal{T}}$ has an $F^\sharp$-guided cofinal branch. The generic extension $H^{\mathrm{Col}(\omega,\bar{\mathcal{T}})}$ contains a real coding $F^\sharp(M(\bar{\mathcal{T}}))$, so by absoluteness for Σ^1_1 formulas it also contains an $F^\sharp$-guided cofinal branch for $\bar{\mathcal{T}}$. By the elementarity of π, the generic extension $V^{\mathrm{Col}(\omega,\mathcal{T})}$ contains an $F^\sharp$-guided cofinal branch for $\mathcal{T}$. Because the $F^\sharp$-guided branch is unique even in generic extensions, it is in V by the homogeneity of $\mathrm{Col}(\omega, \mathcal{T})$. □

As an immediate consequence of Corollary 2.3.11 and Lemma 2.3.14, we have:

LEMMA 2.3.15. *Let ν be an uncountable cardinal, let F be a model operator with real parameter z on H_ν that condenses well, and let $\mathcal{P}$ be a countable model over z. Assume that the $F^\sharp$ operator is total on H_ν. Let $\mathcal{N}_\xi$ be a model of a $K^{c,F}(\mathcal{P})$ construction. Suppose that either $\mathcal{N}_\xi$ has no Woodin cardinal and is F-small, or $\mathfrak{C}_\omega(\mathcal{N}_\xi) = M_1^{F,\sharp}(\mathcal{P})$. Then $\mathfrak{C}_\omega(\mathcal{N}_\xi)$ is (ω, ν, ν)-iterable via the $F^\sharp$-guided iteration strategy.*

We shall now be interested in isolating $K^F(\mathcal{P})$, the "true" F-core model over $\mathcal{P}$. In order for this to work out as in [**40**] we need to see that $K^{c,F}(\mathcal{P})|\nu$ is "fully iterable" in a sense to be made precise. In order to develop the theory of $K^F(\mathcal{P})$, it is useful (but not necessary, see [**10**]) to assume that ν, henceforth written Ω, has large cardinal properties. For simplicity we will assume that Ω is measurable, but in our applications it is important that the argument can be done inside an active premouse where Ω is the critical point of the top extender. We will need the following general lemma, which is easy to prove using the condensation property of F.

LEMMA 2.3.16. *If Ω is measurable and F is a model operator on H_Ω with parameter $a \in H_\Omega$ that condenses well, then*

(1) F has a unique extension to H_{Ω^+} (uniqueness is guaranteed by Lemma 2.1.11,) and
(2) if $\mathcal{N}$ is an F-premouse with length $l(\mathcal{N}) \leq \Omega$ and base model $\mathcal{N}|0 \in H_\Omega$, and $\mathcal{N}$ is (ω, Ω, Ω)-iterable by the $F^\sharp$-guided strategy, then $\mathcal{N}$ is $(\omega, \Omega, \Omega + 1)$-iterable by the $F^\sharp$-guided strategy.

THEOREM 2.3.17 (K^F existence dichotomy). *Let Ω be a measurable cardinal. Let F be a model operator with real parameter z on H_Ω that condenses well. Let $\mathcal{P}$ be a countable model with parameter z. Let $K^{c,F}(\mathcal{P})$ denote the model $K^{c,F}(\mathcal{P})|\Omega$. Then the following statements hold true.*

(1) If the $K^{c,F}(\mathcal{P})$ construction reaches $M_1^{F,\sharp}(\mathcal{P})$ then $M_1^{F,\sharp}(\mathcal{P})$ is $(\omega, \Omega, \Omega + 1)$-iterable via the $F^\sharp$-guided strategy.
(2) If the $K^{c,F}(\mathcal{P})$ construction does not reach $M_1^{F,\sharp}(\mathcal{P})$, then the model $K^{c,F}(\mathcal{P})$ itself is $(\omega, \Omega, \Omega + 1)$-iterable via the $F^\sharp$-guided strategy.

PROOF. First notice that by Lemma 2.3.16 we can extend F to H_{Ω^+}. Then using the measurability of Ω it is easy to show that the $F^\sharp$ operator is total on H_Ω. Applying Lemma 2.3.16 again, we can extend $F^\sharp$ to H_{Ω^+}.

(1) If the $K^{c,F}(\mathcal{P})$ construction reaches $M_1^{F,\sharp}(\mathcal{P})$ then $M_1^{F,\sharp}(\mathcal{P})$ is (ω, Ω, Ω)-iterable via the $F^\sharp$-guided strategy by Lemma 2.3.15, so by Lemma 2.3.16 it is in fact $(\omega, \Omega, \Omega+1)$-iterable by the $F^\sharp$-guided strategy. (In fact, any iteration strategy for $M_1^{F,\sharp}(\mathcal{P})$ must be $F^\sharp$-guided.)

(2) If the $K^{c,F}(\mathcal{P})$ construction does not reach $M_1^{F,\sharp}(\mathcal{P})$ then each of its models is F-small. Every stack in H_Ω of normal trees on $K^{c,F}(\mathcal{P})$ is based on $K^{c,F}(\mathcal{P})|\nu$ for some inaccessible cardinal $\nu < \Omega$. Because the $F^\sharp$ operator is total, the model $\mathcal{N}_\nu = K^{c,F}(\mathcal{P})|\nu$ satisfies $F^\sharp(\mathcal{N}_\nu|\xi) \lhd \mathcal{N}_\nu$ for all $\xi < \nu$ and so by F-smallness it has no Woodin cardinals. Applying Lemma 2.3.15 to $\mathcal{N}_\nu$ we see that $\mathcal{N}_\nu$, and therefore $K^{c,F}(\mathcal{P})$, is (ω, Ω, Ω)-iterable via the $F^\sharp$-guided iteration strategy. Then Lemma 2.3.16 shows that $K^{c,F}(\mathcal{P})$ is $(\omega, \Omega, \Omega+1)$-iterable by the $F^\sharp$-guided iteration strategy. □

If (2) holds, then following [**40**] one can define the true F-core model $K^F(\mathcal{P})$.

2.4. $M_1^{F,\sharp}$ from a strong pseudo-homogeneous ideal

Let F be a model operator with real parameter z on H_{ω_1} that condenses well and is definable from a countable sequence of ordinals. In this section we will need a variant of the "lower part" F mouse Lp^F.

Definition 2.4.1 (Lp'_F). Let F be a model operator with real parameter z on H_{ω_1} that condenses well and is definable from a countable sequence of ordinals. Assume that ω_1^V is measurable in HOD_s for any countable sequence of ordinals s. For example, this holds if there is a pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$.

Let $\mathcal{P}$ be a model over z. We define $\mathrm{Lp}'_F(\mathcal{P})$ as the union of all ω-sound hybrid premice $\mathcal{M}$ with base model $\mathcal{P}$, projecting to $\mathcal{P}$, and such that whenever $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ is an elementary embedding such that $\bar{\mathcal{M}}$ is countable we have that $\bar{\mathcal{M}}$ is an F-premouse that is ω_1-iterable via a strategy definable from a countable sequence of ordinals s. Such models can be co-iterated in the model HOD_s, which thinks that ω_1^V is a measurable cardinal, so they are comparable and $\mathrm{Lp}'_F(\mathcal{P})$ may be reorganized as a hybrid premouse with base model $\mathcal{P}$. We will tend to confuse it with its reorganization as such.

We remark that $\mathrm{Lp}^F(\mathcal{P})$ is contained in $\mathrm{Lp}'_F(\mathcal{P})$, because a countable ω-sound F-premouse $\bar{\mathcal{M}}$ that projects to its base model $\bar{\mathcal{P}}$ and is (ω_1+1)-iterable has a unique (ω_1+1)-iteration strategy, which is definable from F, and therefore definable from a countable sequence of ordinals. In all cases that interest us, Lp' and Lp will turn out to be the same.

This definition will be used in conjunction with several abuses of notation. First, $\mathrm{Lp}'_F(\mathbb{R})$ we will mean $\mathrm{Lp}'_F(\mathcal{P})$ where $\mathcal{P} = (V_{\omega+1}, \in, \emptyset, \emptyset, \emptyset, \emptyset, z)$. Second, if J is an F-mouse operator we will write $\mathrm{Lp}'_J(\mathbb{R})$ for $\mathrm{Lp}'_{F_J}(\mathbb{R})$ where F_J is the model operator coding J. Third, we will contradict our earlier definition of the $M_n^{F,\sharp}$ operator by writing:

- $M_n^{F,\sharp}(\mathcal{P})$ is the least F-premouse $\mathcal{M} \lhd \mathrm{Lp}'_F(\mathcal{P})$, if it exists, that is active and has n many Woodin cardinals that are witnessed to be Woodin by extenders on its E-sequence.

The reason for this complication is that we do not know in advance that F and $F^\sharp$ can be extended to act on $H_{\mathfrak{c}^+}$. Eventually it will turn out that F and $F^\sharp$ can be so extended. Then because $M_n^{F,\sharp}(\mathcal{P})$ has an $F^\sharp$-guided iteration strategy, the Q-structure reflection lemma 2.3.14 will show that it is $\mathfrak{c}^+$-iterable. In particular it will be (ω_1+1)-iterable, so it will be an initial segment of $\mathrm{Lp}^F(\mathcal{P})$ and the definitions will coincide after all.

To get Woodin cardinals (and thereby determinacy) from a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$ we will use the following theorem. The argument, which is similar to that from the hypothesis "CH + there is a homogeneous presaturated ideal on ω_1" in [**32**], will take the remainder of this section.

Definition 2.4.2. Let F be a model operator on H_ν with real parameter z. We say that F *relativizes well* if there is a formula θ such that for every pair of models $\mathcal{P}$ and $\mathcal{Q}$ in H_ν with parameter z such that $\mathcal{P} \in \mathcal{Q}$, and every transitive model M of ZF^- containing $F(\mathcal{Q})$, we have $F(\mathcal{P}) \in \mathcal{M}$ and the formula θ defines $F(\mathcal{P})$ from $F(\mathcal{Q})$ in M.

Definition 2.4.3. Let F be a model operator on H_ν with real parameter z. We say that F *determines itself on generic extensions* if there is a formula θ such that whenever M is a countable transitive model of ZF^- that is closed under F, and $g \in V$ is an M-generic filter on a poset in M, the generic extension $M[g]$ is also closed under F, and the formula θ defines $F \restriction M[g]$ in $M[g]$ from $F \restriction M$.

The notions of "condenses well" and "determines itself on generic extensions" can be defined for (hybrid) mouse operators J as well as for model operators, either by substituting J or F_J for F in the definitions. If F condenses well, relativizes well, and determines itself on generic extensions, then so do the F-mouse operators $F^\sharp$ and $M_n^{F,\sharp}$, if they exist. (To show this for $M_n^{F,\sharp}$ one uses $L^F[\vec{E}]$ constructions, which are the fully backgrounded versions of $K^{c,F}$ constructions.)

Theorem 2.4.4 (ZFC). *Let F be a model operator with real parameter z on H_{ω_1} that condenses well, relativizes well, determines itself on generic extensions, and is definable from a countable set of ordinals. Let $\mathcal{I}$ be a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$. Then the following statements hold.*

(1) $F^\sharp(\mathcal{P})$ exists for every countable model $\mathcal{P}$ with parameter z.
(2) $F^{\sharp^\sharp}(\mathcal{P})$ exists for every countable model $\mathcal{P}$ with parameter z.
(3) $M_1^{F,\sharp}(\mathcal{P})$ exists for every countable model $\mathcal{P}$ with parameter z.

We will prove that (2) implies (3). The proof of (1) and the proof that (1) implies (2) are both simplifications of this proof based on the fact that if $F^\sharp(\mathcal{P})$ does not exist then the core model $K^F(\mathcal{P})$ is simply $L^F(\mathcal{P})$, and if the $F^\sharp$ operator is total but $F^{\sharp^\sharp}(\mathcal{P})$ does not exist, then $K^F(\mathcal{P})$ is simply $L^{F^\sharp}(\mathcal{P})$.

Suppose that the $F^{\sharp^\sharp}$ operator is defined on every countable model with parameter z. Let $H \subset \mathcal{I}^+ \setminus \mathcal{I}$ be V-generic and let

$$j : V \to \mathrm{Ult}(V, H) \subset V[H]$$

denote the corresponding elementary embedding. Define

$$\mathcal{M} = j(F^{\sharp\sharp})(\mathbb{R}^V).$$

We have $\mathcal{M} \in V$ by pseudo-homogeneity. Furthermore, in V we have $\mathcal{M} \lhd \mathrm{Lp}'_F(\mathbb{R})$, because if $\bar{\mathcal{M}}$ is a countable hybrid premouse that elementarily embeds into $\mathcal{M}$ in V, then $\bar{\mathcal{M}}$ elementarily embeds into $\mathcal{M}$ in $\mathrm{Ult}(V, H)$ also, so it is a $j(F)$-premouse there and is $j(\omega_1)$-iterable. Then we apply the elementarity of j to see that $\bar{\mathcal{M}}$ is an F-premouse in V that is ω_1-iterable by the trivial strategy. Take an $\mathcal{M}$-generic filter $G \subset \mathrm{Col}(\omega_1, \mathbb{R}^V)$ in $\mathrm{Ult}(V, H)$.[8] Notice that the generic extension $\mathcal{M}[G]$ satisfies the Axiom of Choice, and that $\mathbb{R}^{\mathcal{M}[G]} = \mathbb{R}^V$. It has a cardinal Ω that resembles a measurable cardinal, namely the critical point of the top extender of $\mathcal{M}$. Moreover, it is closed under the $j(F^\sharp)$ operator because in the generic ultrapower, $j(F^\sharp)$ relativizes well and determines itself on generic extensions.

Now let $\mathcal{P}$ be a countable model with parameter z. It suffices to show that in $\mathcal{M}[G]$, the $K^{c,j(F)}(\mathcal{P})|\Omega$ construction reaches $M_1^{j(F),\sharp}(\mathcal{P})$. Because $\mathbb{R}^{\mathcal{M}[G]} = \mathbb{R}^V$ and $j(F) \upharpoonright (H_{\omega_1})^V = F$, the hybrid mouse $M_1^{j(F),\sharp}(\mathcal{P})$ would then satisfy the definition of $M_1^{F,\sharp}(\mathcal{P})$ in V, completing the proof.

Therefore we suppose toward a contradiction that the $K^{c,j(F)}(\mathcal{P})|\Omega$ construction of $\mathcal{M}[G]$ does not reach $M_1^{j(F),\sharp}(\mathcal{P})$. Then we can define the core model $K^{j(F)}(\mathcal{P})|\Omega$ in $\mathcal{M}[G]$. We shed some of this notation by writing

$$K^F = (K^{j(F)}(\mathcal{P})|\Omega)^{\mathcal{M}[G]}.$$

This should not cause any confusion because $(K^{j(F)}(\mathcal{P})|\Omega)^{\mathcal{M}[G]}$ is in V by the homogeneity of the forcing $\mathrm{Col}(\omega_1, \mathbb{R}^V)$, and because $j(F)$ condenses well in $\mathrm{Ult}(V, H)$ we can apply Lemma 2.1.11 in $\mathcal{M}[G]$ to show that $j(F) \upharpoonright \mathcal{M}[G]$ is uniquely determined by $j(F) \upharpoonright (H_{\omega_1})^{\mathcal{M}[G]}$, which is simply $F \upharpoonright (H_{\omega_1})^V$. Therefore K^F is independent of the choice of pseudo-homogeneous ideal $\mathcal{I}$ and of the choice of generic filter $H \subset \mathcal{I}^+/\mathcal{I}$.

The following claim will lead to a contradiction because a Shelah cardinal is stronger than a Woodin cardinal.

Claim 2.4.5. *ω_1^V is a Shelah cardinal in $j(K^F)$.*

Proof. By homogeneity of the forcing poset $\mathrm{Col}(\omega_1, \mathbb{R})$ we have $j(K^F) \in \mathrm{HOD}^{\mathrm{Ult}(V,H)}_{\mathcal{P},j(F)}$, so because F is definable from a countable sequence of ordinals, by the pseudo-homogeneity of $\mathcal{I}$ we have $j(K^F) \in V$.

Let $\kappa = \omega_1^V$ and let E be the $(\kappa, j(\kappa))$-extender on $j(K^F)$ derived from j. We will show that $E \in \mathrm{Ult}(V, H)$ and that the fragments $E \upharpoonright \alpha$ for $\alpha < j(\kappa)$ are in $j(K^F)$ and witness that κ is Shelah in $j(K^F)$. Because $j(K^F) \in V$ we have that κ is inaccessible in K^F, so $j(\kappa)$ is inaccessible in $j(K^F)$. In particular $\kappa^{+j(K^F)} < j(\kappa) = \mathfrak{c}^+$.[9] Take a surjection

[8]The author does not know if this step is really necessary. Perhaps the facts about $K^{c,F}$ and K^F that we use can be proved without the Axiom of Choice.

[9]Because $\kappa^{+j(K^F)} \leq \omega_2$ is automatic, if CH fails then we do not need to use the fact that $\mathcal{I}$ is strong here and a precipitous pseudo-homogeneous ideal would suffice to give PD. However in general the gap case of the core model induction will need the strength of $\mathcal{I}$ regardless of CH.

$\pi : \mathbb{R} \to \wp(\kappa) \cap j(K^F)$ in V. Now an observation due to Kunen gives that $j \restriction (\wp(\kappa) \cap j(K^F))$ is in $\mathrm{Ult}(V, H)$: It is represented in the ultrapower by the function

$$\sigma \mapsto \{(\pi(x) \cap \kappa, \pi(x)) : x \in \sigma\}.$$

Therefore $E \in \mathrm{Ult}(V, H)$ as desired. Let $f : \kappa \to \kappa$ be a function in $j(K^F)$. Take an ordinal $\alpha < j(\kappa)$ such that $\alpha > j(f)(\kappa)$ and $\alpha > \kappa$. Because K^F and $j(K^F)$ agree up to κ, we have that $j(K^F)$ and $j(j(K^F))$ agree up to $j(\kappa)$. The natural factor map $k : \mathrm{Ult}(j(K^F), E \restriction \alpha) \to j(j(K^F))$ has $\mathrm{crit}(k) \geq \alpha$, so $j(K^F)$ and $\mathrm{Ult}(j(K^F), E \restriction \alpha)$ agree up to α and in particular up to $j(f)(\kappa)$. In fact they agree up to $i_{E\restriction\alpha}(f)(\kappa)$ because

$$i_{E\restriction\alpha}(f)(\kappa) \leq k(i_{E\restriction\alpha}(f)(\kappa)) = j(f)(k(\kappa)) = j(f)(\kappa).$$

So it remains to show that the extender $E \restriction \alpha$ is in $j(K^F)$. Notice that $j(K^F)$ is the core model $K^{j(F)}(\mathcal{P})$ as defined in some (equivalently, in any) generic extension $j(\mathcal{M})[G']$, where $G' \subset j(\mathrm{Col}(\omega_1, \mathbb{R}^V))$ is $j(\mathcal{M})$-generic. By increasing α we may assume it is a cardinal of $j(K^F)$. For technical reasons, rather than working with $j(K^F)$ itself we will work with the canonical soundness witness W for $j(K^F) \restriction \alpha$ that is obtained by the iteration of K^F that hits the order zero measure on each measurable cardinal $\geq \alpha$ of K^F exactly once. (Notice that W is also in V.)

The first step is to show that the phalanx $(W, \mathrm{Ult}(W, E \restriction \alpha), \alpha)$ is $(j(\Omega) + 1)$-iterable in $j(\mathcal{M})[G']$. Because $j(\mathcal{M})[G']$ is closed under the $j(j(F^\sharp))$ operator, by a Q-structure reflection argument (like Lemma 2.3.14 but for phalanxes) it's enough to show that if $\mathcal{Q}$ and $\mathcal{R}$ are countable $j(F)$-premice over $\mathcal{P}$ admitting elementary embeddings

$$\sigma : \mathcal{Q} \to W, \qquad \mathrm{crit}(\sigma) \geq \alpha,$$
$$\tau : \mathcal{R} \to \mathrm{Ult}(W, E \restriction \alpha), \qquad \mathrm{crit}(\tau) \geq \alpha,$$

then the phalanx $(\mathcal{Q}, \mathcal{R}, \alpha)$ is ω_1-iterable in $j(\mathcal{M})[G']$ via the strategy guided by $j(j(F^\sharp))$, or equivalently by $j(F^\sharp)$. By the inductive definition of K^F in $j(\mathcal{M})[G']$ it suffices to show that $\mathcal{Q}$ and $\mathcal{R}$ are both α-strong in $j(\mathcal{M})[G']$. Here the inductive definition of K^F, and the definition of α-strong, are just as for K in [**40**].

The fact that $\mathcal{Q}$ is α-strong in $j(\mathcal{M})[G']$ follows immediately from the definition. It remains to show that $\mathcal{R}$ is α-strong in $j(\mathcal{M})[G']$. In $\mathrm{Ult}(V, H)$ there is an elementary embedding

$$\tau' : \mathcal{R} \to j(W), \qquad \mathrm{crit}(\tau') \geq \alpha.$$

This is because in $V[H]$ composing τ with the natural factor map $\mathrm{Ult}(W, E \restriction \alpha) \to j(W)$ gives such an embedding, so by absoluteness there is also such an embedding in $\mathrm{Ult}(V, H)$. In V, take functions $(\alpha_\rho : \rho \in \wp_{\omega_1}(\mathbb{R}))$ and $(\tau'_\rho : \rho \in \wp_{\omega_1}(\mathbb{R}))$ representing α and τ' respectively. For H-almost-every $\rho \in \wp_{\omega_1}(\mathbb{R})$ we have that τ'_ρ is an elementary embedding $\mathcal{R}_\rho \to W$ with critical point $\geq \alpha_\rho$. For such ρ, we have such an elementary embedding in $j(\mathcal{M})[G']$ by absoluteness, so W witnesses that $\mathcal{R}_\rho$ is α_ρ-strong in $j(\mathcal{M})[G']$. So by the elementarity of j

and the homogeneity of the forcings to wellorder the reals, we have that $\mathcal{R}_\rho$ is α_ρ-strong in $\mathcal{M}[G]$ as well. It follows that $\mathcal{R}$ is α-strong in $j(\mathcal{M})[G']$.

Now we will use this iterability to show that the extender fragment $E \restriction \alpha$ is in W and therefore in $j(K^F)$. This part is exactly the same as in the proof of Theorem 7.1 of [**40**], where the hypothesis is the existence of a generic almost huge embedding, except that we have replaced K by K^F. Nevertheless we will reproduce that argument here for the convenience of the reader. In $j(\mathcal{M})[G']$, compare W with the phalanx $(W, \mathrm{Ult}(W, E \restriction \alpha), \alpha)$. The comparison terminates at some stage θ and we get iteration trees $\mathcal{T}$ on W and $\mathcal{U}$ on $(W, \mathrm{Ult}(W, E \restriction \alpha), \alpha)$ of length $\theta + 1$. So $\mathcal{U}_0 = W$ and $\mathcal{U}_1 = j(W)$.

We claim that $\mathcal{M}_\theta^{\mathcal{U}}$ is above $\mathrm{Ult}(W, E \restriction \alpha)$ in the tree $\mathcal{U}$. If it is above W instead, then because W is universal the branch embeddings $i_{0,\theta}^{\mathcal{U}}$ and $i_{0,\theta}^{\mathcal{T}}$ are defined and the branch models are equal, say $\mathcal{M}_\theta = \mathcal{M}_\theta^{\mathcal{T}} = \mathcal{M}_\theta^{\mathcal{U}}$. Let Δ be the set of common fixed points of $i_{0,\theta}^{\mathcal{U}}$ and $i_{0,\theta}^{\mathcal{T}}$, so Δ is thick in W and in $\mathcal{M}_\theta$. The construction of $\mathcal{U}$ guarantees that $\mathrm{crit}\, i_{0,\theta}^{\mathcal{U}} < \alpha$, so by the definability property of W at all $\gamma < \alpha$ we have that $\mathrm{crit}\, i_{0,\theta}^{\mathcal{U}}$ is the least γ not in the hull $H^{\mathcal{M}_\theta}(\Delta)$. We have $\mathrm{crit}\, i_{0,\theta}^{\mathcal{T}} = \gamma$ also for the same reason, so by the hull property of W at γ we have $i_{0,\theta}^{\mathcal{T}}(A) = i_{0,\theta}^{\mathcal{U}}(A)$ for all $A \subset \gamma$ in W. Therefore the first extenders used in each branch are equal, a contradiction.

So $\mathcal{M}_\theta^{\mathcal{U}}$ is above $\mathrm{Ult}(W, E \restriction \alpha)$ in $\mathcal{U}$. Now $\mathrm{Ult}(W, E \restriction \alpha)$ is universal in $j(\mathcal{M})[G']$ because the set of fixed points of $i_{E\restriction\alpha}$ is β-club in $j(\Omega)$ for all sufficiently large regular β, so that $\mathrm{Ult}(W, E \restriction \alpha)$ computes β^+ correctly for stationary many $\alpha < \Omega$. Therefore $i_{0,\theta}^{\mathcal{U}}$ and $i_{0,\theta}^{\mathcal{T}}$ are defined and the branch models are equal, say $\mathcal{M}_\theta = \mathcal{M}_\theta^{\mathcal{T}} = \mathcal{M}_\theta^{\mathcal{U}}$. Let Γ be the set of common fixed points of $i_{0,\theta}^{\mathcal{T}}$ and $i_{1,\theta}^{\mathcal{U}} \circ i_{E\restriction\alpha}$, so Γ is thick in W and in $\mathcal{M}_\theta$. Now $\kappa = \mathrm{crit}(i_{1,\theta}^{\mathcal{U}} \circ i_{E\restriction\alpha})$, so by the definability property of W at all $\gamma < \alpha$ we have that κ is the least γ not in $H^{\mathcal{M}_\theta}(\Gamma)$. We have $\mathrm{crit}\, i_{0,\theta}^{\mathcal{T}} = \kappa$ also for the same reason, so by the hull property of W at κ we have $i_{0,\theta}^{\mathcal{T}}(A) = i_{1,\theta}^{\mathcal{U}} \circ i_{E\restriction\alpha}(A)$ for all $A \subset \kappa$ in W.

Let $\mathcal{M}_{\eta+1}^{\mathcal{T}}$ be the successor of the root node W in $\mathcal{T}$. Then we have $\mathrm{crit}(i_{\eta+1,\theta}^{\mathcal{T}}) \geq \alpha$ because $E_\eta^{\mathcal{T}}$, like all extenders used in the comparison, has length $> \alpha$ and sup of generators $\geq \alpha$. Also by the construction of $\mathcal{U}$ we have $\mathrm{crit}(i_{1,\theta}^{\mathcal{U}}) \geq \alpha$. So $i_{0,\eta+1}^{\mathcal{T}}(A) \cap \alpha = i_{E\restriction\alpha}(A) \cap \alpha$ for all $A \subset \kappa$ in W. Therefore the extender fragment $E \restriction \alpha$ is equal to $E_\eta^{\mathcal{T}} \restriction \alpha$, which is in $\mathcal{M}_\eta^{\mathcal{T}}$ and therefore in W by coherence. $\square$

It remains to show that our $M_1^{F,\sharp}(\mathcal{P})$ is the real $M_1^{F,\sharp}(\mathcal{P})$, that is, it is (ω_1+1)-iterable. To do this, by a Q-structure reflection argument it suffices to show that the $F^\sharp$ operator can be extended to act on $H_{\mathfrak{c}^+}$. We will do this by showing that the $F^\sharp$ operator is $\mathfrak{c}^+$-universally Baire.

First we show that the $F^\sharp$ operator is ω_1-universally Baire. Letting $\mathcal{M}_\infty$ be the direct limit of all countable iterates of $M_1^{F,\sharp}$ in V, a boolean-valued comparison argument shows that $\mathcal{M}_\infty$ is also the direct limit of all countable iterates of $M_1^{F,\sharp}$ in $V^{\mathrm{Col}(\omega,\omega)}$. We can define an $<\omega_1$-absolutely complementing pair of trees S and T that project to the code sets of the $F^\sharp$ operator and its complement respectively: a branch looks for an integer n, a real x coding a countable model $\mathcal{P}$, an elementary embedding of a countable model $\mathcal{N}$ into $\mathcal{M}_\infty$, an $\mathcal{N}$-generic filter g for $\mathrm{Col}(\omega, \delta^{\mathcal{N}})$ such that $x \in \mathcal{N}[g]$, and a verification that n is in (resp. is not in) the code of $F^\sharp(\mathcal{P})$ relative to x, where $F^\sharp(\mathcal{P})$ is defined in $\mathcal{N}[g]$ using the fact that the $F^\sharp$ operator relativizes well. Using genericity iterations of $M_1^{F,\sharp}$ in $V^{\mathrm{Col}(\omega,\omega)}$ we can see that these trees have enough branches.

To show that the $F^\sharp$ operator is $\mathfrak{c}^+$-universally Baire using our pseudo-homogeneous ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$, we take a V-generic filter $H \subset \mathcal{I}^+/\mathcal{I}$ and let $j : V \to \mathrm{Ult}(V, H) \subset V[H]$ be the corresponding generic elementary embedding. The trees $j(S)$ and $j(T)$ are in V by pseudo-homogeneity. Let $g \subset \mathrm{Col}(\omega, \mathbb{R}^V)$ be a $V[H]$-generic filter. Then $\mathrm{Ult}(V, H)[g]$ is just the extension of $\mathrm{Ult}(V, H)$ by a Cohen real, so $j(S)$ and $j(T)$ project to complements there. The model $\mathrm{Ult}(V, H)[g]$ contains all the reals of $V[g]$, so the trees $j(S)$ and $j(T)$ also project to complements in $V[g]$, as desired.

2.5. The coarse and fine-structural mouse witness conditions

The following definition is standard (see, e.g., [**32**].)

DEFINITION 2.5.1 (W^*_γ). We say that the coarse mouse witness condition W^*_γ holds if, whenever $U \subset \mathbb{R}$ and both U and its complement have scales in $\mathrm{Lp}(\mathbb{R})|\gamma$,[10] then for all $k < \omega$ and $x \in \mathbb{R}$ there is a coarse (k, U)-Woodin mouse containing x with an $(\omega_1 + 1)$-iteration strategy whose restriction to H_{ω_1} is in $\mathrm{Lp}(\mathbb{R})|\gamma$.

If W^*_γ holds then one can prove exactly as in [**32**] that every set of reals in $\mathrm{Lp}(\mathbb{R})|\gamma$ is determined.

DEFINITION 2.5.2. An ordinal γ is a *critical ordinal in* $\mathrm{Lp}(\mathbb{R})$ if there is some $U \subset \mathbb{R}$ such that U and $\mathbb{R} \setminus U$ have scales in $\mathrm{Lp}(\mathbb{R})|(\gamma + 1)$ but not in $\mathrm{Lp}(\mathbb{R})|\gamma$. In other words, γ is critical in $\mathrm{Lp}(\mathbb{R})$ just in case $W^*_{\gamma+1}$ does not follow trivially from W^*_γ.

Next we define a notion of mouse witnesses for Σ^2_1 formulas. Our definition is equivalent to [**36**, Def. 9.2].

DEFINITION 2.5.3. Suppose that $\theta(v)$ is a Σ^2_1 formula saying that v has a $\Sigma^1_k(A)$ property φ for some set of reals A. A (θ, z)-*prewitness* is an ω-sound premouse $\mathcal{N}$ over z, projecting to z, in which there are ordinals $\delta_0 < \cdots < \delta_k$ and trees S and T such that $\mathcal{N}$ satisfies

- ZFC and $\delta_0, \ldots, \delta_k$ are Woodin cardinals,
- S and T are δ_k-absolutely complementing trees on $\omega \times \mathrm{Ord}$, and
- $\mathcal{N} \models \varphi[p[T], z]$.

If in addition there we have $\mathcal{N} \lhd \mathrm{Lp}(z)$, we call $\mathcal{N}$ a (θ, z)-*witness*.

The following lemma, which can be proved using genericity iterations, justifies the name *witness*. It is phrased in terms of the notion of a good iteration strategy defined in [**36**, p. 10]. The iteration strategies arising in practice are all good.

[10] That is, the *sequence* of norms is in $\mathrm{Lp}(\mathbb{R})|\gamma$

Lemma 2.5.4. *Let $\theta(v)$ be a Σ^2_1 formula and let z be a real. If there is a (θ, z)-witness whose $(\omega_1 + 1)$-iteration strategy is good, then $\theta[z]$ holds.*

We will be proving a sort of converse to this as we go along in a core model induction, namely

Definition 2.5.5 (W_γ). We say that the (fine-structural) *mouse witness condition* W_γ holds if, whenever $\theta(v)$ is a Σ^2_1 formula, $z \in \mathbb{R}$, and $\mathrm{Lp}(\mathbb{R})|\gamma \models \theta[z]$, there is a (θ, z)-witness with an $(\omega_1 + 1)$-iteration strategy whose restriction to H_{ω_1} is in $\mathrm{Lp}(\mathbb{R})|\gamma$.

We will need the following result (see [**32**]) to convert our coarse mice into fine-structural mice as we go along.

Lemma 2.5.6. *Assume that there is no inner model of $\mathsf{AD} + \theta_0 < \Theta$ containing all the reals and ordinals. Let γ be a limit ordinal. If W^*_γ holds, then W_γ holds.*

By [**30**] the hypothesis can be weakened to "there is no inner model of the theory $\mathsf{AD}_\mathbb{R}$ + "Θ is regular" containing all the reals and ordinals, but we will not need this because we are only trying to get a model of $\mathsf{AD} + \theta_0 < \Theta$.

2.6. Hyperprojective determinacy from a strong pseudo-homogeneous ideal

In this section we will prove the following proposition.

Proposition 2.6.1 (ZFC). *Assume that there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$. Then every hyperprojective set of reals is determined.*

In order to prove Proposition 2.6.1 we will argue to establish a more general fact that implies it, namely Proposition 2.6.2 below. To state this more general fact we need to define an ordinal that measures our progress in the core model induction. We let α be the strict supremum of the ordinals γ such that

(1) The coarse mouse witness condition $W^*_{\gamma+1}$ holds,
(2) γ is a critical ordinal in $\mathrm{Lp}(\mathbb{R})$, and
(3) $\gamma + 1$ begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$.

We have AD in $\mathrm{Lp}(\mathbb{R})|\alpha$ because if γ is a critical ordinal in $\mathrm{Lp}(\mathbb{R})$ and $W^*_{\gamma+1}$ holds then the coarse mice can be used to prove determinacy in $\mathrm{Lp}(\mathbb{R})|(\gamma + 1)$ as in [**32**]. The level $\mathrm{Lp}(\mathbb{R})|\alpha$ is a passive premouse over $\mathbb{R}$. The ordinal α is a limit of ordinals beginning Σ_1-gaps in $\mathrm{Lp}(\mathbb{R})$, so α itself begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$.

Proposition 2.6.1 will follow from the following proposition because the hyperprojective sets of reals are exactly the ones in the least admissible level of $\mathrm{Lp}(\mathbb{R})$, namely $L_{\kappa^\mathbb{R}}(\mathbb{R})$.

Proposition 2.6.2 (ZFC). *Assume that there is no inner model of $\mathsf{AD} + \theta_0 < \Theta$ containing all the reals and ordinals. Let α be the strict supremum of the ordinals γ such that $W^*_{\gamma+1}$ holds, γ is critical in $\mathrm{Lp}(\mathbb{R})$, and $\gamma + 1$ begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$. If there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$, then $\mathrm{Lp}(\mathbb{R})|\alpha$ is admissible.*

Proof. Suppose to the contrary that $\mathrm{Lp}(\mathbb{R})|\alpha$ is inadmissible. We begin by constructing the "next operator" J, which will be an F-mouse operator for some model operator F. (In contrast to the "gap case" this F will exist already in $\mathrm{Lp}(\mathbb{R})|\alpha$.) We define F and J according to cases. If $\alpha = 0$, we simply let F and J be the rudimentary closure operator coded in the appropriate ways.

If $\alpha = \gamma + 1$ then we may assume that the coarse mice witnessing $W^*_{\gamma+1}$ actually come from F-mouse operators $M_n^{F,\sharp}$, $n < \omega$ for some model operator F on H_{ω_1} with some real parameter z that condenses well. This assumption is harmless because whenever we prove the coarse witness condition we prove it this way.[11] We define a new F-mouse operator J as follows. For every countable model $\mathcal{P}$ with parameter z, we define $J(\mathcal{P})$ to be the least F-premouse $\mathcal{M} \lhd \mathrm{Lp}^F(\mathcal{P})$ such that $M_n^{F,\sharp}(\mathcal{P}|\xi) \lhd \mathcal{P}$ for every $n < \omega$ and $\xi < l(\mathcal{M})$.

If α is a limit ordinal (of either countable or uncountable cofinality) then from W^*_α we can get the fine-structural mouse witness condition W_α. In this case our next operator J will be an ordinary mouse operator (the "diagonal operator") so we let $F = \mathrm{rud}$. Because α begins a Σ_1-gap there is a partial surjection $\mathbb{R} \to \mathrm{Lp}(\mathbb{R})|\alpha$ that is Σ_1-definable over $\mathrm{Lp}(\mathbb{R})|\alpha$. So some real parameter witnesses the failure of admissibility. That is, there is a real z and a Σ_1 formula $\varphi(v_0, v_1)$ such that α is least with the property that $\mathrm{Lp}(\mathbb{R})|\alpha \models (\forall y \in \mathbb{R})\,(\varphi[z, y])$. Given a model $\mathcal{P}$ with parameter z, we define J to be approximately[12] the least premouse $\mathcal{M} \lhd \mathrm{Lp}(\mathcal{P})$ such that every $\mathrm{Col}(\omega, \mathcal{P})$-generic extension $\mathcal{M}[g]$ is a witness for the formula $(\forall y \in L_1[x_g] \cap \mathbb{R})\,(\varphi[z, y])$ where x_g is the generic code of $\mathcal{P}$ relative to g.

Now that we have our next F-mouse operator J one can show as in [**39**] (see also [**32**]) one can show that the associated model operator F_J condenses well, relativizes well, and determines itself on generic extensions, and apply Theorem 2.4.4 repeatedly to get more F-mouse operators $M_n^{J,\sharp}$, $n < \omega$. These operators $M_n^{J,\sharp}$ can also be considered as F_J-mouse operators—recall that $M_n^{J,\sharp}$ is a synonym for $M_n^{F_J,\sharp}$. Note that applying Theorem 2.4.4 to the model operator coding $M_n^{J,\sharp}$ actually gives us something a bit stronger than $M_{n+1}^{J,\sharp}$, but this is okay. Each operator $M_n^{F_J,\sharp}$ condenses well, relativizes well, and determines itself on generic extensions because F_J has these properties. As in [**39**] (see also [**32**]) one can show that the coarse mice given by the $M_n^{J,\sharp}$ operators witness that $W^*_{\alpha+1}$ holds and that $\mathrm{Lp}(\mathbb{R})|(\alpha + 1)$ satisfies AD. They also witness that α is critical. The ordinal α begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$ because it is a limit of ordinals beginning Σ_1-gaps in $\mathrm{Lp}(\mathbb{R})$. The gap is trivial because α is inadmissible (cf. [**32**].) That is, the gap has the form $[\alpha, \alpha]$, so $\alpha + 1$ also begins a Σ_1-gap in $L(\mathbb{R})$. This is a contradiction, because α is defined to be greater than every ordinal γ such that $W^*_{\gamma+1}$ holds, γ is critical in $\mathrm{Lp}(\mathbb{R})$, and $\gamma + 1$ begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$. □

For the "gap in scales" case we will need the following theorem.

Theorem 2.6.3 (Steel [**37**]). *Let $\mathcal{M}$ be a passive level of* $\mathrm{Lp}(\mathbb{R})$ *such that* $\mathcal{M} \models \mathsf{AD}$. *Then the pointclass consisting of all $\Sigma_1^{\mathcal{M}}$ sets of reals has the scale property.*

[11] We will later see that the model operator F will code either an ordinary mouse operator, or an $\vec{A}$-mouse operator for some self-justifying system $\vec{A}$. We will define the notion of $\vec{A}$-mouse operator in Chapter 4.

[12] The correct definition can be found in [**32**, §4.2] or [**39**, §1.3].

If there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$, then either there is already an inner model of $\mathsf{AD} + \theta_0 < \Theta$ containing all the reals and ordinals, or $\mathrm{Lp}(\mathbb{R})|\alpha$ is admissible by Proposition 2.6.2. In the latter case the pointclass $\Gamma = \Sigma_1^{\mathrm{Lp}(\mathbb{R})|\alpha}$ is closed under real quantifiers, so although it has the scale property we cannot use the second periodicity theorem to propogate scales on Γ sets. In fact by a theorem of Martin that we will mention in the next chapter, there is no scale on a universal $\check{\Gamma}$ set in $\mathrm{Lp}(\mathbb{R})|(\alpha+1)$. Hence we have a "gap in scales" and α is not a critical ordinal in $\mathrm{Lp}(\mathbb{R})$.

We say a pointclass is *inductive-like* if it is ω-parameterized, closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, and recursive substitution, and has the scale property.

PROPOSITION 2.6.4. *Assume that there is no inner model of* $\mathsf{AD} + \theta_0 < \Theta$ *containing all the reals and ordinals. Let* α *be the strict supremum of the ordinals* γ *such that* $W^*_{\gamma+1}$ *holds,* γ *is critical in* $\mathrm{Lp}(\mathbb{R})$*, and* $\gamma+1$ *begins a* Σ_1*-gap in* $\mathrm{Lp}(\mathbb{R})$*. If there is a strong pseudo-homogeneous ideal on* $\wp_{\omega_1}$ *then the pointclass* $\Gamma = \mathrm{Lp}(\mathbb{R})|\alpha$ *is inductive-like, and its boldface ambiguous part* $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ *is determined.*

PROOF. By Proposition 2.6.2 the model $\mathrm{Lp}(\mathbb{R})|\alpha$ is admissible, so $\Sigma_1^{\mathrm{Lp}(\mathbb{R})|\alpha}$ is closed under real quantifiers. It is clear that the pointclass $\Gamma = \Sigma_1^{\mathrm{Lp}(\mathbb{R})|\alpha}$ is ω-parameterized and closed under recursive substitution. The scale property follows from Theorem 2.6.3. The pointclass $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is equal to $\mathrm{Lp}(\mathbb{R})|\alpha \cap \wp(\mathbb{R})$ by admissibility, so it is determined by Proposition 2.6.2. $\square$

The next chapter will consist of a general analysis of inductive-like pointclasses Γ such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined.

CHAPTER 3

The next Suslin cardinal in $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$

In this chapter we reformulate the notion of the envelope of an inductive-like pointclass Γ in such a way that many of its essential properties can by derived without assuming full AD. This generalizes classical results proved under AD in [**21**]. A modern reference for this material is [**7**]. By an argument similar to that for the Kechris–Woodin transfer theorem in [**17**], we prove that if $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined then so is the envelope $\underset{\sim}{\mathbf{Env}}(\Gamma)$—this extra determinacy is enough for the analysis.

A *product space* is a space of the form

$$\mathscr{X} = X_1 \times \cdots \times X_n, \quad X_i = \mathbb{R} \text{ or } X_i = \omega \text{ for all } i \leq n.$$

A *pointset* is a subset of a product space. A *pointclass* is a collection of pointsets, typically an initial segment of some complexity hierarchy for pointsets.

Definition 3.0.5. A pointclass Γ is *inductive-like* if it is ω-parameterized, closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, and recursive substitution, and has the scale property.

For the rest of this chapter we let Γ be an inductive-like pointclass, although some of the results can be proved even if the scale property of Γ is weakened to the pre-wellordering property. As usual we denote the dual of Γ and the ambiguous part of Γ by

$$\check{\Gamma} = \{\neg A : A \in \Gamma\} \quad \text{and}$$
$$\Delta_\Gamma = \Gamma \cap \check{\Gamma}$$

respectively. We also define the boldface pointclasses $\underset{\sim}{\boldsymbol{\Gamma}} = \bigcup_{x \in \mathbb{R}} \Gamma(x)$ and $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma = \bigcup_{x\in\mathbb{R}} \Delta_\Gamma(x)$. We will be operating under the following determinacy assumption:

$$\underset{\sim}{\boldsymbol{\Delta}}_\Gamma \text{ is determined.}$$

Examples are $\Gamma = \text{IND}$ assuming $\underset{\sim}{\mathbf{HYP}}$ determinacy [**25**], and $\Gamma = (\Sigma^2_1)^{L(\mathbb{R})}$ assuming $(\underset{\sim}{\boldsymbol{\Delta}}^2_1)^{L(\mathbb{R})}$ determinacy [**23**]. Let T be the tree of a Γ-scale on a universal Γ set. Let κ be the supremum of the lengths of the $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ prewellorderings of $\mathbb{R}$.

3.1. A local notion of ordinal definability

Let Γ be an inductive-like pointclass. We define the pointclass $\mathrm{OD}^{<\Gamma}$ as follows.

Definition 3.1.1. A pointset $A \subset \mathscr{X}$ is in $\mathrm{OD}^{<\Gamma}$ if there are Γ pointsets $U, W \subset \mathbb{R} \times \mathscr{X}$, a Γ-norm φ, and an ordinal $\alpha < \kappa$ such that $A = U_x = \neg W_x$ for every $x \in \mathrm{dom}(\varphi)$ with $\varphi(x) = \alpha$.

In terms of a future definition, Definition 3.4.3, we remark that a set $A \subset \omega$ of *integers* is in $\mathrm{OD}^{<\Gamma}$ if and only if it is in C_Γ, the largest countable Γ set of reals. Indeed, the pointclass $\mathrm{OD}^{<\Gamma}$ can be considered as a natural generalization of C_Γ from sets of integers to sets of reals.

For complexity calculations involving sets in $\mathrm{OD}^{<\Gamma}$ it is helpful to use the notion of the companion of Γ from Mochovakis in [**27**]. Notice that in Moschovakis, the companion is defined for $\underset{\sim}{\boldsymbol{\Gamma}}$ where Γ is an inductive-like pointclass, but the adaptation to our lightface situation is straightforward.

A structure $(M; \in, R_1, \ldots, R_n)$, where $R_1, \ldots, R_n$ are relations on M, is called *admissible* if it is transitive, nonempty, closed under pairing and union, and satisfies the Δ_0-separation and Δ_0-collection axiom schemas. In contrast to Moschovakis, we allow all formulas to refer to the relations $R_1, \ldots, R_n$. In particular we have Δ_0-separation and Δ_0-collection for formulas containing $R_1, \ldots, R_n$. Note that an admissible structure must in fact satisfy the ostensibly stronger axiom schemas of Δ_1-separation and Σ_1-collection.

Also in contrast to Moschovakis we do not allow arbitrary reals as unstated parameters in our formulas. We do, however, allow $\mathbb{R}$ itself as an unstated parameter in our formulas, so that real quantification counts as bounded quantification.

Definition 3.1.2. A *companion* of an inductive-like pointclass Γ is a structure $\mathscr{M} = (M; \in, \vec{R})$ such that

- M is a transitive set and $\mathbb{R} \in M$,
- $\vec{R} = R_1, \ldots, R_n$ is a finite sequence of relations on M,
- $\mathscr{M}$ is admissible,
- $\mathscr{M}$ is *projectible* on $\mathbb{R}$, that is, there is a $\Delta_1^{\mathscr{M}}$ partial surjection $\mathbb{R} \to M$,[1]
- $\mathscr{M}$ is *resolvable*, that is, there is a $\Delta_1^{\mathscr{M}}$ sequence $(M_\alpha : \alpha < \mathrm{Ord}^M)$ called a *resolution* such that $M = \bigcup_\alpha M_\alpha$, and
- Γ is the pointclass of all $\Sigma_1^{\mathscr{M}}$ relations on product spaces.

We may speak loosely of "the" companion of Γ by the following theorem.

Theorem 3.1.3 ([**27**]). *Let Γ be an inductive-like pointclass.*

(1) Γ has a companion $\mathscr{M} = (M; \in, \vec{R})$.
(2) If $\mathscr{M} = (M; \in, \vec{R})$ and $\mathscr{M}' = (M'; \in, \vec{R}')$ are companions of Γ, then $M = M'$ and moreover the $\Sigma_1^{\mathscr{M}}$ subsets of M are exactly the $\Sigma_1^{\mathscr{M}'}$ subsets of M'.

For any a companion $\mathscr{M}_\Gamma = (M_\Gamma; \in, \vec{R}_\Gamma)$, the set M_Γ consists of the transitive collapses of all wellfounded binary relations in $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$. Here we relax the definition of "transitive collapse" to apply to non-extensional relations, so many relations will collapse to the same set. The set M_Γ can also be characterized as the smallest admissible set such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma \subset M_\Gamma$. The ordinal height Ord^{M_Γ} of the companion is equal to κ.

The following lemma is essential to many complexity calculations.

[1] The domain of the partial surjection need not be $\Delta_1^{\mathscr{M}}$, only $\Sigma_1^{\mathscr{M}}$.

Lemma 3.1.4. *If Γ is an inductive-like pointclass and φ is a Γ-norm on a Γ set, then φ is $\Delta_1^{\mathscr{M}_\Gamma}$ for any companion $\mathscr{M}_\Gamma$ of Γ. (By this we mean that the graph of φ is $\Delta_1^{\mathscr{M}_\Gamma}$, although in general the domain of φ is only $\Sigma_1^{\mathscr{M}_\Gamma}$.)*

Proof. The proper initial segments of the norm relation $\leq_\varphi$ are all in $\boldsymbol{\Delta}_\Gamma$ and therefore in $\mathscr{M}_\Gamma$, so by admissibility all of the corresponding initial segments of the norm φ are all in $\mathscr{M}_\Gamma$ as well. We have $\varphi(x) = \alpha$ if and only if $f(x) = \alpha$ for some—or equivalently for every—initial segment of φ of length $\alpha + 1$. A function f from a set of reals onto $\alpha + 1$ is an initial segment of φ if and only if for some $x \in \mathbb{R}$ we have

$$y_1, y_2 \in \operatorname{dom} f \;\;\&\;\; f(y_1) \leq f(y_2) \iff y_1, y_2 \in \operatorname{dom} \varphi \;\;\&\;\; \varphi(y_1) \leq \varphi(y_2) \leq \varphi(x).$$

Because φ is a Γ-norm and every Γ pointset is $\Sigma_1^{\mathscr{M}_\Gamma}$, this calculation shows that φ is a $\Delta_1^{\mathscr{M}_\Gamma}$ relation. □

The notion of the companion will be useful in complexity calculations involving $\mathrm{OD}^{<\Gamma}$ pointsets because it allows us to gloss over the details of coding ordinals $\alpha < \kappa$ by reals. We will use the following equivalent characterizations of $\mathrm{OD}^{<\Gamma}$ without further comment throughout the chapter.

Proposition 3.1.5. *Let Γ be an inductive-like pointclass and let $\mathscr{M}_\Gamma = (M_\Gamma; \in, R_\Gamma)$ be a companion of Γ. Take a resolution $(M_\alpha : \alpha < \kappa)$ of $\mathscr{M}_\Gamma = (M_\Gamma; \in, R_\Gamma)$ and define the structure $\mathscr{M}_\alpha = (M_\alpha; \in, R_\Gamma \cap M_\alpha)$. Then for a pointset $A \subset \mathscr{X}$, the following statements are equivalent.*

(1) $A \in \mathrm{OD}^{<\Gamma}$,
(2) A is Δ_1-definable over $\mathscr{M}_\Gamma$ from ordinal parameters,
(3) A is Δ_1-definable over $\mathscr{M}_\alpha$ from ordinal parameters for some $\alpha < \kappa$,
(4) A is definable over $\mathscr{M}_\alpha$ from ordinal parameters for some $\alpha < \kappa$,
(5) A is in $\mathscr{M}_\Gamma$ and $\{A\}$ is Σ_1-definable over $\mathscr{M}_\Gamma$ from ordinal parameters.
(6) A is in $\mathscr{M}_\Gamma$ and $\{A\}$ is Δ_1-definable over $\mathscr{M}_\Gamma$ from ordinal parameters.

Proof. The implication (2) $\implies$ (3) can be seen by using the admissibility of $\mathscr{M}_\Gamma$ to bound the indices α of models α where the witnesses to the Σ_1 formulas defining A and its complement appear. The chain of implications (3) $\implies$ (4) $\implies$ (5) $\implies$ (6) $\implies$ (2) is easy to check, using the fact that the resolution sequence $(M_\alpha : \alpha < \kappa)$ is $\Delta_1^{\mathscr{M}_\Gamma}$. It remains to see the equivalence (1) $\iff$ (2).

Suppose that (1) holds, that is, $A \in \mathrm{OD}^{<\Gamma}$. Take a Γ-norm φ, a pair of Γ pointsets $U, W \subset \mathbb{R} \times \mathscr{X}$, and an ordinal $\alpha < \kappa$ such that $A = U_x = \neg W_x$ for all $x \in \operatorname{dom}(\varphi)$ such that $\varphi(x) = \alpha$. Then the pointset A is defined over the structure $\mathscr{M}_\Gamma$ by the formula $\exists x \in \operatorname{dom}(\varphi)\,(\varphi(x) = \alpha \;\;\&\;\; y \in U_x)$ and also by the formula $\forall x \in \operatorname{dom}(\varphi)\,(\varphi(x) = \alpha \implies y \notin W_x)$. By Lemma 3.1.4 these formulas are equivalent to a Σ_1 formula and a Π_1 formula respectively, so (2) holds.

Conversely, if (2) holds we can take an ordinal $\alpha < \kappa$ and Σ_1 formulas θ and ψ such that $y \in A$ is equivalent to $\mathscr{M}_\Gamma \models \theta(\alpha, y)$ and also to $\mathscr{M}_\Gamma \models \neg\psi(\alpha, y)$. Then the pointsets

$U, W \subset \mathbb{R} \times \mathscr{X}$ defined by

$$(x,y) \in U \iff x \in \mathrm{dom}(\varphi) \ \& \ \mathscr{M}_\Gamma \models \theta(\varphi(x), y)$$
$$(x,y) \in W \iff x \in \mathrm{dom}(\varphi) \ \& \ \mathscr{M}_\Gamma \models \psi(\varphi(x), y),$$

are $\Sigma_1^{\mathscr{M}_\Gamma}$ by Lemma 3.1.4, so they are in Γ. We have $A = U_x = \neg W_x$ for every real $x \in \mathrm{dom}(\varphi)$ with $\varphi(x) = \alpha$. Therefore (1) holds. $\square$

The pointclass of $\mathrm{OD}^{<\Gamma}$ sets is Σ_1-definable over the model $\mathscr{M}_\Gamma$, just as full ordinal-definability is Σ_1-definable in the presence of the full Axiom of Replacement. This suggests that $\mathrm{OD}^{<\Gamma}$ is the natural notion of ordinal-definability for the structure $\mathscr{M}_\Gamma$. Using a resolution sequence of $\mathscr{M}_\Gamma$, an easy computation establishes the following useful result.

Proposition 3.1.6. *There is an enumeration of the* $\mathrm{OD}^{<\Gamma}$ *pointsets in order type* κ *that is* $\Delta_1^{\mathscr{M}_\Gamma}$-good, *meaning that the sequence of its initial segments is* $\Delta_1^{\mathscr{M}_\Gamma}$.

3.2. The envelope of a pointclass

Under AD the envelope of the boldface pointclass $\underset{\sim}{\mathbf{\Gamma}}$ is defined as follows.

Definition 3.2.1 (Martin, AD). For a product space $\mathscr{X}$ and a pointset $A \subset \mathscr{X}$, we say $A \in \underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Gamma}}, \kappa)$ if there is a sequence $(A_\alpha : \alpha < \kappa)$ of $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ subsets of $\mathscr{X}$ such that for every countable $\sigma \subset \mathscr{X}$ there is an $\alpha < \kappa$ with $A \cap \sigma = A_\alpha \cap \sigma$.[2]

Under ZFC, if $\kappa \geq \mathfrak{c}$ then we trivially have every pointset in $\underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Gamma}}, \kappa)$. Even if the continuum is large, this seems unlikely to be an interesting definition in the absence of AD. We give another definition of the envelope, Env, that differs from Martin's in two ways. First, Γ and $\mathrm{Env}(\Gamma)$ are lightface rather than boldface pointclasses, and second, our definition of Env makes sense without full AD—in particular, in an ambient universe satisfying ZFC.

Definition 3.2.2. For a product space $\mathscr{X}$ and a pointset $A \subset \mathscr{X}$, define

- $A \in \mathrm{Env}(\Gamma)$ if for every countable set $\sigma \subset \mathscr{X}$ there is a set $A' \in \mathrm{OD}^{<\Gamma}$ such that $A \cap \sigma = A' \cap \sigma$.
- $A \in \underset{\sim}{\mathbf{Env}}(\Gamma)$ if $A \in \mathrm{Env}(\Gamma(x))$ for some $x \in \mathbb{R}$.

When the choice of product space is not important, we will sometimes speak of pointsets $A \subset \mathbb{R}$ and leave the generalization to other product spaces to the reader. Notice that for a set $A \subset \omega$ of *integers*, the statements $A \in \mathrm{Env}(\Gamma)$, $A \in \mathrm{OD}^{<\Gamma}$, and $A \in C_\Gamma$ are all equivalent. Our definition of $\underset{\sim}{\mathbf{Env}}(\Gamma)$ generalizes Martin's definition of $\underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Gamma}}, \kappa)$:

Proposition 3.2.3 (AD). *For a pointset* $A \subset \mathscr{X}$*, the following are equivalent:*

(1) $A \in \underset{\sim}{\mathbf{Env}}(\Gamma)$*, and*
(2) $A \in \underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Gamma}}, \kappa)$.

[2]The original definition may have used sequences of $\underset{\sim}{\mathbf{\Gamma}}$ sets, but this is equivalent: see [**7**].

Proof. ($1 \implies 2$) If $A \in \mathrm{Env}(\Gamma(x))$ for $x \in \mathbb{R}$, then taking any enumeration $(A_\alpha : \alpha < \kappa)$ of the $\mathrm{OD}^{<\Gamma}(x)$ subsets of $\mathscr{X}$ shows that $A \in \underset{\sim}{\boldsymbol{\Lambda}}(\underset{\sim}{\boldsymbol{\Gamma}}, \kappa)$.

($2 \implies 1$) Let $(A_\alpha : \alpha < \kappa)$ be a sequence of $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ subsets of $\mathscr{X}$ witnessing $A \in \underset{\sim}{\boldsymbol{\Lambda}}(\underset{\sim}{\boldsymbol{\Gamma}}, \kappa)$. Take a Γ-norm φ onto κ. Let $U \subset \mathbb{R} \times \mathscr{X}$ be a universal Γ set. By the Moschovakis Coding Lemma (see [**28**]) the relations

$$\{(x, y) : x \in \mathrm{dom}(\varphi) \;\;\&\;\; A_{\varphi(x)} = U_y\} \text{ and}$$
$$\{(x, y) : x \in \mathrm{dom}(\varphi) \;\;\&\;\; A_{\varphi(x)} = \neg U_y\}$$

have choice sets in $\underset{\sim}{\boldsymbol{\Gamma}}$, say in $\Gamma(z)$ where $z \in \mathbb{R}$. So the relation $\{(x, \alpha) : x \in A_\alpha\}$ is $\Delta_1(z)$ over the expanded structure $(M_\Gamma; \in, R_\Gamma, \varphi)$, which is also a companion of Γ. Therefore each A_α is $\mathrm{OD}^{<\Gamma}(z)$, so $A \in \mathrm{Env}(\Gamma(z))$. $\square$

The following argument is derived from Kechris–Woodin [**17**]. It is also similar to the proof in [**15**] that Δ^1_2-determinacy implies that OD-determinacy holds in $L[x]$ for a cone of reals x.

Theorem 3.2.4. *Let Γ be an inductive-like pointclass*[3] *and suppose that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined. Then $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is determined.*

Proof. Let $A \in \mathrm{Env}(\Gamma)$; the proof will easily relativize to any real. Let $(A_\alpha : \alpha < \kappa)$ be a $\Sigma_1^{\mathscr{M}_\Gamma}$-good enumeration of the $\mathrm{OD}^{<\Gamma}$ sets of reals. Suppose our set A is not determined. Given $t \in \mathbb{R}$, by $\mathsf{DC}_\mathbb{R}$ there is a countable Turing ideal $M \subset \mathbb{R}$ containing t such that for all $\sigma \in M$ there are $x, y \in M$ such that $\sigma * y \notin A_\alpha$ and $x * \sigma \in A$. That is, in the game G_A neither player has a strategy in M (coded by a real in M, technically) that wins against all plays in M. Because M is countable there is an $\alpha < \kappa$ such that $A_\alpha \cap M = A \cap M$.

Define $\alpha(t)$ as the least $\alpha < \kappa$ such that there is a countable Turing ideal $M \ni t$ with the property that for all $\sigma \in M$ there are $x, y \in M$ such that $\sigma * y \notin A_\alpha$ and $x * \sigma \in A_\alpha$. The function $\mathbb{R} \to \kappa$ given by $t \mapsto \alpha(t)$ is total and Δ_1-definable over $\mathscr{M}_\Gamma$, so the following game is in Δ and is therefore determined:

$$\text{(G)} \qquad \begin{array}{lllll} \mathrm{I} & x_0, z_0 & & x_1, z_1 \ldots & \\ \mathrm{II} & & y_0, s_0 & & y_1, s_1, \ldots \end{array} \qquad \mathrm{I} \text{ wins} \iff x \oplus y \in A_{\alpha(z \oplus s)}.$$

Assume that player I has a winning strategy σ_G in G. (The other case is similar.) Fix a real s coding σ_G. By the definition of $\alpha(s)$ we may fix a countable Turing ideal $M \ni s$ such that for every $\sigma \in M$ there is $y \in M$ such that $\sigma * y \notin A_{\alpha(s)}$. Let σ be the strategy for player I in the game $G_{A_{\alpha(s)}}$ that is obtained from σ_G by pretending that s was played alongside y by player II in G, and ignoring the z produced alongside x by σ_G. That is, for $\bar{y} \in \omega^n$ we have

$$\sigma(\bar{y}) = \bar{x} \iff \sigma_G(\bar{y}, s \restriction n) = (\bar{x}, \bar{z}) \text{ for some } \bar{z} \in \omega^n.$$

This strategy σ can be computed from s, so it is in M. Therefore there is some $y \in M$ with $\sigma * y \notin A_{\alpha(s)}$. On the other hand, because σ_G is a winning strategy for player I in G

[3]The proof shows that the scale property of Γ is not needed, only the pre-wellordering property.

there is a real $z \in M$ such that $\sigma * y \in A_{\alpha(z \oplus s)}$. We will derive a contradiction by showing that $\alpha(z \oplus s) = \alpha(s)$. The function α is increasing by definition so we have $\alpha(z \oplus s) \geq \alpha(s)$. But $z \oplus s \in M$, so $\alpha(z \oplus s) \leq \alpha(s)$ by the minimization in the definition of α. This is a contradiction. $\square$

The following proposition shows that sets in $\mathrm{Env}(\Gamma)$, being pieced together from $\mathrm{OD}^{<\Gamma}$ sets, themselves satisfy a form of ordinal-definability related to Γ:

PROPOSITION 3.2.5. *Let Γ be an inductive-like pointclass and suppose that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined. Then for any universal Γ set U, there is a well-ordering of $\mathrm{Env}(\Gamma)$ of order type $\leq \Theta$ definable from U.*

PROOF. Let $(A_\alpha : \alpha < \kappa)$ be a $\Sigma_1^{\mathscr{M}_\Gamma}$-good enumeration of the $\mathrm{OD}^{<\Gamma}$ sets of reals. For $A \in \mathrm{Env}(\Gamma)$ and d a Turing degree define

$$\alpha_d(A) = \text{the least } \alpha \text{ such that } (\forall x \leq_T d)\,(x \in A \iff x \in A_\alpha).$$

for $A', B' \in \mathrm{OD}^{<\Gamma}$ the sets

$$\{d : \alpha_d(A') < \alpha_d(B')\}, \{d : \alpha_d(A') = \alpha_d(B')\}, \text{ and } \{d : \alpha_d(A') > \alpha_d(B')\}$$

are in $\mathrm{OD}^{<\Gamma}$, so for $A, B \in \mathrm{Env}(\Gamma)$, the sets

$$\{d : \alpha_d(A) < \alpha_d(B)\}, \{d : \alpha_d(A) = \alpha_d(B)\}, \text{ and } \{d : \alpha_d(A) > \alpha_d(B)\}$$

are in $\mathrm{Env}(\Gamma)$ and by Theorem 3.2.4 exactly one of them contains a cone. Therefore we can define a prewellordering of $\mathrm{Env}(\Gamma)$ by

$$\begin{aligned} A < B \iff & A <_{\mathrm{W}} B, \text{ or} \\ & A \equiv_{\mathrm{W}} B \;\&\; \alpha_d(A) < \alpha_d(B) \text{ for a Turing cone of } d. \end{aligned}$$

This is clearly a pre-wellordering. Because it refines the Wadge pre-wellordering, each of its proper initial segments is a surjective image of $\mathbb{R}$ and therefore its length is at most Θ. If $A \neq B$ then $\alpha_d(A) \neq \alpha_d(B)$ on a cone, so it is in fact a wellordering. It is definable from $\mathscr{M}_\Gamma$, which in turn is definable from any universal Γ set. $\square$

The following theorem is due to Martin in the context of AD. We follow the proof given in [**7**], checking that it works under our limited determinacy hypothesis with our modified definition of the envelope.

THEOREM 3.2.6. *Let Γ be an inductive-like pointclass and suppose that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined. Then $\mathrm{Env}(\Gamma)$ is closed under real quantification.*

PROOF. Let $A \in \mathrm{Env}(\Gamma)$, say $A \subset \mathscr{X} \times \mathbb{R}$. We will show that the set $B = \exists^{\mathbb{R}} A \subset \mathscr{X}$ is in $\mathrm{Env}(\Gamma)$. The case of universal quantification is completely analogous. Let $(A_\alpha : \alpha < \kappa)$ be a $\Sigma_1^{\mathscr{M}_\Gamma}$-good enumeration of the $\mathrm{OD}^{<\Gamma}$ subsets of $\mathscr{X} \times \mathbb{R}$. We consider a real z as coding a countable partial function f_z with $\mathrm{dom}(f_z) \subset \mathscr{X}$ and $\mathrm{ran}(f_z) \subset \{0, 1\}$. Given a Turing degree d let $\alpha_z(d)$ be the least $\alpha < \kappa$, if it exists, such that for all $x \in \mathrm{dom}(f_z)$ we have

$$f_z(x) = 1 \iff \exists y \leq_T d\, (x, y) \in A_\alpha.$$

Define $C \subset \mathbb{R}$ to be the set of $z \in \mathbb{R}$ such that $\alpha_z(d)$ is defined for a cone of d. We have $C \in \Sigma_1^{\mathscr{M}_\Gamma}$ by the admissibility of the companion, so $C \in \Gamma$. Given $z, z' \in C$, take a Turing degree d_0 be such that $\alpha_z(d)$ and $\alpha_{z'}(d)$ are defined for all $d \geq_T d_0$. The sets

$$\{d \geq_T d_0 : \alpha_z(d) < \alpha_{z'}(d)\},$$
$$\{d \geq_T d_0 : \alpha_z(d) = \alpha_{z'}(d)\}, \text{ and}$$
$$\{d \geq_T d_0 : \alpha_z(d) > \alpha_{z'}(d)\}$$

are all $\Delta_1^{\mathscr{M}_\Gamma}(z, z', d_0)$ and therefore in $\underset{\sim}{\mathbf{\Delta}}_\Gamma$, so by $\underset{\sim}{\mathbf{\Delta}}_\Gamma$-determinacy exactly one of them contains a cone. So we can define a regular norm φ on C by $\varphi(z) \leq \varphi(z')$ if $\alpha_z(d) \leq \alpha_{z'}(d)$ for a cone of d. The relations $\leq_\varphi^*$ and $<_\varphi^*$ defined by

$$z \leq_\varphi^* z' \iff z \in C \;\&\; (z' \in C \implies \varphi(z) \leq \varphi(z')) \text{ and}$$
$$z <_\varphi^* z' \iff z \in C \;\&\; (z' \in C \implies \varphi(z) < \varphi(z'))$$

are Σ_1-definable over $\mathscr{M}_\Gamma$ and are therefore in Γ, so φ is a Γ-norm. Therefore $\mathrm{ran}(\varphi) \subset \kappa$ and we assume that the companion $\mathscr{M}_\Gamma$ has been expanded by a relation for φ—in particular, this does not change its notion of Σ_1-definability.

Now let $\sigma \subset \mathscr{X}$ be countable. Take $z \in \mathbb{R}$ with $\mathrm{dom}(f_z) = \sigma$ and $\forall x \in \sigma\, f_z(x) = 1 \iff x \in B$. Notice that we have $z \in C$. Let $\alpha = \varphi(z)$. Notice that for every $x \in \mathbb{R}$ there is a $z' \in C$ with $\varphi(z') = \alpha$ and $x \in \mathrm{dom}(f_{z'})$. So the set $B' \subset \mathscr{X}$ defined by

$$\begin{aligned} x \in B' &\iff \exists z' \in C\, (\varphi(z') = \alpha \;\&\; x \in \mathrm{dom}(f_{z'}) \;\&\; f_{z'}(x) = 1) \\ &\iff \forall z' \in C\, (\varphi(z') = \alpha \;\&\; x \in \mathrm{dom}(f_{z'}) \implies f_{z'}(x) = 1), \end{aligned}$$

is $\Delta_1^{\mathscr{M}_\Gamma}(\alpha)$ and satisfies $B' \cap \sigma = B \cap \sigma$. □

We haven't actually showed that the envelope contains any sets beyond $\mathrm{OD}^{<\Gamma}$ itself. The following lemma does this and has a few other uses as well.

Lemma 3.2.7. *If $A \subset \mathbb{R}$ is $\Sigma_1^{\mathscr{M}_\Gamma}(\vec{\alpha})$ for some $\vec{\alpha} \in \kappa^{<\omega}$ then $A \in \mathrm{Env}(\Gamma)$.*

Proof. Let $\sigma \subset \mathbb{R}$ be countable and take a Σ_1 formula θ such that

$$x \in A \iff \mathscr{M}_\Gamma \models \theta[x, \vec{\alpha}].$$

Let $(M_\xi : \xi < \kappa)$ be a resolution of the companion $\mathscr{M}_\Gamma = (M_\Gamma; \in, R_\Gamma)$. We may assume that $M_\xi \subseteq M_{\xi'}$ for $\xi < \xi'$. Let

$$\mathscr{M}_\xi = (M_\xi; \in, R_\Gamma \cap M_\xi).$$

Because σ is countable and $\mathrm{cof}(\kappa) > \omega$, if we choose a sufficiently large $\xi < \kappa$—in particular larger than $\max(\vec{\alpha})$—and we define a set of reals A' by

$$x \in A' \iff \mathscr{M}_\xi \models \theta[x, \vec{\alpha}],$$

then we have $A \cap \sigma = A' \cap \sigma$. The set A' is $\mathrm{OD}^{<\Gamma}$, so it witnesses that $A \in \mathrm{Env}(\Gamma)$. $\square$

Next we establish a few more closure properties of the envelope.

THEOREM 3.2.8. *Let Γ be an inductive-like pointclass and suppose that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined.*

(1) $\mathrm{Env}(\Gamma)$ *contains* Γ *and is closed under* $\neg$, $\wedge$, $\vee$, $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, *and recursive substitution.*

(2) $\utilde{\mathbf{Env}}(\Gamma)$ *contains* $\utilde{\boldsymbol{\Gamma}}$ *and is closed under* $\neg$, $\wedge$, $\vee$, $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, *and Wadge reducibility.*

PROOF. We have $\Gamma \subset \mathrm{Env}(\Gamma)$ by the special case of Lemma 3.2.7 where $\vec{\alpha} = \emptyset$. The closure of $\mathrm{Env}(\Gamma)$ under Boolean combinations is immediate from the closure of $\mathrm{OD}^{<\Gamma}$ under Boolean combinations, and the closure of $\mathrm{Env}(\Gamma)$ under real quantifiers is Theorem 3.2.6.

Now let $A \in \mathrm{Env}(\Gamma)$ and let $g : \mathbb{R} \to \mathbb{R}$ be a recursive function. Given a countable $\sigma \subset \mathbb{R}$ the image $g``\sigma$ is also countable, so we can take $A' \in \mathrm{OD}^{<\Gamma}$ with $A \cap g``\sigma = A' \cap g``\sigma$. Then $g^{-1}(A') \in \mathrm{OD}^{<\Gamma}$ and $g^{-1}(A) \cap \sigma = g^{-1}(A') \cap \sigma$, so we have shown that $g^{-1}(A) \in \mathrm{Env}(\Gamma)$.

Part (2) follows easily from part (1). $\square$

Lastly we use the following theorem to place a limitation on the extent of the envelope.

THEOREM 3.2.9 (Martin [**20**]). *The $\check{\Gamma}$ relation $\{(x, y) : y \notin C_\Gamma(x)\}$ cannot be uniformized by a set in $\utilde{\mathbf{Env}}(\Gamma)$.*

PROOF. The relation $\{(x, y) : y \notin C_\Gamma(x)\}$ is $\Pi_1^{\mathscr{M}_\Gamma}$ and is therefore indeed in $\check{\Gamma}$. Let $f : \mathbb{R} \to \mathbb{R}$ be in $\utilde{\mathbf{Env}}(\Gamma)$, say in $\mathrm{Env}(\Gamma(x))$ where $x \in \mathbb{R}$. By the closure properties of $\mathrm{Env}(\Gamma(x))$ in Theorem 3.2.8 relativized to x, we have $f(x) \in \mathrm{Env}(\Gamma(x))$ and if we let $A \subset \omega$ code $f(x)$ in any natural way then $A \in \mathrm{Env}(\Gamma(x))$. But A is a set of integers, so this is equivalent to saying that $A \in \mathrm{OD}^{<\Gamma}$ and also to saying that $A \in C_\Gamma$. So $x \in C_\Gamma$, a contradiction. $\square$

3.3. Digression: gaps in $L(\mathbb{R})$

Following [**35**] we define the notions of weak and strong gaps in $L(\mathbb{R})$. We use the Jensen hierarchy $(J_\alpha : \alpha \in \mathrm{Ord})$ for $L(\mathbb{R})$ but the reader would not miss much by pretending that $J_\kappa(\mathbb{R}) = L_\kappa(\mathbb{R})$. However we should note here that the ordinal height of $J_\alpha(\mathbb{R})$ is $\omega\alpha$ and not α.

DEFINITION 3.3.1. A Σ_1-*gap*—or simply a *gap*—in $L(\mathbb{R})$ is a maximal interval of ordinals $[\kappa, \beta]$ such that $J_\kappa(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$ and $\beta \leq \Theta^{L(\mathbb{R})}$.

The superscript $\mathbb{R}$ means that real parameters are allowed. Recall that by convention $\mathbb{R}$ itself is allowed as an unstated parameter in all formulas. The gaps partition the interval $[0, \Theta^{L(\mathbb{R})}]$. We say κ *begins a gap in* $L(\mathbb{R})$ if $[\kappa, \beta]$ is a gap for some β. This means that new Σ_1 facts about reals are witnessed cofinally often below κ in the Jensen hierarchy. If κ begins a gap then there is a $\Sigma_1^{J_\kappa(\mathbb{R})}$ partial surjection $\mathbb{R} \mapsto J_\kappa(\mathbb{R})$. In general, we take the expression $\rho_n^{J_\kappa(\mathbb{R})} = \mathbb{R}$ ("the n^{th} projectum of $J_\kappa(\mathbb{R})$ is $\mathbb{R}$") to mean that there is a $\Sigma_n^{J_\kappa(\mathbb{R})}$ partial surjection $\mathbb{R} \mapsto J_\kappa(\mathbb{R})$. This is equivalent to the existence of a $\Sigma_n^{J_\kappa(\mathbb{R})}$ set of reals that is not in $J_\kappa(\mathbb{R})$.

If κ begins a gap and $J_\kappa(\mathbb{R})$ satisfies AD then the pointclass $\Sigma_1^{J_\kappa(\mathbb{R})}$ has the scale property by [**35**]. If $J_\kappa(\mathbb{R})$ is not admissible (does not satisfy Σ_1-collection) then we have $\Pi_2^{J_\kappa(\mathbb{R})} = \forall^{\mathbb{R}}\Sigma_1^{J_\kappa(\mathbb{R})}$, $\Sigma_3^{J_\kappa(\mathbb{R})} = \exists^{\mathbb{R}}\Pi_2^{J_\kappa(\mathbb{R})}$ and so on, and if these pointclasses are determined then they all have the scale property by the second periodicity theorem.

Assume now that κ begins a gap and $J_\kappa(\mathbb{R})$ satisfies AD and is admissible. Define the pointclass

$$\Gamma = \Sigma_1^{J_\kappa(\mathbb{R})}.$$

This pointclass is closed under real quantifiers and therefore is inductive-like, and the corresponding pointclass $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is equal to $J_\kappa(\mathbb{R}) \cap \wp(\mathbb{R})$ and therefore is determined. The prewellordering ordinal of $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is κ itself by admissibility.

The underlying set of the companion $\mathscr{M}_\Gamma$ is given by $M_\Gamma = J_\kappa(\mathbb{R})$ because this is the least admissible set containing all $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ sets of reals.

So in this case we have

$$\mathrm{OD}^{<\Gamma} = \mathrm{OD}^{<\kappa}$$

where for an ordinal β we say a pointset A is in $\mathrm{OD}^{<\beta}$ if it is ordinal-definable over the structure $(J_\alpha(\mathbb{R}); \in)$ for some $\alpha < \beta$. Similarly, we write OD^{β} for $\mathrm{OD}^{<(\beta+1)}$, the class of sets $A \subset \mathbb{R}$ that are definable over $L_\beta(\mathbb{R})$ itself from ordinal parameters by any formula (not necessarily Δ_1.) So in this notation we have $A \in \mathrm{Env}(\Gamma)$ if and only if for every countable $\sigma \subset \mathbb{R}$ there is a set $A' \in \mathrm{OD}^{<\kappa}$ with $A \cap \sigma = A' \cap \sigma$.

We would like to analyze the extent of the envelope in terms of the Jensen hierarchy. To do this we need to use a reflection property introduced by Steel in [**35**].

Definition 3.3.2 ([**35**]). The Σ_1-gap $[\kappa, \beta]$ in $L(\mathbb{R})$ is *strong* if $J_\kappa(\mathbb{R})$ is admissible, and letting $n < \omega$ be least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, every Σ_n-type realized in $J_\beta(\mathbb{R})$ is realized in $J_\alpha(\mathbb{R})$ for some $\alpha < \beta$ (and therefore for some $\alpha < \kappa$ by the definition of a gap.) Otherwise we say the gap is *weak*.

We remark that this definition differs from the standard one in that it considers an "improper" admissible gap $[\kappa, \kappa]$ to be strong. Under this definition, the first example of a strong gap is $[\kappa^{\mathbb{R}}, \kappa^{\mathbb{R}}]$ where $J_{\kappa^{\mathbb{R}}}(\mathbb{R})$ is the first admissible level of $L(\mathbb{R})$. In this example the pointclass $\Sigma_1^{J_\kappa(\mathbb{R})}$ is equal to IND, the pointclass of sets definable by positive elementary induction on $\mathbb{R}$.

Proposition 3.3.3. *If $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying AD and $[\kappa, \beta]$ is a gap, then*

(1) $\mathrm{OD}^{<\beta} \subset \mathrm{Env}(\Sigma_1^{J_\kappa(\mathbb{R})})$*, and*
(2) $\mathrm{OD}^{\beta} \subset \mathrm{Env}(\Sigma_1^{J_\kappa(\mathbb{R})})$ *if $[\kappa, \beta]$ is a strong gap.*

Proof. (1) Given a set of reals $A \in \mathrm{OD}^{<\beta}$ and a countable set $\sigma \subset \mathbb{R}$, let $a = A \cap \sigma$. Then in $J_\beta(\mathbb{R})$ the pair (σ, a) satisfies the Σ_1 formula saying "there is a set of reals A' that is ordinal-definable over some level, and such that $A' \cap \sigma = a$." Applying the definition of a Σ_1-gap to some real coding the pair (σ, a), this formula is also true of (σ, a) in $J_\kappa(\mathbb{R})$, so there is a set $A' \in \mathrm{OD}^{<\kappa}$ with $A' \cap \sigma = a = A \cap \sigma$.

(2) Let n be least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. Averaging over real parameters, we get a partial surjection $\mathbb{R} \to J_\beta(\mathbb{R})$ that is Σ_n-definable over $J_\beta(\mathbb{R})$ from ordinal parameters only, so the OD^β sets of reals are generated under Boolean combinations and real quantification from sets of reals that are $\Sigma_n^{J_\beta(\mathbb{R})}$ in ordinal parameters. The envelope is closed under Boolean combinations and real quantification by Theorem 3.2.8, so it is enough to show that it contains the set $A \subset \mathbb{R}$ given by

$$x \in A \iff J_\beta(\mathbb{R}) \models \theta[x, \bar{\beta}]$$

where $\bar{\beta} \in (\omega\beta)^{<\omega}$ and θ is a Σ_n formula.

Given a countable set of reals σ, take a real z coding σ. By the definition of a strong gap the Σ_n-type of $(z, \bar{\beta})$ in $J_\beta(\mathbb{R})$ is realized in some $J_\alpha(\mathbb{R})$ with $\alpha < \kappa$. The real z is determined by its type, so there is a finite sequence of ordinals $\bar{\alpha} \in (\omega\alpha)^{<\omega}$ such that the Σ_n-type of $(z, \bar{\alpha})$ in $J_\alpha(\mathbb{R})$ is equal to the Σ_n-type of $(z, \bar{\beta})$ in $J_\beta(\mathbb{R})$. Then we have $A \cap \sigma = A' \cap \sigma$ where $A' \in \mathrm{OD}^{<\kappa}$ is the set defined by $x \in A' \iff J_\alpha(\mathbb{R}) \models \theta[x, \bar{\alpha}]$. This shows that $A \in \mathrm{Env}(\Gamma)$. □

Relativizing Lemma 3.3.3 to arbitrary reals, we see that if $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying AD and $[\kappa, \beta]$ is a gap, then

(1′) $J_\beta(\mathbb{R}) \cap \wp(\mathbb{R}) \subset \underset{\sim}{\mathbf{Env}}(\Sigma_1^{J_\kappa(\mathbb{R})})$, and

(2′) $J_{\beta+1}(\mathbb{R}) \cap \wp(\mathbb{R}) \subset \underset{\sim}{\mathbf{Env}}(\Sigma_1^{J_\kappa(\mathbb{R})})$ if $[\kappa, \beta]$ is a strong gap.

Part (1′) does not give us any more information about the extent of determinacy in $L(\mathbb{R})$—we already know that AD holds in $J_\beta(\mathbb{R})$ by the definition of a gap because it is a Π_1 statement. However part (2′) gives us a nontrivial determinacy transfer theorem, whose original proof interleaved proofs of Theorems 3.2.4 and 3.2.6 without explicitly using the notion of the envelope.

Corollary 3.3.4 (Kechris–Woodin [**17**]). *If $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying AD and beginning a strong gap $[\kappa, \beta]$, then $J_{\beta+1}(\mathbb{R}) \models \mathsf{AD}$.*

Proof. Given a set of reals $A \in J_{\beta+1}(\mathbb{R})$, we have $A \in \underset{\sim}{\mathbf{Env}}(\Sigma_1^{J_\kappa(\mathbb{R})})$ by Proposition 3.3.3, so A is determined by Theorem 3.2.4. □

As another corollary we can show that no new reals become ordinal-definable at the end of a strong gap:

Corollary 3.3.5 (Martin [**20**]). *If $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying AD and beginning a strong gap $[\kappa, \beta]$, then every real $x \in \mathrm{OD}^\beta$ is in $\mathrm{OD}^{<\kappa}$, or equivalently, is in C_Γ where $\Gamma = \Sigma_1^{J_\kappa(\mathbb{R})}$.*

Proof. Code $x \in \mathrm{OD}^\beta$ as a set of integers $A \subset \omega$. By Proposition 3.3.3 we have $A \in \mathrm{Env}(\Gamma)$. Because $A \subset \omega$ this means $A \in \mathrm{OD}^{<\Gamma}$ by definition, or equivalently, $A \in \mathrm{OD}^{<\kappa}$. □

3.4. The models $L[T, x]$

Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined. Let T be the tree of a Γ-scale on a universal Γ set. For some applications of the envelope we will need to relate $\operatorname{Env}(\Gamma)$ to the models $L[T, x]$ where x is a real. For the most part the proofs in this section are not new; we just have to check that our assumption of $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ determinacy is enough.

Lemma 3.4.1. *Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined. Let κ be the pre-wellordering ordinal of Γ and let T on $\omega \times \kappa$ be the tree of a Γ-scale on a universal Γ set. Then the ordinal κ is a regular cardinal in $L[T]$.*

Proof. This is a straightforward adaptation of the "projective set theory" introduced in Harrington–Kechris [**6**], which is used to show under PD that the projective ordinal $\underset{\sim}{\boldsymbol{\delta}}^1_{2n+1}$ is a regular cardinal in $L[T_{2n+1}]$ where T_{2n+1} is the tree of a Π^1_{2n+1}-scale on a universal Π^1_{2n+1} set. We sketch the adaptation of that argument to the following projective-like hierarchy (a type IV hierarchy in the terminology of [**16**]):

- $\underset{\sim}{\mathbf{\Gamma}}_0 = \underset{\sim}{\mathbf{\Gamma}}$,
- $\underset{\sim}{\mathbf{\Gamma}}_1$ is the class of sets $A \cap B$ where $A \in \underset{\sim}{\mathbf{\Gamma}}$ and $B \in \check{\underset{\sim}{\mathbf{\Gamma}}}$,
- $\underset{\sim}{\mathbf{\Gamma}}_{2n+2} = \exists^{\mathbb{R}} \underset{\sim}{\mathbf{\Gamma}}_{2n+1}$, and
- $\underset{\sim}{\mathbf{\Gamma}}_{2n+3} = \forall^{\mathbb{R}} \underset{\sim}{\mathbf{\Gamma}}_{2n+2}$.

These pointclasses are contained in $\underset{\sim}{\mathbf{Env}}(\Gamma)$ by Theorem 3.2.8 so they are all determined. By [**16**] they all have the prewellordering property. Let $\underset{\sim}{\boldsymbol{\delta}}_n$ be the supremum of the lengths of the $\underset{\sim}{\mathbf{\Delta}}_{\Gamma_n}$ prewellorderings. As in [**6**] one can show that each $\underset{\sim}{\boldsymbol{\delta}}_n$ cannot be singularized by functions coded by pointsets in $\bigcup_{i<\omega} \underset{\sim}{\mathbf{\Gamma}}_i$ and that there is a measure defined on the subsets of $\underset{\sim}{\boldsymbol{\delta}}_n$ that are coded by sets of reals in $\bigcup_{i<\omega} \underset{\sim}{\mathbf{\Gamma}}_i$. Moreover, sufficiently large $\underset{\sim}{\boldsymbol{\delta}}_n$'s are Silver indiscernibles for $L[T]$, and $T^\sharp$ exists and is coded by an ω-sequence of sets in $\bigcup_{i<\omega} \underset{\sim}{\mathbf{\Gamma}}_i$. This shows that every subset of κ in $L[T]$ is coded by a set in $\bigcup_{i<\omega} \underset{\sim}{\mathbf{\Gamma}}_i$ and therefore cannot singularize κ. $\square$

We collect some related lemmas on $L[T]$ and $\mathrm{OD}^{<\Gamma}$ here:

Lemma 3.4.2. *Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined. Let κ be the pre-wellordering ordinal of Γ and let T on $\omega \times \omega \times \kappa$ be the tree of a Γ-scale on a universal Γ set.*

(1) If $y \in L[T] \cap \mathbb{R}$ then $y \in \mathrm{OD}^{<\Gamma}$,
(2) the set of $\mathrm{OD}^{<\Gamma}$ reals is a countable Γ set, and
(3) every countable Γ set of reals is contained in $L[T]$.

Proof. (1) Because κ is regular in $L[T]$ by Lemma 3.4.1, a Skolem hull argument shows that $y \in L_\alpha[T \restriction \gamma]$ for some ordinals α and γ with $\gamma < \alpha < \kappa$. Say T is the tree of the Γ-scale $\vec{\varphi}$. By Lemma 3.1.4 the norms of $\vec{\varphi}$ are all $\Delta_1^{\mathscr{M}_\Gamma}$, and the uniformity in the proof shows that the scale $\vec{\varphi}$ itself is $\Delta_1^{\mathscr{M}_\Gamma}$. Therefore the tree T is $\Delta_1^{\mathscr{M}_\Gamma}$ and we can calculate that y is $\Delta_1^{\mathscr{M}_\Gamma}$.

(2) The set of $\mathrm{OD}^{<\Gamma}$ reals is a $\Sigma_1^{\mathscr{M}_\Gamma}$ pointset and is therefore in Γ. To see that it is countable, let $(M_\alpha : \alpha < \kappa)$ be a resolution of the companion $\mathscr{M}_\Gamma = (M_\Gamma; \in, R_\Gamma)$ of Γ, so that the $\mathrm{OD}^{<\Gamma}$ reals are exactly the reals that are OD over one of the structures $\mathscr{M}_\alpha = (M_\alpha; \in, R_\Gamma \cap M_\alpha)$ for some $\alpha < \kappa$.

For any $\beta < \kappa$, the set $\mathrm{OD}^{<\beta}$ of reals that are OD over $\mathscr{M}_\alpha$ for some $\alpha < \beta$ has a natural wellordering definable from x and β inside the AD model $\mathscr{M}_\Gamma$, so it is countable. By the admissibility of $\mathscr{M}_\Gamma$, the nondecreasing $\Delta_1^{\mathscr{M}_\Gamma}$ function $\kappa \to \omega_1$ taking β to the length of this wellordering of $\mathrm{OD}^{<\beta}$ must be eventually constant or else we would get a $\Delta_1^{\mathscr{M}_\Gamma}$ function $\omega_1 \to \kappa$ singularizing κ. This shows that the entire set of $\mathrm{OD}^{<\Gamma}$ reals is countable.

(3) If A is a countable Γ set of reals, or in fact any thin Γ set of reals—that is, not containing a perfect set—we have $A = p[S]$ for some tree $S \in L[T]$ and therefore $A = p[S] \subset L[T]$ by the Mansfield–Solovay theorem [**19, 34**]. $\square$

Therefore the following notion from Kechris [**13**] is well-defined in our context.

Definition 3.4.3. C_Γ denotes the largest countable Γ set of reals.

We can use this definition of C_Γ to rewrite Lemma 3.4.2 as an equivalence. We emphasize that the proof of this equivalence was already in [**6**] and [**13**] and we are just checking that it goes through under our limited determinacy hypothesis.

Corollary 3.4.4. *Let Γ be an inductive-like pointclass and suppose that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined. Let κ be the pre-wellordering ordinal of Γ and let T on $\omega \times \kappa$ be the tree of a Γ-scale on a universal Γ set. The we have*

$$\begin{aligned} C_\Gamma &= L[T] \cap \mathbb{R} \\ &= \mathrm{OD}^{<\Gamma} \cap \mathbb{R}. \end{aligned}$$

In particular, the set $L[T] \cap \mathbb{R}$ is countable.

The preceding results about C_Γ, $\mathrm{OD}^{<\Gamma}$, and $L[T]$ all relativize in a straightforward way to $C_\Gamma(x)$, $\mathrm{OD}^{<\Gamma}(x)$, and $L[T,x]$ for any real x. We can use the models $L[T,x]$ to formulate and prove an equivalent definition of the envelope. We use the proof from [**8**], checking that $\utilde{\boldsymbol{\Delta}}_\Gamma$ determinacy is enough.

Theorem 3.4.5. *Let Γ be an inductive-like pointclass and suppose that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined. Let κ be the pre-wellordering ordinal of Γ and let T on $\omega \times \kappa$ be the tree of a Γ-scale on a universal Γ set. Then for any set $A \subset \mathbb{R}$ the following conditions are equivalent.*

(1) $A \in \mathrm{Env}(\Gamma)$, and
(2) $A \cap L[T,x] \in \mathrm{OD}_T^{L[T,x]}$ for every $x \in \mathbb{R}$.

Proof. ($1 \implies 2$) Let $x \in \mathbb{R}$. The set $L[T,x] \cap \mathbb{R}$ is countable by Lemma 3.4.2, so we can take $A' \in \mathrm{OD}^{<\Gamma}$ with $A \cap L[T,x] = A' \cap L[T,x]$. We may take our companion of Γ to be $\mathscr{M}_\Gamma = (M_\Gamma; \in, R_\Gamma, \varphi)$ where φ is the first norm of the scale giving T. Let $(A_\alpha : \alpha < \kappa)$ be a $\Delta_1^{\mathscr{M}_\Gamma}$ enumeration of the $\mathrm{OD}^{<\Gamma}$ sets of reals and fix $\eta < \kappa$ such that $A' = A_\eta$. The relation $B = \{(z, y) : z \in A_{\varphi(y)}\}$ on $\mathbb{R} \times \mathbb{R}$ is Σ_1 over $\mathscr{M}_\Gamma$, so it is in Γ and there is a tree $S \in L[T]$ definable from T with $p[S] = B$. The proof of the Becker–Kechris theorem [**2**] shows that in $L[T]$ we may use T (and S, which is definable from T) to define a family of games $\{G^\eta(z) : z \in \mathbb{R}\}$ on κ such that

- $G^\eta(z)$ is closed uniformly in z,[4] and
- $z \in A_\eta$ if and only if player II has a winning strategy for $G^\eta(z)$.

That is, there is a set $X_\eta \subset \kappa^{<\omega} \times \omega^{<\omega}$ in $\mathrm{OD}_T^{L[T,x]}$ such that

$$z \in A_\eta \iff (\Game f \in \kappa^\omega)(\forall n < \omega)\,(f \restriction n, z \restriction n) \in X_\eta.$$

The right-hand side is absolute to $L[T, x]$, so $A_\eta \cap L[T, x] \in \mathrm{OD}_T^{L[T,x]}$.

($2 \implies 1$) By Lemma 3.4.1 and a Skolem hull argument every set of reals $A \in \mathrm{OD}_T^{L[T,x]}$ is $\mathrm{OD}_{T\restriction\beta}^{L_\gamma[T\restriction\beta,x]}$ for some ordinals β and γ with $\beta < \gamma < \kappa$. So for a Turing degree d, taking $\mathscr{M}_\Gamma = (M_\Gamma; \in, R_\Gamma, T)$ as our companion shows that there is an enumeration $(A_{\alpha,d} : \alpha < \kappa)$ of all $\mathrm{OD}_T^{L(T,d)}$ sets of reals that is $\Sigma_1^{\mathscr{M}_\Gamma}$-good uniformly in d. (There are really only countably many such sets of reals, but this doesn't matter at the moment.)

We consider a real z to code a countable partial function f_z with $\mathrm{dom}(f_z) \subset \mathbb{R}$ and $\mathrm{ran}(f_z) \subset \{0, 1\}$. Given a Turing degree $d \geq_T z$ let $\alpha_z(d)$ by the least $\alpha < \kappa$ such that

$$\forall x \in \mathrm{dom}(f_z)\,(f_z(x) = 1 \iff x \in A_{\alpha,d}),$$

if it exists. Let $C = \{x \in \mathbb{R} : \forall^* d\, \alpha_z(d) \text{ exists}\}$.

For $z, z' \in C$ we define $z \preceq z'$ if $\alpha_d(z) \leq \alpha_d(z')$ for a cone of d. To see that the relation $\preceq$ is a prewellordering it suffices to verify that any elements z and z' of C are comparable. Indeed, if we take d_0 such that $\alpha_z(d)$ and $\alpha_{z'}(d)$ are defined for all $d \geq_T d_0$ then the sets $\{d \geq_T d_0 : \alpha_z(d) < \alpha_{z'}(d)\}$, $\{d \geq_T d_0 : \alpha_z(d) = \alpha_{z'}(d)\}$, and $\{d \geq_T d_0 : \alpha_z(d) > \alpha_{z'}(d)\}$ are all $\Delta_1^{\mathscr{M}_\Gamma}(z, z', d_0)$ and therefore $\utilde{\boldsymbol{\Delta}}_\Gamma$, so by $\utilde{\boldsymbol{\Delta}}_\Gamma$-determinacy exactly one of them contains a cone. The sets

$$\{(z, z') : z \in C \;\&\; (z' \in C \implies z \preceq z')\}$$
$$\{(z, z') : z \in C \;\&\; (z' \in C \implies z \prec z')\}$$

are $\Sigma_1^{\mathscr{M}_\Gamma}$ and are therefore in Γ. So we get a Γ-norm $\varphi : C \to \kappa$ with $\varphi(z) \leq \varphi(z') \iff z \preceq z'$. We can add expand the companion by a predicate for φ.

Let $\sigma \subset \mathbb{R}$ be countable. Take $z \in \mathbb{R}$ with $\mathrm{dom}(f_z) = \sigma$ and $f_z(x) = 1 \iff x \in A$. By our assumption on A we have $z \in C$. Let $\alpha = \varphi(z)$. Notice that for every $x \in \mathbb{R}$ there is a $z' \in C$ with $\varphi(z') = \alpha$ and $x \in \mathrm{dom}\, f_{z'}$. So the set $A' \subset \mathbb{R}$ defined by

$$\begin{aligned} x \in A' &\iff \exists z' \in C\,(\varphi(z') = \alpha \;\&\; x \in \mathrm{dom}(f_{z'}) \;\&\; f_{z'}(x) = 1) \\ &\iff \forall z' \in C\,(\varphi(z') = \alpha \;\&\; x \in \mathrm{dom}(f_{z'}) \implies f_{z'}(x) = 1), \end{aligned}$$

is $\Delta_1^{\mathscr{M}_\Gamma}(\alpha)$ and satisfies $A' \cap \sigma = A \cap \sigma$, witnessing that $A \in \mathrm{Env}(\Gamma)$. $\square$

[4] It is closed uniformly in z and η, but the Becker–Kechris theorem used only the uniformity in η and the present argument uses only the uniformity in z.

Remark 3.4.6. Under AD the weaker property that $A \cap L[T, x] \in L[T, x]$ for a cone of x is also equivalent to $A \in \underset{\sim}{\mathbf{Env}}(\Gamma)$. See [**8**] for a proof. We do not know if the equivalence is valid in $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$ assuming only $\underset{\sim}{\mathbf{\Delta}}_{\Gamma}$ determinacy.

We can also characterize the subsets of κ that lie in $L[T, z]$ for some real z. Here it is important that an arbitrary real parameter is allowed—we do not know whether there is a useful "lightface" class of subsets of κ.

Lemma 3.4.7. *Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_{\Gamma}$ is determined. Let κ be the pre-wellordering ordinal of Γ and let T on $\omega \times \kappa$ be the tree of a Γ-scale on a universal Γ set. Then for any set of ordinals $Y \subset \kappa$, the following statements are equivalent:*
(1) $Y \in \Delta_1^{\mathscr{M}_{\Gamma}}(z)$ for some $z \in \mathbb{R}$.
(2) $Y \in \Sigma_1^{\mathscr{M}_{\Gamma}}(z)$ for some $z \in \mathbb{R}$.
(3) $Y \in L[T, z]$ for some $z \in \mathbb{R}$.

Proof. Property (1) trivially implies property (2). If property (2) holds, then letting φ be a Γ-norm onto κ the set $\{x \in \mathrm{dom}(\varphi) : \varphi(x) \in Y\}$ is $\Sigma_1^{\mathscr{M}_{\Gamma}}(z)$ and therefore $\Gamma(z)$, so property (3) holds by the Becker–Kechris theorem of [**2**].

Now assume that property (3) holds. As in the proof of Lemma 3.4.1, one can use the projective set theory of [**6**] to show that the sets Y and $\kappa \setminus Y$ in $L[T, z]$ are each coded by a set in the projective-like hierarchy $\underset{\sim}{\mathbf{\Gamma}}_0, \underset{\sim}{\mathbf{\Gamma}}_1, \dots$ over $\underset{\sim}{\mathbf{\Gamma}}$. Therefore the sets $\{x \in \mathrm{dom}(\varphi) : \varphi(x) \in Y\}$ and $\{x \in \mathrm{dom}(\varphi) : \varphi(x) \notin Y\}$ are in $\underset{\sim}{\mathbf{\Gamma}}_n$ for some n. The pointclasses $\underset{\sim}{\mathbf{\Gamma}}_n$ are determined, so these sets are actually in $\underset{\sim}{\mathbf{\Gamma}}$ by the Coding Lemma. This shows that property (1) holds, although not necessarily with the same real parameter z. □

3.5. Countably complete measures and towers

As usual, in this section we let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_{\Gamma}$ is determined. We let T be the tree of a Γ-scale on a universal Γ set.

Definition 3.5.1. $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$ is the σ-algebra consisting of subsets $Y \subset \kappa$ that are $\Delta_1^{\mathscr{M}_{\Gamma}}(z)$ for some real z.

By Lemma 3.4.7 it is equivalent to require that Y is $\Sigma_1^{\mathscr{M}_{\Gamma}}(z)$ for some real z or that $Y \in L[T, x]$ for some real z.

Definition 3.5.2. $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$ is the set of countably complete measures on $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$.

By part 2 of Lemma 3.4.7—the "Σ_1" characterization of $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$—we can code elements of $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$ by reals via a map $x \mapsto Y_x$ such that the relation $\{(x, \alpha) : \alpha \in Y_x\}$ is $\Sigma_1^{\mathscr{M}_{\Gamma}}$. For an example of such a coding, fix a Γ-norm φ onto κ and a universal Γ set $U \subset \mathbb{R} \times \mathbb{R}$ and say $\alpha \in Y_x$ if there is $y \in \mathrm{dom}(\varphi)$ with $\varphi(y) = \alpha$ and $(x, y) \in U$. This coding induces a coding of measures:

Definition 3.5.3. For a measure $\mu \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$, the *code set* C_μ of μ is defined by

$$C_\mu = \{x \in \mathbb{R} : Y_x \in \mu\}.$$

The following fact was proved by Martin in the AD context, where $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa) = \wp(\kappa)$ by the Coding Lemma and $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$ is the set of all measures in κ because all measures are countably complete.

Lemma 3.5.4. *Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined. Then the code set C_μ is in $\mathrm{Env}(\Gamma)$ for every measure $\mu \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa)$.*

Proof. Let $\sigma \subset \mathbb{R}$ be countable. By the countable completeness of μ we can take an ordinal $\alpha < \kappa$ such that

$$\forall x \in \sigma\, (Y_x \in \mu \iff \alpha \in Y_x).$$

Then we have $C_\mu \cap \sigma = C_{\mu_\alpha} \cap \sigma$ where μ_α is the principal measure generated by α. We have $C_{\mu_\alpha} = \{x \in \mathbb{R} : \alpha \in Y_x\}$, which is $\Sigma_1^{\mathscr{M}_\Gamma}(\alpha)$, so in turn by Lemma 3.2.7 there is a set $A' \in \mathrm{OD}^{<\Gamma}$ with $C_{\mu_\alpha} \cap \sigma = A' \cap \sigma$. $\square$

Note that everything in this section can be generalized from κ to $\kappa^{<\omega}$ in a routine way using a $\Delta_1^{\mathscr{M}_\Gamma}$ bijection $\kappa \to \kappa^{<\omega}$. So we can define $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^n)$, $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^n)$, $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$, and $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$. By countable completeness, every measure in $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ concentrates on $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^n)$ for some n. Combining Lemma 3.5.4 with Proposition 3.2.5 on the wellordering of the envelope yields the following result.

Proposition 3.5.5. *Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined. For any universal Γ set U, there is a well-ordering of $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ of order type $\leq \Theta$ that is definable from U.*

In fact we could improve this result a bit to get a well-ordering of the measures that is definable from $\underset{\sim}{\mathbf{\Gamma}}$, using the proof of Kunen's theorem that under AD all measures are ordinal-definable, which can be found in [**12**]. The pointclass $\underset{\sim}{\mathbf{\Gamma}}$ itself might not be OD if there are divergent models of AD.

In [**14**] Kechris gives a different way of coding measures under AD, which assumes the stronger hypothesis of determinacy for real-integer games whose payoffs are $\underset{\sim}{\mathbf{\Gamma}}$ subsets of $\mathbb{R}^\omega \times \omega^\omega$, and proves the stronger conclusion that $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ has size strictly less than Θ.

We use the following notation for basic functions on sets and measures:

Definition 3.5.6.

- For $X \in \wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^i)$ and $j \geq i$, define the *extension*

  $\mathrm{ext}_{i,j}(X) = \{s \in \kappa^j : s \restriction i \in X\}$.

- For $F : \kappa^i \to \mathrm{Ord}$ in $\underset{\sim}{\mathbf{L}}[T]$ and $j \geq i$, define the *extension* $\mathrm{ext}_{i,j}(F) : \kappa^j \to \mathrm{Ord}$ by

  $\mathrm{ext}_{i,j}(F)(s) = F(s \restriction i)$.

- For $\mu \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^j)$ and $i \leq j$, define the *projection*

  $\mathrm{proj}_{j,i}(\mu) = \{X \in \wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^i) : \mathrm{ext}_{i,j}(X) \in \mu\}$.

Definition 3.5.7.

- A *tower of measures* on $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ is a sequence $\bar{\mu} = (\mu_i : i < \omega)$ such that $\mu_i \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^i)$ for all $i < \omega$, and $\mu_i = \mathrm{proj}_{j,i}\,\mu_j$ when $i < j$.
- A tower of measures $\bar{\mu}$ on $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ is *countably complete* if for all sequences $(X_i : i < \omega)$ with $X_i \in \mu_i$ for all i, there is a function $f \in \kappa^\omega$ such that $f \restriction i \in X_i$ for all $i < \omega$.

For a measure $\mu \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^i)$, we define the ultrapower $\mathrm{Ult}(\mathrm{Ord}, \mu)$ using functions $\kappa^i \to \mathrm{Ord}$ in $\underset{\sim}{\mathbf{L}}[T]$. Because μ is countably complete, this ultrapower is wellfounded—$\mathsf{DC}_{\mathbb{R}}$ is enough to show this because every element of $\underset{\sim}{\mathbf{L}}[T]$ is ordinal-definable from T and a real. So let

$$j_\mu : \mathrm{Ord} \to \mathrm{Ord}$$

denote the μ-ultrapower map. If $\mu_i = \mathrm{proj}_{j,i}\,\mu_j$ then we let j_{μ_i,μ_j} be the factor map defined by

$$j_{\mu_i,\mu_j}([F]_{\mu_i}) = [\mathrm{ext}_{i,j}(F)]_{\mu_j}.$$

We have $j_{\mu_j} = j_{\mu_i,\mu_j} \circ j_{\mu_i}$.

Lemma 3.5.8. *For a tower of measures $\bar{\mu} = (\mu_i : i < \omega)$ on $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ the following statements are equivalent:*

(1) $\bar{\mu}$ is not countably complete,
(2) there is a function $h : \omega \to \mathrm{Ord}$ with $j_{\mu_i,\mu_{i+1}}(h(i)) > h(i+1)$ for all $i < \omega$, and
(3) the direct limit of Ord *under the ultrapower maps $(j_{\mu_i,\mu_j} : i \leq j < \omega)$ is illfounded.*

Proof. (1) $\implies$ (2) If the tower $\bar{\mu}$ is not countably complete, take a sequence of measure one sets $(X_i : i < \omega)$ witnessing this. By shrinking the sets we may assume that $X_j \subset \mathrm{ext}_{i,j} X_i$ whenever $i < j$, so the set $U = \bigcup_{i<\omega} X_i$ is a wellfounded tree with μ_i-measure one for each μ_i. We have $U \in \underset{\sim}{\mathbf{L}}[T]$, so its rank function rank_U is in $\underset{\sim}{\mathbf{L}}[T]$ and we can define a function $h : \omega \to \mathrm{Ord}$ by $h(i) = [\mathrm{rank}_U]_{\mu_i}$. Then we have $j_{\mu_i,\mu_{i+1}}(h(i)) > h(i+1)$ for all $i < \omega$.

(2) $\implies$ (3) Given such a function h, the direct limit of the ordinals contains an infinite decreasing sequence $j_{\mu_0;\infty}(h(0)) > j_{\mu_1;\infty}(h(1)) > \cdots$.

(3) $\implies$ (1) If the direct limit of the ordinals is illfounded, then we may use $\mathsf{DC}_{\mathbb{R}}$ to choose a sequence of functions $F_i : \kappa^{n_i} \to \mathrm{Ord}$ in $\underset{\sim}{\mathbf{L}}[T]$ such that for each $i < \omega$ we have $n_{i+1} > n_i$ and $j_{\mu_{n_i},\mu_{n_{i+1}}}([F_i]_{\mu_{n_i}}]) > [F_{i+1}]_{\mu_{n_{i+1}}}$. If we let

$$X_n = \{s \in \kappa^n : \forall i < \omega\, (n_{i+1} \leq n \implies F_i(s \restriction n_i) > F_{i+1}(s \restriction n_{i+1}))\},$$

then the sequence $(X_n : n < \omega)$ witnesses that $\bar{\mu}$ is not countably complete. Each X_n has μ_n-measure one, but if $f \in \kappa^\omega$ were to satisfy $f \restriction n \in X_n$ for all $n < \omega$ then $(F_i(f \restriction n_i) : i < \omega)$ would be a decreasing sequence in κ, a contradiction. □

We may therefore call a tower of measures on $\wp^{\boldsymbol{\Gamma}}_{\sim}(\kappa^{<\omega})$ *wellfounded* if it is countably complete and *illfounded* if it is not countably complete when we wish to emphasize property (3). We call a function h as in statement (2) a *witness to the illfoundedness of* $\bar{\mu}$.

For every illfounded tower $\bar{\mu}$, the pointwise minimum of any family of functions $\omega \to \mathrm{Ord}$ witnessing the illfoundedness of the tower $\bar{\mu}$ itself witnesses the illfoundedness of $\bar{\mu}$, so there is a pointwise least witness to the illfoundedness of $\bar{\mu}$. Next we prove that the least witness has a special form that will occasionally be useful.

LEMMA 3.5.9. *If $\mu = (\mu_i : i < \omega)$ is an illfounded tower of measures on $\wp^{\boldsymbol{\Gamma}}_{\sim}(\kappa^{<\omega})$ then there is a wellfounded tree $U \in \wp^{\boldsymbol{\Gamma}}_{\sim}(\kappa^{<\omega})$ on which each measure μ_i concentrates, such that the function $g : \omega \to \mathrm{Ord}$ defined by $g(i) = [\mathrm{rank}_U]_{\mu_i}$ is a pointwise least witness to the illfoundedness of $\bar{\mu}$.*

PROOF. Take a pointwise least witness $h : \omega \to \mathrm{Ord}$ to the illfoundedness of $\bar{\mu}$. We can use countable choice for reals to choose for each $i < \omega$ a function $F_i : \kappa^i \to \mathrm{Ord}$ such that $[F_i]_{\mu_i} = h(i)$. Defining the tree U by

$$s \in U \iff (\forall i < |s|)\big(F_i(s \restriction i) > F_{|s|}(s)\big),$$

we have that $U \in \mu_i$ for all $i < \omega$ and we have a well-defined function $F = \bigcup_{i<\omega} F_i \restriction U$ that is order-preserving as a function $(U, \supsetneq) \to (\mathrm{Ord}, <)$. Therefore for all $s \in U \cap \kappa^i$ we have $\mathrm{rank}_U(s) \le F(s) = F_i(s)$, so $[\mathrm{rank}_U]_{\mu_i} \le h(i)$ as desired. □

3.6. Semi-scales with norms in the envelope

As usual we let Γ be an inductive-like pointclass such that $\boldsymbol{\Delta}_\Gamma$ is determined, and we let T be the tree of a Γ-scale on a universal Γ set.

DEFINITION 3.6.1. Given a tree $S \in L[T]$ on $\omega \times \kappa$ and a set of measures $\sigma \subset \mathrm{meas}^{\boldsymbol{\Gamma}}_{\sim}(\kappa^{<\omega})$, the *putative semi-scale on $\mathbb{R} \setminus p[S]$ given by σ* is the set of norms $\{\varphi_\mu : \mu \in \sigma\}$ on $\mathbb{R} \setminus p[S]$ defined by

$$\varphi_\mu(x) = [\mathrm{rank}_{S_x}]_\mu$$

where $\mathrm{rank}_{S_x}(t)$ denotes the rank of the node t in the tree S_x, and is considered to be zero if $t \notin S_x$ and undefined if S_x is illfounded below t.

This is an abuse of terminology because the set of norms is indexed by σ and not ω. However, the ordering of the norms does not matter for our purposes. We will say that the putative semi-scale $\{\varphi_\mu : \mu \in \sigma\}$ is a semi-scale just in case for some (equivalently, for any) enumeration $(\mu_i : i < \omega)$ of σ the sequence of norms $(\varphi_{\mu_i} : i < \omega)$ is a semi-scale on $\mathbb{R} \setminus p[S]$ in the usual sense.

LEMMA 3.6.2. *Given a tree $S \in L[T]$ on $\omega \times \kappa$ and a measure $\mu \in \mathrm{meas}^{\boldsymbol{\Gamma}}_{\sim}(\kappa^{<\omega})$, the norm relation $\le_\mu$ defined by $x \le_\mu y \iff \varphi_\mu(x) \le \varphi_\mu(y)$ is in $\mathrm{Env}(\Gamma)$.*

PROOF. By Theorem 3.4.5 it suffices to show that given any $z \in \mathbb{R}$, the restriction $\leq_\mu \cap L[T, z]$ of the norm relation $\leq_\mu$ is $\mathrm{OD}_T^{L[T,z]}$. Indeed, there are only countably many reals in $L[T, z]$ by Lemma 3.4.2 and the measure μ is countably complete, so we can take $s \in \kappa^{<\omega}$ such that for all reals $x, y \in L[T, z]$ we have

$$[\mathrm{rank}_{S_x}]_\mu \leq [\mathrm{rank}_{S_y}]_\mu \iff \mathrm{rank}_{S_x}(s) \leq \mathrm{rank}_{S_y}(s).$$

This shows that the restriction $\leq_\mu \cap L[T, z]$ of the norm relation $\leq_\mu$ is definable in $L[T, z]$ from s and S and it remains to note that $S \in OD_T^{L[T,z]}$. $\square$

DEFINITION 3.6.3. Given a tree $S \in L[T]$ on $\omega \times \kappa$ and a set of measures $\sigma \subset \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$, define the following game, which is closed for player I:

$$(G_S^\sigma) \qquad \begin{array}{llllll} \mathrm{I} & n_0, \alpha_0, h_0 & & n_1, \alpha_1, h_1 \ldots & & \\ \mathrm{II} & & \mu_1 & & \mu_2 \ldots \end{array}$$

We let μ_0 denote the trivial measure on κ^0.

Rules for I: $((n_0, \ldots, n_i), (\alpha_0, \ldots, \alpha_i)) \in S$ and $j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1}$.

Rules for II: $\mu_{i+1} \in \sigma$ is a measure on κ^{i+1} projecting to μ_i and concentrating on the set $S_{(n_0, \ldots, n_i)} \subset \kappa^{i+1}$.

The first player to deviate from these rules loses, and if both players follow the rules for all ω moves then player I wins.

In other words, player I is trying to build a real x and a branch $\vec{\alpha}$ of S_x, player II is trying to build a tower $\vec{\mu}$ of measures in σ concentrating on S_x, and player I is trying to build a continuous witness h to the illfoundedness of $\vec{\mu}$. Notice that no relation between $\vec{\alpha}$ and $\vec{\mu}$ is required by the rules.

The game G_S^σ is a closed game on a wellordered set, so by the Gale–Stewart theorem it is determined. The following lemma was proved by Woodin for the ordinary measures on $\wp(\kappa^{<\omega})$, but easily adapts in our situation to the $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ version we state below, which has somewhat wider applicability.

LEMMA 3.6.4 (Woodin [**46**]). *Let $S \in L[T]$ be a tree on $\omega \times \kappa$, and let $\sigma \subset \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ be a countable set of measures. If player II has a winning strategy in the game G_S^σ then the putative semi-scale on $\mathbb{R} \setminus p[S]$ given by σ is a semi-scale.*

PROOF. Suppose it is not a semi-scale. We will describe a winning strategy for player I. Take a convergent sequence of reals $(x_k : k < \omega)$ witnessing that the norms $(\varphi_\mu : \mu \in \sigma)$ do not form a semiscale. That is, $x_k \notin p[S]$ for each $k < \omega$, and the sequence $(\varphi_\mu(x_k) : k < \omega)$ has an eventually constant value $h(\mu)$ for each $\mu \in \sigma$, but the limit $x = \lim_{k<\omega} x_k$ is in $p[S]$. Take a branch $\vec{\alpha}$ of S_x.

A winning strategy for player I is immediately suggested by the notation: on the i^{th} turn play $(x(i), \alpha_i, h(\mu_i))$. The rule $j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1}$ is satisfied because $\mu_{i+1} \in \sigma$ is a measure on κ^{i+1} projecting to μ_i and concentrating on the set $S_{x \restriction (i+1)} \subset \kappa^{i+1}$, then for sufficiently large k we have

$$\begin{aligned} j_{\mu_i,\mu_{i+1}}(h(\mu_i)) &= j_{\mu_i,\mu_{i+1}}([\mathrm{rank}_{S_{x_k}}]_{\mu_i}) \\ &= [\mathrm{ext}_{i,i+1}\,\mathrm{rank}_{S_{x_k}}]_{\mu_{i+1}} > [\mathrm{rank}_{S_{x_k}}]_{\mu_{i+1}} = h(\mu_{i+1}). \end{aligned} \qquad \square$$

This lemma is used to prove the following theorem, which was first proved by Martin using a different method under the additional assumption of a Suslin cardinal above κ. See also the paper [**22**] of Martin and Woodin for a proof that is similar to the one below, but uses a *normal* fine measure on $\wp_{\omega_1}(\mathbb{R} \cup \mathrm{Env}(\Gamma))$ rather than just a fine measure on $\wp_{\omega_1}(\mathrm{Env}(\Gamma))$.

THEOREM 3.6.5 (Woodin [**46**], AD). *Let* Γ *be an inductive-like pointclass. If* $\underset{\sim}{\mathbf{Env}}(\Gamma) \neq \wp(\mathbb{R})$ *then every* $\check{\Gamma}$ *set has a semi-scale with norms in* $\mathrm{Env}(\Gamma)$.

PROOF. If $\underset{\sim}{\mathbf{Env}}(\Gamma) \neq \wp(\mathbb{R})$ then by Wadge's lemma $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is a surjective image of $\mathbb{R}$. So by the coding of measures in Lemma 3.5.4 there is a surjection $F : \mathbb{R} \to \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$, and by Turing determinacy we have a fine, countably complete measure $\mathcal{U}$ on $\wp_{\omega_1}(\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}))$ defined by $\mathcal{Z} \in \mathcal{U}$ if $\{F(y) : y \leq_{\mathrm{T}} x\} \in \mathcal{Z}$ for a cone of $x \in \mathbb{R}$.

Let $A \subset \mathbb{R}$ be a $\check{\Gamma}$ set and take a tree $S \in L[T]$ on $\omega \times \kappa$ with $p[S] = \mathbb{R} \setminus A$. To get the desired semi-scale on A it suffices by Lemma 3.6.4 to show that player II has a winning strategy in the game G^σ_S for some countable set of measures σ. In fact we will show that player II has a winning strategy in $\mathcal{U}$-almost every σ.

Assuming the contrary, for $\mathcal{U}$-almost every σ the proof of the Gale–Stewart theorem gives a canonical winning strategy F^σ for player I: always play the least move leading to a subgame where player II still has no winning strategy. We define a tower of measures $\vec{\mu}$ from σ that is a valid play against F^σ for $\mathcal{U}$-almost every σ.

Let μ_0 be the trivial measure on κ^0. Let n^σ_0 and α^σ_0 denote the moves played as "n_0" and "α_0" respectively by the strategy F^σ on turn zero. Define a measure $\mu_1 \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ by

$$X \in \mu_1 \iff \forall^*_{\mathcal{U}}\sigma\,(\alpha^\sigma_0 \in X).$$

More generally, let n^σ_i and α^σ_i denote the moves played as "n_i" and "α_i" respectively by the strategy F^σ on turn i against the play $(\mu_1, \ldots, \mu_i)$ by player II, and define a measure $\mu_{i+1} \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ by

$$X \in \mu_{i+1} \iff \forall^*_{\mathcal{U}}\sigma\,((\alpha^\sigma_0, \ldots, \alpha^\sigma_i) \in X).$$

By the countable completeness of $\mathcal{U}$ there is a real x such that $\mathcal{U}$-almost every σ have the property that for every $i < \omega$ we have $n^\sigma_i = x(i)$. Each measure μ_i concentrates on S_x because $(\alpha^\sigma_0, \ldots, \alpha^\sigma_i) \in S_x$ for $\mathcal{U}$-almost every σ. It's easy to check that μ_{i+1} projects to μ_i, so the measures form a tower $\vec{\mu} = (\mu_i : i < \omega)$.

For any σ with the above property and such that $\mu_i \in \sigma$ for every $i < \omega$, the sequence $\vec{\mu}$ is a legal play by player II in the game G^σ_S, so the moves h^σ_i played as "h_i" by the strategy F^σ on turn i against the play $(\mu_1, \ldots, \mu_i)$ by player II form a sequence witnessing that the tower $\vec{\mu}$ is illfounded.

On the other hand, for any sequence of sets $(X_i : i < \omega)$ with $X_i \in \mu_i$ the countable completeness of $\mathcal{U}$ and the definition of μ_{i+1} give a countable set $\sigma \subset \operatorname{meas}^{\boldsymbol{\Gamma}}(\kappa^{<\omega})$ such that $(\alpha_0^\sigma, \ldots, \alpha_i^\sigma) \in X_i$ for all $i < \omega$. This shows that the tower $\vec{\mu}$ is countably complete, a contradiction. $\square$

Notice that the semi-scale itself generally cannot be in $\operatorname{Env}(\Gamma)$ or even in $\underset{\sim}{\mathbf{Env}}(\Gamma)$. If $\underset{\sim}{\mathbf{Env}}(\Gamma)$ were to contain a semi-scale on the $\check{\Gamma}$ set $A = \{(x, y) : y \notin C_\Gamma(x)\}$, then the uniformization given by leftmost branches would also be in $\underset{\sim}{\mathbf{Env}}(\Gamma)$, contradicting Martin's anti-uniformization theorem (Theorem 3.2.9.) Therefore the norm relations are generally Wadge-cofinal in $\underset{\sim}{\mathbf{Env}}(\Gamma)$.

3.7. Digression: $\mathsf{AD}_{\mathbb{R}}$ from divergent models of AD^+

We can use Theorem 3.2.4 and Lemma 3.5.4 to give a simpler proof of a theorem of Woodin. The original proof, which is unpublished, used Sacks forcing. We say that models M_0 and M_1 of AD^+ with $\mathbb{R} \cup \mathrm{Ord} \subset M_0, M_1$ are *divergent* if neither $M_0 \cap \wp(\mathbb{R})$ nor $M_1 \cap \wp(\mathbb{R})$ contains the other. (By Wadge's Lemma, this implies that no model of determinacy contains both M_0 and M_1.)

Theorem 3.7.1 (Woodin, $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$). *If M_0 and M_1 are divergent models of AD^+, then the model $M = L(M_0 \cap M_1 \cap \wp(\mathbb{R}))$ satisfies "every set of reals is Suslin" (and therefore $\mathsf{AD}_{\mathbb{R}}$.)*

Proof. Notice that M satisfies AD^+, and also that $M \cap \wp(\mathbb{R}) = M_0 \cap M_1 \cap \wp(\mathbb{R})$ because both M_0 and M_1 are closed under constructibility. Suppose toward a contradiction that M does not satisfy "all sets are Suslin." Then M has a largest Suslin cardinal κ by AD^+. Let $S(\kappa)$ denote the pointclass of Suslin sets in M, or equivalently by the Coding Lemma, the pointclass of κ-Suslin sets in M_0 or in M_1. The pointclass $S(\kappa)$ is $\mathbb{R}$-parameterized, non-selfdual, closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, and Wadge reducibility, and has the scale property. (See [**7**].) By a theorem of Becker we have $S(\kappa) = \underset{\sim}{\boldsymbol{\Gamma}}$ for some inductive-like pointclass Γ. We could also define the companion $\mathscr{M}$ of the boldface pointclass $S(\kappa)$ as in Moschovakis and then define the lightface pointclass $\Gamma = \Sigma_1^{\mathscr{M}}$.

Notice that $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_0}$ and $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_1}$ are both contained in $\underset{\sim}{\mathbf{Env}}(\Gamma)^V$. All three envelopes are determined and closed under continuous reducibility, so by Wadge's Lemma we may assume without loss of generality that $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_0} \subset \underset{\sim}{\mathbf{Env}}(\Gamma)^{M_1}$. Because M_0 and M_1 are divergent we have $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_0} \neq \wp(\mathbb{R}) \cap M_0$ and by Theorem 3.6.5 there is a semi-scale $\vec{\varphi} \in M_0$ on a universal $\check{\Gamma}$ set B whose norm relations are in $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_0}$. We also have $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_0} \neq \wp(\mathbb{R}) \cap M_1$, so by Wadge's Lemma M_1 contains all countable sequences from $\underset{\sim}{\mathbf{Env}}(\Gamma)^{M_0}$ and our semi-scale $\vec{\varphi}$ is in M_1 as well. So B is Suslin in the intersection model M, a contradiction. $\square$

3.8. Digression: $\mathsf{AD}_{\mathbb{R}}$ in certain derived models

This section is devoted to the proof of the following proposition.

Proposition 3.8.1 (ZFC). *If δ is a limit of Woodin cardinals and either*

(1) δ is δ^+-strongly compact, or

(2) δ is weakly compact in $V^{\mathrm{Col}(\delta,\delta^+)}$,

then the derived model $L(\mathbb{R}^*, \mathrm{Hom}^*)$ at δ satisfies $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Similar results are obtained using the theory of K^c in [**11**]. The conclusion there is slightly stronger in terms of consistency strength, and the hypothesis is weaker, lacking the assumption that δ is a limit of Woodin cardinals. On the other hand, Proposition 3.8.1 gives $\mathsf{AD}_{\mathbb{R}}$ in the derived model itself, which may lie beyond the reach of current methods of inner model theory. (For example, it seems possible that the derived model could contain an iteration strategy for a mouse with a superstrong cardinal.)

We say a tree T is $<\delta$-absolutely complemented, or $<\delta$-a.c., if there is a tree S such that $p[S] = \mathbb{R} \setminus p[T]$ in any generic extension of V by a poset of size less than δ. If δ is a limit of Woodin cardinals and $G \subset \mathrm{Col}(\omega, <\delta)$ is a V-generic filter, then we define the sets

$$\mathbb{R}^*_G = \bigcup_{\alpha<\delta} \mathbb{R} \cap V[G \restriction \alpha] \quad \text{and}$$

$$\mathrm{Hom}^*_G = \{p[T] \cap \mathbb{R}^*_G : \exists \alpha < \delta\, (T \in V[G \restriction \alpha] \;\&\; V[G \restriction \alpha] \models T \text{ is } <\delta\text{-a.c.}\}$$

The model $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$ is called a *derived model at δ*. The forcing poset $\mathrm{Col}(\omega, <\delta)$ is homogeneous, so when the particular generic G is not important we may speak of "the" derived model $L(\mathbb{R}^*, \mathrm{Hom}^*)$ at δ. The general facts we will use about derived models can all be found in [**42**]. Most importantly, the derived model at a limit of Woodin cardinals always satisfies $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. If δ is regular—and therefore inaccessible—then the derived model at δ satisfies DC.

The derived model satisfies $\mathsf{AD}_{\mathbb{R}}$ if and only if, for every set of reals $A \in \mathrm{Hom}^*$, every $\Sigma^2_1(A)^{L(\mathbb{R}^*,\mathrm{Hom}^*)}$ set of reals is also in Hom^*. We will show this to be the case assuming the hypotheses of Proposition 3.8.1. Let $A \in \mathrm{Hom}^*$, say $A = p[T_0]$ where T_0 is in $V[G \restriction \alpha]$ and is $<\delta$-absolutely complemented there. The pointclass

$$\Gamma = \Sigma^2_1(A)^{L(\mathbb{R}^*,\mathrm{Hom}^*)}$$

is inductive-like; in particular, it has the scale property. Notice that $\mathbb{R}^* = \mathbb{R}^{V[G]}$ because δ is inaccessible. Let T be the tree of a Γ-scale on a universal Γ set. The tree T is definable from T_0 in $V[G]$, so by the homogeneity of the factor forcing we have $T \in V[G \restriction \alpha]$. We want to show that T is $<\delta$-absolutely complemented in $V[G \restriction \beta]$ for some $\beta < \delta$.

In $V[G]$ we have $\mathfrak{c}^+ = \delta^+ = \delta^{+V}$, and by Proposition 3.5.5 on the wellordering of measures there is a surjection

$$\pi : \delta^+ \to \mathrm{meas}^{\underset{\sim}{\Gamma}}(\kappa^{<\omega})$$

that is definable from T. We remark that if $\mathrm{meas}^{\underset{\sim}{\Gamma}}(\kappa^{<\omega})$ has size $\leq \delta$ and not δ^+, then the weak compactness of δ suffices for the following argument. Proposition 3.8.1 now follows from the following claim.

Claim 3.8.2. *Assume either*

(1) δ is δ^+-strongly compact, or

(2) δ is weakly compact in $V^{\mathrm{Col}(\delta,\delta^+)}$.

Then T is $<\delta$-absolutely complemented in $V[G \restriction \beta]$ for some $\beta < \delta$.

Proof from (1). This can be proved by the argument of Theorem 3.6.5 except that our fine, countably complete measure on $\wp_{\omega_1}(\mathbb{R}^*)^{V(\mathbb{R}^*)}$ comes from the strong compactness measure in the ground model via the Levy–Solovay theorem rather than from Turing determinacy. However, for the sake of variety we will give a proof that instead uses the embedding characterization of strong compactness.

Let $\mathcal{U}$ be a countably complete fine measure on $\wp_\delta(\delta^+)$ and let $j : V \to N = \mathrm{Ult}(V,\mathcal{U})$ denote the correponding ultrapower embedding. We can extend j to $j^* : V[G] \to N[H]$ where $H \subset \mathrm{Col}(\omega, < j(\delta))$ is $V[G]$-generic. Define the set of measures $\sigma = j^*(\pi)``([\mathrm{id}]_{\mathcal{U}})$ where id denotes the identity function on $\wp_\delta(\delta^+)$. The relevant properties of this set σ are that it is countable in $N[H]$ and we have

$$j^* `` \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}) \subset \sigma.$$

We will show that in $N[H]$, player II has a winning strategy in the game $G^\sigma_{j(T)}$. By the absoluteness of the existence of winning strategies for closed games it is okay to step outside of $N[H]$ while describing our strategy.

If player I has played integers $n_0, \ldots, n_i < \omega$, ordinals $\alpha_0, \ldots, \alpha_i < j(\kappa)$, and ordinals $h_0, \ldots, h_i$, we define the measure $\bar{\mu}_{i+1} \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{V[G]}$ by

$$X \in \bar{\mu}_{i+1} \iff (\alpha_0, \ldots, \alpha_i) \in j^*(X) \quad \text{for all } X \in \wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}).$$

Notice that $j^*(X)$ does not depend on H because $j^* \restriction \underset{\sim}{\mathbf{L}}[T]$ does not depend on H. So we have $\bar{\mu}_{i+1} \in V[G]$. Clearly the measure $\bar{\mu}_{i+1}$ is countably complete in $V[G]$. So we let player II play the measure $\mu_{i+1} = j^*(\bar{\mu}_{i+1})$.

To see that this describes a winning strategy for player II in the game $G^\sigma_{j(T)}$, suppose toward a contradiction that player I is able to follow the rules for all ω rounds. Then the sequence of ordinals $(h_i : i < \omega)$ played by player I witnesses that the tower of measures $(\mu_i : i < \omega)$ is illfounded. By the elementarity of j^* the tower of measures $(\bar{\mu}_i : i < \omega)$ is also illfounded. Take a sequence of sets $(X_i : i < \omega)$ with $X_i \in \bar{\mu}_i$ witnessing that $(\bar{\mu}_i : i < \omega)$ is not countably complete. Then by the elementarity of j^* the sequence of sets $(j^*(X_i) : i < \omega)$ witnesses that $(\mu_i : i < \omega)$ is not countably complete. But $(\alpha_0, \ldots, \alpha_{i-1}) \in j^*(X_i)$ for each $i < \omega$, a contradiction.

Now by the elementarity of j^* there is a countable set of measures

$$\bar{\sigma} \in \wp_{\omega_1}(\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}))^{V[G]}$$

such that player II wins the game $G^{\bar{\sigma}}_T$ as defined in the model $V[G]$. So by Lemma 3.6.4 the putative semi-scale defined from $\bar{\sigma}$ is in fact a semi-scale on $\mathbb{R}^{V[G]} \setminus p[T]$. The tree $\tilde{T}$ of this semi-scale is definable in $V[G]$ from T and a countable set of ordinals, namely $\pi^{-1}``\bar{\sigma}$. So we have $\tilde{T} \in V[G \restriction \beta]$ for some $\beta < \delta$. In $V[G \restriction \beta]$, the trees T and $\tilde{T}$ are $<\!\delta$-absolutely complementing. $\square$

Proof from (2). Let $h \subset \mathrm{Col}(\delta, \delta^+)^V$ be a $V[G]$-generic filter. In $V[G]$, the forcing poset $\mathrm{Col}(\delta, \delta^+)^V$ is homogeneous and does not add reals. So every element of $\mathrm{Env}(\Gamma)^{V[G][h]}$ is OD_T in $V[G][h]$ and is therefore in $V[G]$. Membership in $\mathrm{Env}(\Gamma)$ is absolute between transitive models containing Γ with the same reals and ordinals, so $\mathrm{Env}(\Gamma)^{V[G]} = \mathrm{Env}(\Gamma)^{V[G][h]}$. In particular we have

$$\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{V[G]} = \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{V[G][h]}.$$

In $V[G][h]$ the set $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ has size δ because we have collapsed δ^{+V} without adding any measures on $\wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$. Our assumption is that δ is weakly compact in $V[h]$, and weak compactness is preserved by small forcing, so δ is weakly compact in $V[G \restriction \alpha][h]$ as well.

Working in $V[G \restriction \alpha][h]$, let $M \prec H(\delta^+)$ be a transitive model of size δ with $V_\delta \cup \{T\} \subset M$ and $M^{<\delta} \subset M$. Because $\delta^{+V} < \delta^+$ we have $\delta^{+V} \subset M$, so

$$\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{V[G][h]} = \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{M[G]}.$$

Take an elementary embedding $j : M \to N$ with N transitive and $\mathrm{crit}(j) = \delta$. Taking a $V[G][h]$-generic filter $H \subset \mathrm{Col}(\omega, <\!j(\delta))$ with $G \subset H$, we can extend j to an elementary embedding

$$\begin{aligned} j^* : M[G] &\to N[H] \\ \tau_G &\mapsto j(\tau)_H. \end{aligned}$$

The set of measures

$$\sigma = j^{*\,\text{“}}(\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{M[G]})$$

is in $N[H]$ and is countable there. We will show that in $N[H]$, player II has a winning strategy in the game $G^\sigma_{j(T)}$. By the absoluteness of the existence of winning strategies for closed games it is okay to step outside of $N[H]$ while describing our strategy.

If player I has played integers $n_0, \ldots, n_i < \omega$, ordinals $\alpha_0, \ldots, \alpha_i < j(\kappa)$, and ordinals $h_0, \ldots, h_i$, we define the measure $\bar{\mu}_{i+1}$ by

$$X \in \bar{\mu}_{i+1} \iff (\alpha_0, \ldots, \alpha_i) \in j^*(X) \quad \text{for all } X \in \wp^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}).$$

Notice that $j^*(X)$ does not depend on H because $j^* \restriction \underset{\sim}{\mathbf{L}}[T]$ does not depend on H. So we have $\bar{\mu}_{i+1} \in V[G][h]$. In fact we have $\bar{\mu}_{i+1} \in M[G]$ because $\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})^{V[G][h]} \subset M[G]$. Clearly the measure $\bar{\mu}_{i+1}$ is countably complete in $M[G]$. So we let player II play the measure $\mu_{i+1} = j^*(\bar{\mu}_{i+1})$.

To see that this describes a winning strategy for player II in the game $G^\sigma_{j(T)}$, suppose toward a contradiction that player I is able to follow the rules for all ω rounds. Then the sequence of ordinals $(h_i : i < \omega)$ played by player I witnesses that the tower of measures $(\mu_i : i < \omega)$ is illfounded. By the elementarity of j^* the tower of measures $(\bar{\mu}_i : i < \omega)$ is also illfounded. Take a sequence of sets $(X_i : i < \omega)$ with $X_i \in \bar{\mu}_i$ witnessing that $(\bar{\mu}_i : i < \omega)$ is not countably complete. Then by the elementarity of j^* the sequence of sets $(j^*(X_i) : i < \omega)$ witnesses that $(\mu_i : i < \omega)$ is not countably complete. But $(\alpha_0, \ldots, \alpha_{i-1}) \in j^*(X_i)$ for each $i < \omega$, a contradiction.

Now by the elementarity of j^* there is a countable set of measures

$$\bar{\sigma} \in \wp_{\omega_1}(\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}))^{M[G]}$$

such that player II wins the game $G_T^{\bar{\sigma}}$ as defined in the model $M[G]$. So by Lemma 3.6.4 the putative semi-scale defined from $\bar{\sigma}$ is in fact a semi-scale on $\mathbb{R}^{V[G]} \setminus p[T]$. The tree $\tilde{T}$ of this semi-scale is definable in $V[G]$ from T and a countable set of ordinals, namely $\pi^{-1}``\bar{\sigma}$. So we have $\tilde{T} \in V[G \upharpoonright \beta]$ for some $\beta < \delta$. In $V[G \upharpoonright \beta]$, the trees T and $\tilde{T}$ are $<\delta$-absolutely complementing. $\square$

CHAPTER 4

Sealing the envelope

In this chapter we work under the hypotheses of Chapter 3. Namely, Γ is an inductive-like pointclass whose boldface ambiguous part $\undertilde{\mathbf{\Delta}}_\Gamma$ is determined. Let κ be the supremum of the lengths of the $\undertilde{\mathbf{\Delta}}_\Gamma$ prewellorderings of $\mathbb{R}$. By T we denote the tree of a Γ-scale on a universal Γ set. As before, we assume the base theory $\mathsf{ZF} + \mathsf{DC}_\mathbb{R}$.

We give a general method for constructing a self-justifying system $\vec{A} \in \undertilde{\mathbf{Env}}(\Gamma)$ such that A_0 is a universal Γ set. We say that such a self-justifying system *seals the envelope* of Γ. Under a mouse capturing hypothesis, this allows us to construct a term relation hybrid mouse operator beyond the envelope. Applying this method to the inductive-like pointclass arising in the "gap in scales" case of the core model induction, we complete the proof of the Main Theorem.

4.1. Scales with norms in the envelope

To construct a model operator beyond $\undertilde{\mathbf{\Delta}}_\Gamma$, the semi-scales given by the technique of Section 3.6 are not good enough and we need to construct a *scale* on a universal $\check{\Gamma}$ set with norms in the envelope. To do this we use a combination of Woodin's argument in Section 3.6 with an idea of Jackson in [**9**]. The reader should be aware that our definition of stability is a bit different from Jackson's.

We recall Definition 3.6.1. Given a tree $S \in L[T]$ on $\omega \times \kappa$ and a set of measures $\sigma \subset \mathrm{meas}^{\undertilde{\mathbf{\Gamma}}}(\kappa^{<\omega})$, the *putative semi-scale on* $\mathbb{R} \setminus p[S]$ *given by* σ is the set of norms $\{\varphi_\mu : \mu \in \sigma\}$ on $\mathbb{R} \setminus p[S]$ defined by

$$\varphi_\mu(x) = [\mathrm{rank}_{S_x}]_\mu$$

where $\mathrm{rank}_{S_x}(t)$ denotes the rank of the node t in the tree S_x, and is considered to be zero if $t \notin S_x$ and undefined if S_x is illfounded below t.

This is an abuse of terminology because no enumeration of the norms is given, and indeed our definition does not even require σ to be countable.

Definition 4.1.1. Given a tree $S \in L[T]$ on $\omega \times \kappa$, a set of measures $\sigma \subset \mathrm{meas}^{\undertilde{\mathbf{\Gamma}}}(\kappa^{<\omega})$, and a measure $\mu \in \sigma$, we say that σ *stabilizes* μ if, whenever $(x_k : k < \omega)$ is a sequence of reals in $\mathbb{R} \setminus p[S]$ converging to a limit x and such that for each $\mu' \in \sigma$, the ordinals $\varphi_{\mu'}(x_k)$ are eventually constant, we have $\varphi_\mu(x) \leq \lim_{k\to\omega} \varphi_\mu(x_k)$. (In particular, $\varphi_\mu(x) < \infty$.)

Some remarks on the definition:

- If σ stabilizes the trivial measure μ_0, then the putative semi-scale $\{\varphi_\mu : \mu \in \sigma\}$ on $\mathbb{R} \setminus p[T]$ is in fact a semi-scale.

- The putative semi-scale $\{\varphi_\mu : \mu \in \sigma\}$ is a scale if and only if σ stabilizes every measure $\mu \in \sigma$.

To get a set of measures σ stabilizing a given measure μ, we define a variant of Woodin's game from Definition 3.6.3:

Definition 4.1.2. Given a countable set of measures $\sigma \subset \mathrm{meas}^{\underset{\sim}{\Gamma}}(\kappa^{<\omega})$ and a measure $\mu \in \sigma$, let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order. We define the following game, which is closed for player I:

$$(G_S^{\sigma,\mu}) \quad \begin{array}{llllll} \mathrm{I} & m_0, \ldots, m_n, s_n, h_n & & m_{n+1}, s_{n+1}, h_{n+1} \ldots & \\ \mathrm{II} & & \mu_{n+1} & & \mu_{n+2} \cdots \end{array}$$

Rules for I:

- $m_i < \omega$
- $S_{(m_0,\ldots,m_{n-1})} \in \mu$
- $s_i \in j_{\mu_i}(S_{(m_0,\ldots,m_i)})$, and in particular $s_i \in j_{\mu_i}(\kappa)^{i+1}$
- $s_n \supsetneq [\mathrm{id}]_{\mu_n}$
- $j_{\mu_i,\mu_{i+1}}(s_i) \subsetneq s_{i+1}$
- $h_i \in \mathrm{Ord}$
- $j_{\mu_i,\mu_{i+1}}(h_i) > h_{i+1}$

Rules for II:

- $\mu_{i+1} \in \sigma$ is a measure on κ^{i+1} projecting to μ_i
- μ_{i+1} concentrates on the set $S_{(m_0,\ldots,m_i)} \subset \kappa^{i+1}$

The first player to deviate from these rules loses, and if both players follow the rules for all ω moves then player I wins.

In other words, player I builds a real $x = (m_0, m_1, \ldots)$, player II is trying to build a tower $\vec{\mu}$ of measures in σ concentrating on S_x, and player I is trying to build a continuous witness $\vec{h}$ to the illfoundedness of $\vec{\mu}$ as well a special kind of branch $(j_{i,\infty}(s_i) : i \geq n)$ through the direct limit $j_{0,\infty}(S_x)$ of S_x along $\vec{\mu}$.

The game $G_S^{\sigma,\mu}$ is a closed game on a wellordered set, so by the Gale–Stewart theorem it is determined. The following lemma adapts Woodin's Lemma 3.6.4 from the game G_S^σ to the game $G_S^{\sigma,\mu}$.

Lemma 4.1.3. *Let $S \in L[T]$ be a tree on $\omega \times \kappa$, let $\sigma \subset \mathrm{meas}^{\underset{\sim}{\Gamma}}(\kappa^{<\omega})$ be a countable set of measures, and let $\mu \in \sigma$ be a measure. If player II has a winning strategy in the game $G_S^{\sigma,\mu}$ then σ stabilizes μ.*

Proof. Suppose not. We will describe a winning strategy for player I. Take a convergent sequence of reals $(x_k : k < \omega)$ witnessing that σ does not stabilize μ. That is, $x_k \notin p[S]$ for each $k < \omega$, and the sequence of ordinals $(\varphi_\nu(x_k) : k < \omega)$ has an eventually constant value $h(\nu)$ for each measure $\nu \in \sigma$, but the limit x of the sequence of reals satisfies $\varphi_\mu(x) > \lim_{k\to\infty} \varphi_\mu(x_k)$. (This includes the possibility that $\varphi_\mu(x) = \infty$.)

Define the integers $m_i = x(i)$ and the ordinals $h(\nu) = \lim_{k\to\infty} \varphi_\nu(x_k)$. Let n be such that the measure μ concentrates on κ^n, and let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order. We have $\mathrm{rank}_{j_{\mu_n}(T_x)}([\mathrm{id}]_{\mu_n}) = \varphi_{\mu_n}(x)$ by definition, and this is strictly greater than $h(\mu_n)$, so there is a successor $s_n \supsetneq [\mathrm{id}]_{\mu_n}$ with rank at least $h(\mu_n)$ in the tree $j_{\mu_n}(T_x)$. On the first turn, player I plays integers $m_0 = x(0), \ldots, m_n = x(n)$, the least such finite sequence s_n, and the ordinal $h_n = h(\mu_n)$. For $i \geq n$ player I inductively maintains the inequality

$$\mathrm{rank}_{j_{\mu_i}(T_x)}(s_i) \geq h(\mu_i).$$

When player II plays a measure μ_{i+1} according to the rules of the game, we have the inequality

$$\mathrm{rank}_{j_{\mu_{i+1}}(T_x)}(j_{\mu_i,\mu_{i+1}}(s_i)) = j_{\mu_i,\mu_{i+1}}(\mathrm{rank}_{j_{\mu_i}(T_x)}(s_i)) \geq j_{\mu_i,\mu_{i+1}}(h(\mu_i)) > h(\mu_{i+1}),$$

so player I can choose a successor $s_{i+1} \supsetneq j_{\mu_i,\mu_{i+1}}(s_i)$ of rank at least $h(\mu_{i+1})$ in the tree $j_{\mu_{i+1}}(T_x)$, maintaining the inequality. Then player I plays the integer $m_{i+1} = x(i+1)$, the least such finite sequence s_{i+1}, and the ordinal $h_{i+1} = h(\mu_{i+1})$, maintaining the inequality. In this way player I can follow the rules forever. □

We may now improve "semi-scale" to "scale" in Woodin's theorem 3.6.5. The conclusion of the theorem below was proved was proved by Woodin (unpublished) under an additional mouse capturing hypothesis. From just the hypotheses below, Jackson proved in [**9**] a weaker conclusion where the norms are $\mathrm{OD}_{T,x}$ for some real x.

THEOREM 4.1.4 (AD). *Let* Γ *be an inductive-like pointclass. If* $\underset{\sim}{\mathbf{Env}}(\Gamma) \neq \wp(\mathbb{R})$ *then every* $\check{\Gamma}$ *set has a scale with norms in* $\mathrm{Env}(\Gamma)$*.*

PROOF. If $\underset{\sim}{\mathbf{Env}}(\Gamma) \neq \wp(\mathbb{R})$ then by Wadge's lemma $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is a surjective image of $\mathbb{R}$. So by the coding of measures in Lemma 3.5.4 there is a surjection $F : \mathbb{R} \to \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$, and by Turing determinacy we have a fine, countably complete measure $\mathcal{U}$ on $\wp_{\omega_1}(\mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega}))$ defined by $\mathcal{Z} \in \mathcal{U}$ if $\{F(y) : y \leq_{\mathrm{T}} x\} \in \mathcal{Z}$ for a cone of $x \in \mathbb{R}$.

Let $B \in \check{\Gamma}$, say $B = \mathbb{R} \setminus p[S]$ for a tree $S \in L[T]$ on $\omega \times \kappa$. For a measure $\mu \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$, notice that if a set of measures $\sigma \subset \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ stabilizes μ then so does every superset of measures $\sigma' \supseteq \sigma$. So to get a countable set of measures σ stabilizing each of its elements, it suffices by $\mathsf{DC}_{\mathbb{R}}$ to show that any given measure μ is stabilized by some countable set of measures σ. In fact, we will show that $\mathcal{U}$-almost every set of measures σ stabilizes μ.

Assume toward a contradiction that there is a measure μ that $\mathcal{U}$-almost every set σ fails to stabilize. Then Lemma 4.1.3 says that for $\mathcal{U}$-almost every σ, player II has no winning strategy in the game $G_S^{\sigma,\mu}$ and so the proof of the Gale–Stewart theorem gives a canonical winning strategy F^σ for player I. To get a contradiction, we define a single tower of measures $\vec{\mu}$ from σ that is a winning play against the strategy F^σ for $\mathcal{U}$-almost every σ.

Let n be such that the measure μ concentrates on κ^n, and let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order. Let $m_0^\sigma, \ldots, m_n^\sigma$ and s_n^σ denote the moves played as "$m_0, \ldots, m_n$" and "s_n" respectively by the strategy F^σ on the first turn, n, where we number the turns starting with n for notational convenience. Define a measure $\mu_{n+1} \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ by

$$X \in \mu_{n+1} \iff \forall^*_{\mathcal{U}}\sigma\,(s^\sigma_n \in j_{\mu_n}(X)).$$

For $i \geq n$ let m^σ_i and s^σ_i denote the moves played as "m_i" and "s_i" respectively by the strategy F^σ on turn i against the play $(\mu_{n+1},\dots,\mu_i)$ by player II, and define a measure $\mu_{i+1} \in \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ by

$$X \in \mu_{i+1} \iff \forall^*_{\mathcal{U}}\sigma\,(s^\sigma_i \in j_{\mu_i}(X)).$$

By the countable completeness of $\mathcal{U}$ there is a real x such that $\mathcal{U}$-almost every set σ has the property that $m^\sigma_i = x(i)$ for every $i < \omega$. The measures $\mu_0,\dots,\mu_n$ all concentrate on S_x by the rule for player I concerning $m_0,\dots,m_n$, and the measures μ_{i+1} for $i \geq n$ all concentrate on S_x as well because $s^\sigma_i \in j_{\mu_i}(S_x)$ for $\mathcal{U}$-almost every σ by the rules for player II. It's easy to check that μ_{i+1} projects to μ_i, so the measures form a tower $\vec{\mu} = (\mu_i : i < \omega)$.

For any set of measures σ with the above property and such that $\mu_i \in \sigma$ for every $i < \omega$, the tower of measures $\vec{\mu} = (\mu_i : i > n)$ is a legal play by player II in the game $G^{\sigma,\mu}_S$, so the moves h^σ_i played as "h_i" by the strategy F^σ against $\vec{\mu}$ form a sequence witnessing that $\vec{\mu}$ is illfounded.

Take a wellfounded tree $W \in \underset{\sim}{\mathbf{L}}[T]$ on κ on which each measure μ_i in the tower concentrates, and such that the function $h : \omega \to \mathrm{Ord}$ defined by $h(i) = [\mathrm{rank}_W]_{\mu_i}$ is a pointwise minimal witness to the illfoundedness of $\vec{\mu}$ as in Lemma 3.5.9. Actually we only need the minimality of $h(n)$. By the countable completeness of $\mathcal{U}$ and the definition of μ_{i+1}, there is some countable $\sigma \subset \mathrm{meas}^{\underset{\sim}{\mathbf{\Gamma}}}(\kappa^{<\omega})$ with all the properties mentioned above as well as the property that $s^\sigma_i \in j_{\mu_i}(W)$ for every $i < \omega$. Define a function $h' : \omega \to \mathrm{Ord}$ by

$$h'(i) = \mathrm{rank}_{j_{\mu_i}(W)}(s^\sigma_i).$$

Then by the rules for player I concerning the finite sequences s_i we have $j_{\mu_i,\mu_{i+1}}(h'(i)) > h'_{i+1}$ and also $h'(n) < \mathrm{rank}_{j_{\mu_n}(W)}([\mathrm{id}]_{\mu_n}) = h(n)$, contradicting the minimality of $h(n)$. $\square$

4.2. Local term-capturing

Definition 4.2.1. Let c be a countable transitive set, let M be a transitive model of a sufficient fragment of ZFC with $c \in M$, and let A be a set of reals. The *local capturing term* $\tau^M_{A,c}$ is the unique standard $\mathrm{Col}(\omega,c)$-term in M, if it exists, such that

$$(\tau^M_{A,c})_g = A \cap M[g]$$

for every M-generic filter $g \subset \mathrm{Col}(\omega,c)$. If the local capturing term $\tau = \tau^M_{A,c}$ exists then we say that M *locally term-captures* A at c via τ.[1]

As usual, we let Γ be an inductive-like pointclass such that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined, and let T be the tree of a Γ-scale on a universal Γ set. The following lemma shows that constructing from T adds local capturing terms for every set in the envelope. This is not vacuous because

[1]Some authors write "understands" in place of "locally term-captures."

if $M = L(T, c)$ then M-generic filters on $\mathrm{Col}(\omega, c)$ actually exist: taking a real x coding c, there are only countably many reals in $L[T, x]$ by Corollary 3.4.4, so there are only countably many subsets of $\mathrm{Col}(\omega, c)$ in the model M.

LEMMA 4.2.2 (Local term-capturing). *Let Γ be an inductive-like pointclass such that $\utilde{\mathbf{\Delta}}_\Gamma$ is determined, and let T be the tree of a Γ-scale $\vec{\varphi}$ on a universal Γ set. Let c be a countable transitive set and let A be a set of reals. Let $M = L(T, c)$.*

(1) If $A \in \mathrm{OD}^{<\Gamma}$ then the term $\tau^M_{A,c}$ exists.
(2) If $A \in \mathrm{Env}(\Gamma)$ then the term $\tau^M_{A,c}$ exists and is equal to $\tau^M_{A',c}$ for some $A' \in \mathrm{OD}^{<\Gamma}$.

PROOF. (1) Let φ be the first norm of the scale $\vec{\varphi}$. Let $(A_\alpha : \alpha < \kappa)$ be a $\Delta_1^{\mathscr{M}_\Gamma}$ enumeration of the $\mathrm{OD}^{<\Gamma}$ sets of reals and fix $\eta < \kappa$ such that $A = A_\eta$. The norm φ is $\Delta_1^{\mathscr{M}_\Gamma}$, so the relation

$$B = \{(x, y) : y \in \mathrm{dom}(\varphi) \;\;\&\;\; x \in A_{\varphi(y)}\}$$

on $\mathbb{R} \times \mathbb{R}$ is Σ_1 over $\mathscr{M}_\Gamma$. Therefore B is in Γ and there is a tree $S \in L[T]$ with $p[S] = B$. The proof of the Becker–Kechris theorem [**2**] shows that in $L[T]$ we may use T and S to define a family of games $\{G^\eta(x) : x \in \mathbb{R}\}$ on κ such that

- $G^\eta(x)$ is closed uniformly in x,[2] and
- $x \in A_\eta$ if and only if player II has a winning strategy for $G^\eta(x)$.

That is, there is a set $S_\eta \subset \kappa^{<\omega} \times \omega^{<\omega}$ in $L[T]$ such that

$$x \in A_\eta \iff (\Game f \in \kappa^\omega)(\forall n < \omega)\,(f \restriction n, x \restriction n) \in S_\eta.$$

In M we define a $\mathrm{Col}(\omega, c)$ term τ for a set of reals by

$$(p, \dot{x}) \in \tau \iff p \Vdash (\Game f \in \kappa^\omega)(\forall n < \omega)\,(f \restriction n, x \restriction n) \in S_\eta$$

For every M-generic filter $g \subset \mathrm{Col}(\omega, c)$ we have $\tau_g = A \cap M[g]$ by the absoluteness of the existence of winning strategies for closed games. Therefore the term τ satisfies the definition of $\tau^M_{A,c}$.

(2) By part 1 we may define the set $X = \{\tau^M_{A',c} : A' \in \mathrm{OD}^{<\Gamma}\}$ of local capturing terms for sets of reals in $\mathrm{OD}^{<\Gamma}$. Although there are uncountably many $\mathrm{OD}^{<\Gamma}$ sets of reals, the set X is countable by Corollary 3.4.4. Suppose toward a contradiction that none of the terms $\tau \in X$ satisfies the definition of $\tau^M_{A,c}$. Then by countable choice for reals we may choose for each term $\tau \in X$ a real x_τ witnessing this failure to satisfy the definition, that is,

$$x_\tau \in \tau_g \iff x_\tau \notin A \cap M[g].$$

for some M-generic filter $g \subset \mathrm{Col}(\omega, c)$. Let $\sigma \subset \mathbb{R}$ be the countable set $\{x_\tau : \tau \in X\}$ and take $A' \in \mathrm{OD}^{<\Gamma}$ such that $A \cap \sigma = A' \cap \sigma$. Considering the local capturing term $\tau^M_{A',c} \in X$ immediately leads to a contradiction. □

[2]It is closed uniformly in x and η, but the Becker–Kechris theorem used only the uniformity in η and the present argument uses only the uniformity in x.

The term-capturing lemma easily relativizes to $\mathrm{Env}(\Gamma(z))$ for any real $z \in L(T, c)$, and consequently every set of reals $A \in \underset{\sim}{\mathbf{Env}}(\Gamma)$ has all of the classical regularity properties: it has the perfect set property and the property of Baire, and is Lebesgue measurable and completely Ramsey. Of course, all of these properties of A except the last one are already known to follow from the determinacy of $\underset{\sim}{\mathbf{Env}}(\Gamma)$. However the term-capturing lemma gives some extra information in the lightface context; for example, if $A \in \mathrm{Env}(\Gamma)$ then either $A \subset L[T]$ (equivalently $A \subset C_\Gamma$) or else A contains a perfect set.

Our main application of the term-capturing lemma will be to define local capturing terms for the sets in a self-justifying system. In the next section, we will obtain a self-justifying system from a scale.

4.3. Self-justifying systems and condensation

Definition 4.3.1. A pointclass Λ has the *weak scale property* if every set in Λ has a scale $\vec{\varphi} = (\varphi_i : i < \omega)$ whose norms φ_i are all Λ-norms.

Notice that in this definition no uniformity is required—$\vec{\varphi}$ need not be a Λ-scale or even a $\underset{\sim}{\mathbf{\Lambda}}$-scale. The following theorem is proved by the same argument as in [**7**], but as usual we must be careful to check that we have enough determinacy:

Theorem 4.3.2. *Let Γ be an inductive-like pointclass and suppose that $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined. If there is a scale on a universal $\check{\Gamma}$ set whose norm relations are in $\underset{\sim}{\mathbf{Env}}(\Gamma)$, then $\underset{\sim}{\mathbf{Env}}(\Gamma)$ has the weak scale property.*

Proof. Let U be a universal $\check{\Gamma}$ set and let $\vec{\varphi} = (\varphi_i : i < \omega)$ be a scale on U with each norm relation $\leq_i$ in $\underset{\sim}{\mathbf{Env}}(\Gamma)$. By Theorem 3.2.9 there is a $\check{\Gamma}$ relation on $\mathbb{R} \times \mathbb{R}$ with no uniformization in $\underset{\sim}{\mathbf{Env}}(\Gamma)$, so the norm relations must be Wadge-cofinal in $\underset{\sim}{\mathbf{Env}}(\Gamma)$.

Let $A \in \underset{\sim}{\mathbf{Env}}(\Gamma)$ and let $\underset{\sim}{\mathbf{\Gamma}}_0 \subsetneq \underset{\sim}{\mathbf{Env}}(\Gamma)$ be a non-self-dual boldface pointclass with the prewellordering property and closed under $\exists^{\mathbb{R}}$ and $\forall^\omega$, such that A is contained in the ambiguous part $\underset{\sim}{\mathbf{\Delta}}_{\Gamma_0}$ of $\underset{\sim}{\mathbf{\Gamma}}_0$. We can find such a pointclass by analysis of the projective-like hierarchy containing A because $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is projectively closed by Theorem 3.2.8. Then $\underset{\sim}{\mathbf{\Gamma}}_0$ is closed under wellordered unions of sequences that are coded by sets in $\underset{\sim}{\mathbf{Env}}(\Gamma)$ (see [**7**].)

Take $n < \omega$ such that the norm relation $\leq_n$ is not in $\underset{\sim}{\mathbf{\Gamma}}_0$. In particular we have that $\underset{\sim}{\mathbf{\Gamma}}_0$ is closed under wellordered unions of sequences that are coded by sets projective in $\leq_n$. This implies that there is an ordinal $\alpha < \mathrm{ran}(\varphi_n)$ such that the set

$$U_\alpha = \{x \in U : \varphi_n(x) < \alpha\}$$

is not in $\underset{\sim}{\mathbf{\Delta}}_{\Gamma_0}$. Suppose to the contrary that every U_α is in $\underset{\sim}{\mathbf{\Delta}}_{\Gamma_0}$. Then we could show by induction on α that the initial segment $\leq_n \restriction U_\alpha \times U_\alpha$ of $\leq_n$ is in $\underset{\sim}{\mathbf{\Gamma}}_0$. The base case is trivial. If $\leq_n \restriction U_\alpha \times U_\alpha$ is in $\underset{\sim}{\mathbf{\Gamma}}_0$ then so is $\leq_n \restriction U_{\alpha+1} \times U_{\alpha+1}$ because we have

$$\begin{aligned}
&\varphi_n(x) \leq \varphi_n(y) < \alpha + 1 \iff \\
&\varphi_n(x) \leq \varphi_n(y) < \alpha \text{ or } \varphi_n(x) < \alpha \ \& \ \varphi_n(y) = \alpha \text{ or } \varphi_n(x) = \varphi_n(y) = \alpha.
\end{aligned}$$

The limit step follows by closure under wellordered unions. The norm relation $\leq_n$ itself is not in $\undertilde{\Gamma}_0$, so at stage $\alpha = \operatorname{ran}\varphi_n$ we get a contradiction.

So take an ordinal α such that U_α is not in $\undertilde{\mathbf{\Delta}}_{\Gamma_0}$ and therefore $A <_{\mathrm{W}} U_\alpha$. By the lower semi-continuity property of the scale $\vec{\varphi}$, the restrictions $(\varphi_i \restriction U_\alpha : i < \omega)$ give a scale on U_α. For each $i < \omega$ the norm relation of the restriction $\varphi_i \restriction U_\alpha$ is in $\undertilde{\mathbf{Env}}(\Gamma)$. Taking a continuous function f such that $A = f^{-1}(U_\alpha)$, we have a scale on A given by $((\varphi_i \circ f) \restriction A : i < \omega)$ whose norm relations are in $\undertilde{\mathbf{Env}}(\Gamma)$. $\square$

Definition 4.3.3. A sequence of pointsets $\vec{A} = (A_i : i < \omega)$ is a *self-justifying system* if the set $\{A_i : i < \omega\}$ is closed under complements and every set A_n, where $n < \omega$, has a scale whose norm relations are all in the set $\{A_i : i < \omega\}$.

The following corollary of Theorem 4.3.2 is immediate by $\mathsf{DC}_{\mathbb{R}}$. The author does not know whether the corresponding statement holds for the lightface envelope $\mathrm{Env}(\Gamma)$.

Corollary 4.3.4. *Let Γ be an inductive-like pointclass and suppose that $\undertilde{\mathbf{\Delta}}_\Gamma$ is determined. If there is a scale on a universal $\check{\Gamma}$ set U whose norm relations are in $\undertilde{\mathbf{Env}}(\Gamma)$, then there is a self-justifying system $\vec{A} \in \undertilde{\mathbf{Env}}(\Gamma)^\omega$ with $A_0 = U$.*

The primary motivation for the definition and study of self-justifying systems comes from the following condensation property.

Theorem 4.3.5 (Woodin). *Let $(A_i : i < \omega)$ be a self-justifying system. Let c be a countable transitive set and let M be a transitive model of a sufficiently large fragment of ZFC such that $c \in M$. Suppose that M locally term-captures each set A_i at c. Let $\pi : \bar{M} \to M$ be an elementary embedding with $c \in \operatorname{ran}(\pi)$ and $\{\tau^M_{A_i,c} : i < \omega\} \subset \operatorname{ran}(\pi)$, say $\pi(\bar{c}) = c$ and $\pi(\bar{\tau}_i) = \tau^M_{A_i,c}$. Then we have $\bar{\tau}_i = \tau^M_{A_i,\bar{c}}$. That is, $\bar{M}$ locally term-captures A_i at $\bar{c}$ via $\bar{\tau}_i$.*

We need to relativize Lemma 3.4.2 on C_Γ to arbitrary countable transitive sets with the following Lemma. We note that equivalence of (2), (3), and (4) was proved under the assumption of AD in [**44**].

Lemma 4.3.6. *Let Γ be an inductive-like pointclass such that $\undertilde{\mathbf{\Delta}}_\Gamma$ is determined. Let κ be the prewellordering ordinal of Γ and let T on $\omega \times \kappa$ be the tree of a Γ-scale on a universal Γ set. Let a be a countable transitive set. Given a surjection $g : \omega \to a$, let $a_g = \{(m, n) : g(m) \in g(n)\}$ denote the relation on ω coding the set a relative to g. For any subset $b \subset a$, the following statements are equivalent:*

(1) $b \in \mathrm{OD}^{<\Gamma}(a, x)$ for some set $x \in a$.[3]

(2) $g^{-1}``(b) \in C_\Gamma(a_g)$ for every surjection $g : \omega \to a$.

(3) $g^{-1}``(b) \in C_\Gamma(a_g)$ for comeager many surjections $g : \omega \to a$.

(4) $b \in L(T, a)$.

In particular, $\wp(a)^{L(T,a)}$ is countable.

[3] That is, $b \in \Delta_1^{\mathscr{M}_\Gamma}(a, x, \alpha)$ for some $\alpha < \kappa$.

Proof. Let a be a countable transitive set and let $b \subset a$.

(1) implies (2): If $b \in \mathrm{OD}^{<\Gamma}(a, x)$ for some set $x \in a$ then for any surjection $g : \omega \to a$, we have $g^{-1}``(b) \in \mathrm{OD}^{<\Gamma}(a_g)$, and therefore $g^{-1}``(b) \in C_\Gamma(a_g)$ by Lemma 3.4.2.

(2) implies (3): Immediate.

(3) implies (4): The model $L(T, a)$ has only countably many subsets of a by the relativization of Lemma 3.4.2 to any real coding a, so we can build $L(T, a)$-generic filters on $\mathrm{Col}(\omega, a)$. Let $G \subset \mathrm{Col}(\omega, a)$ be a $L(T, a)$-generic filter such that the function $g = \bigcup G$ satisfies $g^{-1}``(b) \in C_\Gamma(a_g)$. Therefore we have $g^{-1}``(b) \in L[T, a_g]$ by Lemma 3.4.2 relativized to the real a_g. The model $L(T, a)[g]$ is equal to $L[T, a_g]$, so it contains $g^{-1}``(b)$ and therefore b itself. Now in the same way we can take an $L(T, a)[g]$-generic filter $h \subset \mathrm{Col}(\omega, a)$ such that $b \in L(T, a)[h]$ also, so b was already in the ground model $L(T, a)$.

(4) implies (1): The ordinal κ is a regular cardinal in $L(T, a)$ by the proof of Lemma 3.4.1. Therefore if $b \in L(T, a)$ a Skolem hull argument shows that $b \in L_\alpha(T \restriction \gamma, a)$ for some ordinals α and γ with $\gamma < \alpha < \kappa$. Take $x \in a$ such that y is ordinal-definable from $T \restriction \gamma$, a, and x in $L_\alpha(T \restriction \gamma, a)$. Then b is Δ_1-definable from α, γ, a, and x over the companion $\mathscr{M}_\Gamma$ of Γ because T is $\Delta_1^{\mathscr{M}_\Gamma}$. $\square$

Definition 4.3.7. Let Γ be an inductive-like pointclass such that $\utilde{\mathbf{\Delta}}_\Gamma$ is determined. Let a be a countable transitive set.

- $C_\Gamma(a)$ denotes the collection of subsets $b \subset a$ satisfying the equivalent conditions of Lemma 4.3.6.
- $\mathrm{Lp}^\Gamma(a)$ is the union of all ω-sound premice over a projecting to a with ω_1-iteration strategies in $\utilde{\mathbf{\Delta}}_\Gamma$, or equivalently in $\utilde{\mathbf{\Gamma}}$, reorganized as a premouse.

Notice that under AD a countable premouse is $(\omega_1 + 1)$-iterable if and only if it is ω_1-iterable, because ω_1 is a measurable cardinal. If in addition every set of reals is Suslin, which is the case in $\mathscr{M}_\Gamma$ by [**7**, p. 1790], this iterability is also equivalent to (ω_1, ω_1)-iterability and also to $(\omega_1, \omega_1 + 1)$-iterability (see [**36**, p. 66].) Next we define two versions of mouse capturing, a global version and a local version.

Definition 4.3.8.

- Assume AD. Then MC, *mouse capturing*, is the statement that for every countable transitive set a and every subset $b \subset a$ that is OD from elements of $a \cup \{a\}$ we have $b \in \mathrm{Lp}(a)$.
- Assume that Γ is an inductive-like pointclass and $\utilde{\mathbf{\Delta}}_\Gamma$ is determined. Then Γ-MC is the statement that for every countable transitive set a and every $b \in C_\Gamma(a)$ we have $b \in \mathrm{Lp}^\Gamma(a)$.

Notice that the converses of MC and Γ-MC are always true, because mice can be compared and mice with iteration strategies in $\mathscr{M}_\Gamma$ can be compared inside $\mathscr{M}_\Gamma$. Therefore Γ-MC implies that $C_\Gamma(a) = \mathrm{Lp}^\Gamma(a) \cap \wp(a)$.

Definition 4.3.9 ($\vec{A}$-mouse, etc.). Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined and Γ-MC holds. Let z be a real such that there is a self-justifying system $\vec{A} = (A_i : i < \omega)$ in $\mathrm{Env}(\Gamma(z))^\omega$ where A_0 is a universal Γ set.

For a countable model $\mathcal{M}$ with parameter z the model $\mathcal{M} \oplus \vec{A}$ is defined (essentially as in [**36**]) as $(\mathcal{M}^+; \in, B)$ where $\mathcal{M}^+ = \mathrm{Lp}^\Gamma_\omega(\mathcal{M})$ and B is the *term relation* for $\vec{A}$, namely the set of pairs (i, τ) where $i < \omega$ and τ is the term that locally captures the pointset $\bigoplus_{i<n} A_i$ over $\mathrm{Lp}^\Gamma_\omega(\mathcal{M})$ at the cardinal δ_i. Note that $\mathcal{M} \oplus \vec{A}$ is an amenable structure.

An $\vec{A}$-*mouse* $\mathcal{N}$ is like an ordinary mouse except that the B-predicates of appropriate levels are allowed to code term relations, and for every level $\mathcal{M} \lhd \mathcal{N}$ we have $\mathcal{M} \oplus \vec{A} \trianglelefteq \mathcal{N}$.

An $\vec{A}$-*mouse operator* is defined by analogy with an ordinary mouse operator. (Recall that all our mouse operators are first-order.) An example of an $\vec{A}$-mouse operator would be the $M_n^{\vec{A},\sharp}$ operator, which maps a countable model $\mathcal{P}$ with parameter z to the $\vec{A}$-mouse $M_n^{\vec{A},\sharp}(\mathcal{P})$ defined as the least $\vec{A}$-mouse with base model $\mathcal{P}$ that is active, ω-sound, projects to $\mathcal{P}$, and has n Woodin cardinals as witnessed by extenders on its E-sequence.

The *model operator* F_J *coding an* $\vec{A}$*-mouse operator* J is defined just like for an ordinary mouse operator. That is, $F_J(\mathcal{M})$ is the model coding the least initial segment of $J(\mathcal{M})$ projecting below $\rho_\omega(\mathcal{M})$ if it exists, and is the model coding $J(\mathcal{M})$ itself if there is no such initial segment.

The following proposition, which can be proved using Woodin's condensation theorem 4.3.5, allows us to construct $\vec{A}$-mice using $K^{c,F}$ constructions.

Proposition 4.3.10. *Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined and Γ-MC holds. Let z be a real and let $\vec{A} = (A_i : i < \omega)$ be a self-justifying system in $\mathrm{Env}(\Gamma(z))^\omega$ such that A_0 is a universal Γ set. Then for any $\vec{A}$-mouse operator J, then the corresponding model operator F_J condenses well.*

4.4. Sealing the envelope with a strong pseudo-homogeneous ideal

Now we apply the techniques developed in this chapter to our proof of the inner model direction (2) of the Main Theorem, and in particular to the "gap in scales" case of the core model induction. That is, once we have used a strong, pseudo-homogeneous ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$ to construct an inductive-like pointclass Γ, we will construct scales beyond Γ.

The first time the "gap in scales" case arises is when Γ is equal to IND, the pointclass of inductive sets. In this case we do not need to use our hypothesis on the ideal $\mathcal{I}$ to get the scales beyond Γ; they exist on general descriptive-set-theoretic grounds by Moschovakis [**26**]. In fact, for a long time the scales are given by Steel's analysis of scales in $L(\mathbb{R})$ and $K(\mathbb{R})$ [**35, 37, 41**] without using our hypothesis. However, the hypothesis will eventually be needed to reach a model of $\mathsf{ZF} + \mathsf{AD} + \theta_0 < \Theta$, so we may as well assume it from the start and use the argument below to handle every gap in scales in a uniform way.

Proposition 4.4.1 (ZFC). *If there is a strong pseudo-homogeneous ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$ and Γ is an inductive-like pointclass such that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined and every set in Γ is definable from a countable sequence of ordinals, then $\utilde{\mathbf{Env}}(\Gamma)$ has the weak scale property.*

Before proving this, we note that the hypothesis is reasonable because all the determined sets of reals that we construct in our proof of the Main Theorem are definable from countable sequences of ordinals. The conclusion of the proposition implies that there is a model operator F with condensation that determines itself on generic extensions and is coded by a set of reals just beyond $\utilde{\mathbf{Env}}(\Gamma)$. We will later use the argument for PD with the core model K replaced by its relativization K^F to construct determined sets of reals beyond $\utilde{\mathbf{Env}}(\Gamma)$.

We begin by proving a lemma on the size of the envelope. Note that the hypothesis follows from the existence of a strong ideal on $\wp_{\omega_1}(\mathbb{R})$.

Lemma 4.4.2 (ZFC). *Suppose that every function $\wp_{\omega_1}(\mathbb{R}) \to \omega_1$ is bounded on a stationary set by a canonical function for some ordinal $\gamma < \mathfrak{c}^+$. If Γ is an inductive-like pointclass such that $\utilde{\boldsymbol{\Delta}}_\Gamma$ is determined, then $\left|\utilde{\mathbf{Env}}(\Gamma)\right| \leq \mathfrak{c}$.*

Proof. We will prove the corresponding statement for the lightface envelope, namely that $|\mathrm{Env}(\Gamma)| \leq \mathfrak{c}$, which is equivalent. Fix a $\Delta_1^{\mathscr{M}_\Gamma}$ enumeration $(A_\alpha : \alpha < \kappa)$ of the $\mathrm{OD}^{<\Gamma}$ sets of reals. For every Turing degree d and every set of reals $A \in \mathrm{Env}(\Gamma)$ define $f_d(A)$ to be the least ordinal $\alpha < \kappa$ such that $x \in A \iff x \in A_\alpha$ for every real $x \leq_{\mathrm{T}} d$. For $A, B \in \mathrm{Env}(\Gamma)$ define $A < B$ if $f_d(A) < f_d(B)$ for a cone of d. By the proof of Proposition 3.2.5, this defines a well-ordering of $\mathrm{Env}(\Gamma)$. Let λ be the length of this well-ordering. We need to see that $\lambda < \mathfrak{c}^+$.

For $\sigma \in \wp_{\omega_1}(\mathbb{R})$ choose any Turing degree $d(\sigma)$ above all the reals in σ and let $F(\sigma)$ be the order type of the subset of κ given by $\{f_{d(\sigma)}(A) : A \in \mathrm{Env}(\Gamma)\}$. We have $F(\sigma) < \omega_1$ because $C_\Gamma(\{x : x \leq_{\mathrm{T}} d\})$ is countable.

Take a countable elementary substructure $X \prec V_{\omega+3}$ containing the well-ordering $<$ of $\mathrm{Env}(\Gamma)$ and let $\sigma = X \cap \mathbb{R}$. For any $A, B \in X \cap \mathrm{Env}(\Gamma)$ we have $A < B \iff f_{d(\sigma)}(A) < f_{d(\sigma)}(B)$: this inequality holds for a cone of d and by elementarity the base of some such cone is in σ and therefore below $d(\sigma)$. Therefore for a club of countable elementary substructures $X \prec V_{\omega+3}$ we have

$$\mathrm{o.t.}(X \cap \lambda) \leq F(X \cap \mathbb{R}).$$

Now taking $\gamma < \mathfrak{c}^+$ such that F is bounded by the γ^{th} canonical function $\wp_{\omega_1}(\mathbb{R}) \to \omega_1$ one can show that $\lambda \leq \gamma$ by standard arguments. $\square$

Proof of Proposition 4.4.1. Let $G \subset \mathcal{I}^+ \setminus \mathcal{I}$ be a V-generic filter and let $j : V \to \mathrm{Ult}(V, G) \subset V[G]$ be the associated elementary embedding. Let T be the tree of a Γ-scale on a universal Γ set. Then T is definable from a countable sequence of ordinals, so by the pseudo-homogeneity of $\mathcal{I}$ its image $j(T)$ is independent of the generic filter G and is therefore in V.

Let λ be the length of the wellordering of $\mathrm{Env}(\Gamma)$ from Proposition 3.2.5. By Lemma 4.4.2 we have $\lambda < \mathfrak{c}^+$, so $j``\lambda \in \mathrm{Ult}(V, G)$. Therefore the set of measures

$$\sigma = j``(\mathrm{meas}^{\underset{\sim}{\boldsymbol{\Gamma}}}(\kappa^{<\omega}))$$

is in $\mathrm{Ult}(V, G)$ and is countable there. We will show that for every measure $\mu \in \sigma$, player II has a winning strategy in the game $G^{\sigma,\mu}_{j(T)}$ from Definition 4.1.2 as defined in the generic ultrapower $\mathrm{Ult}(V, G)$. Then by the elementarity of j there is a countable set of measures $\bar{\sigma} \subset \mathrm{meas}^{\underset{\sim}{\boldsymbol{\Gamma}}}(\kappa^{<\omega})$ stabilizing each of its measures, so $\underset{\sim}{\mathbf{Env}}(\Gamma)$ has the weak scale property by Theorem 4.3.2.

We will describe a winning strategy for player II in the game $G^{\sigma,j(\mu)}_{j(T)}$ as defined in the generic ultrapower $\mathrm{Ult}(V, G)$. By the absoluteness of the existence of winning strategies for closed games, it is okay to work in $V[G]$ when defining the strategy. Let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order and let $\bar{\mu}_0, \ldots, \bar{\mu}_n$ denote their preimages under j, that is, $j(\bar{\mu}_i) = \mu_i$ for all $i \leq n$.

At the beginning of the game, which we will call turn n for notational convenience, player I plays integers $m_0, \ldots, m_n$, a finite sequence $s_n \in j_{\mu_n}(j(\kappa))^{n+1}$ satisfying $s_n \in j_{\mu_n}(j(T_{m_0,\ldots,m_n}))$, and an ordinal h_n. Notice the fact that

$$j_{\mu_i} \circ j = j \circ j_{\bar{\mu}_i}$$

for all i, which we will use without comment several times. Define the measure $\bar{\mu}_{n+1} \in \mathrm{meas}^{\underset{\sim}{\boldsymbol{\Gamma}}}(\kappa^{<\omega})$ by

$$X \in \bar{\mu}_{n+1} \iff s_n \in j_{\mu_n}(j(X)).$$

Notice that $j_{\mu_n}(j(X))$ does not depend on the generic G by pseudo-homogeneity: X is definable from T and a real, $\bar{\mu}_n$ is definable from T and an ordinal by Proposition 3.5.5, and T itself is definable from a countable sequence of ordinals, so $j_{\bar{\mu}_n}(X)$ is definable from a countable sequence of ordinals. Therefore we have $\bar{\mu}_{n+1} \in V$. Also notice that because $s_n \in j(j_{\bar{\mu}_n}(T_{(m_0,\ldots,m_n)}))$ and $s_n \supsetneq j([\mathrm{id}]_{\bar{\mu}_n})$, the new measure $\bar{\mu}_{n+1}$ concentrates on the set $T_{(m_0,\ldots,m_n)}$ and projects to $\bar{\mu}_n$. Player II then plays the measure $\mu_{n+1} = j(\bar{\mu}_{n+1})$.

On the turn numbered i where $i > n$, player I plays an integer m_i, a finite sequence $s_i \in j_{\mu_i}(j(\kappa))^{i+1}$ satisfying $s_i \in j_{\mu_i}(j(T_{m_0,\ldots,m_i}))$, and an ordinal h_i. Define the measure $\bar{\mu}_{i+1} \in \mathrm{meas}^{\underset{\sim}{\boldsymbol{\Gamma}}}(\kappa^{<\omega})$ by

$$X \in \bar{\mu}_{i+1} \iff s_i \in j_{\mu_i}(j(X)).$$

As before, $j_{\mu_i}(j(X))$ does not depend on the generic G by pseudo-homogeneity so we have $\bar{\mu}_{i+1} \in V$. It is easy to check that $\bar{\mu}_{i+1}$ concentrates on the set $T_{(m_0,\ldots,m_i)}$ and projects to $\bar{\mu}_i$. Player II then plays the measure $\mu_{i+1} = j(\bar{\mu}_{i+1})$.

Assume toward a contradiction that player I is able to follow the rules forever against the strategy that we have described. Then we get a real $x = (m_0, m_1, \ldots)$, a tower of measures $(\mu_i : i < \omega)$ from σ, and a sequence of ordinals $(h_i : i < \omega)$ witnessing the illfoundedness of this tower. So by the elementarity of j the corresponding tower $(\bar{\mu}_i : i < \omega)$ of measures in V is also illfounded.

Take a wellfounded tree $W \in \underset{\sim}{\mathbf{L}}[T]$ on κ on which each measure $\bar{\mu}_i$ in this tower concentrates, and such that the function $\bar{h} : \omega \to \mathrm{Ord}$ defined by $\bar{h}(i) = [\mathrm{rank}_W]_{\mu_i}$ is a pointwise minimal witness to the illfoundedness of the tower $(\bar{\mu}_i : i < \omega)$ as in Lemma 3.5.9. Then by the elementarity of j, the function $h = j(\bar{h})$ is a pointwise minimal witness to the illfoundedness of the tower $(\mu_i : i < \omega)$. Actually we only need the minimality of $h(n)$.

Because $\bar{\mu}_i$ concentrates on W we have $s_i \in j_{\mu_n}(j(W))$ for all $i < \omega$. Define a function $h' : \omega \to \mathrm{Ord}$ by $h'(i) = \mathrm{rank}_{j_{\mu_n}(j(W))}(s_i)$. Then from the rules for player I concerning the finite sequences s_i we have $j_{\mu_i,\mu_{i+1}}(h'(i)) > h'_{i+1}$ and also $h'(n) < \mathrm{rank}_{j_{\mu_n}(j(W))}([\mathrm{id}]_{\mu_n}) = h(n)$, contradicting the minimality of $h(n)$. $\square$

4.5. AD from a strong pseudo-homogeneous ideal

Recall the ordinal that we considered in Section 2.6, which measures our progress in the core model induction in $\mathrm{Lp}(\mathbb{R})$. Namely, we let α be the strict supremum of the ordinals γ such that

(1) The coarse mouse witness condition $W^*_{\gamma+1}$ holds,
(2) γ is a critical ordinal in $\mathrm{Lp}(\mathbb{R})$, and
(3) $\gamma + 1$ begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$.

We have AD in $\mathrm{Lp}(\mathbb{R})|\alpha$ because if γ is a critical ordinal in $\mathrm{Lp}(\mathbb{R})$ and $W^*_{\gamma+1}$ holds then the coarse mice can be used to prove determinacy in $\mathrm{Lp}(\mathbb{R})|(\gamma+1)$.

Now assume that there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$. If there is a model of $\mathsf{AD} + \theta_0 < \Theta$ then we are done, so we work under the "smallness assumption" that there is no such model. Then $\mathrm{Lp}(\mathbb{R})|\alpha$ is admissible by Proposition 2.6.2. By Steel's theorem 2.6.3 the pointclass

$$\Gamma = \Sigma_1^{\mathrm{Lp}(\mathbb{R})|\alpha}$$

has the scale property, which combined with the admissibility of $\mathrm{Lp}(\mathbb{R})|\alpha$ shows that Γ is inductive-like. Admissibility also implies that, defining the ambiguous part $\Delta_\Gamma = \Gamma \cap \check{\Gamma}$, the corresponding boldface pointclass $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is equal to $(\mathrm{Lp}(\mathbb{R})|\alpha) \cap \wp(\mathbb{R})$. Therefore $\underset{\sim}{\mathbf{\Delta}}_\Gamma$ is determined, which implies that the envelope $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is determined.

By Proposition 4.4.1 there is a self-justifing system $\vec{A} \in \underset{\sim}{\mathbf{Env}}(\Gamma)^\omega$ containing a universal Γ set. We note that the elements of this self-justifying system must be Wadge-cofinal in $\underset{\sim}{\mathbf{Env}}(\Gamma)$ because in general the envelope of an inductive-like pointclass Γ cannot contain a scale on a universal $\check{\Gamma}$ set. The next step in proving more determinacy is the following lemma.

Lemma 4.5.1. *Let α be the strict supremum of the ordinals γ such that $W^*_{\gamma+1}$ holds, γ is a critical ordinal in* $\mathrm{Lp}(\mathbb{R})$*, and $\gamma+1$ begins a Σ_1-gap in* $\mathrm{Lp}(\mathbb{R})$*. Define the pointclass $\Gamma = \Sigma_1^{\mathrm{Lp}(\mathbb{R})|\alpha}$. Suppose there is no model of $\mathsf{AD} + \theta_0 < \Theta$. If there is a strong pseudo-homogeneous ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$ then $\underset{\sim}{\mathbf{Env}}(\Gamma) = \mathrm{Lp}(\mathbb{R}) \cap \wp(\mathbb{R})$.*

Proof. Let $D \in \underset{\sim}{\mathbf{Env}}(\Gamma)$, say $D \in \mathrm{Env}(\Gamma(x))$ where $x \in \mathbb{R}$. We will show that $D \in \mathrm{Lp}(\mathbb{R})$. Let $j : V \to \mathrm{Ult}(V, G) \subset V[G]$ be the elementary embedding associated to a V-generic filter $G \subset \mathcal{I}^+/\mathcal{I}$. Then the set D, which is equal to $j(D) \cap \mathbb{R}^V$, is in $C_{j(\Gamma)}(\mathbb{R}^V)$. By the mouse capturing principle $j(\Gamma)$-MC in the generic ultrapower, which follows from the coarse mouse witness condition together with our smallness assumption that there is no model of $\mathsf{AD} + \theta_0 < \Theta$, we have $D \in \mathcal{M}$ for some ω-sound premouse $\mathcal{M}$ over $\mathbb{R}^V$ projecting to $\mathbb{R}^V$ with a $j(\omega_1)$-iteration strategy in $j(\mathrm{Lp}(\mathbb{R})|\alpha)$. The least such $\mathcal{M}$ is ordinal-definable from $\mathbb{R}^V$ in the generic ultrapower and so we have $\mathcal{M} \in V$ by the pseudo-homogeneity of $\mathcal{I}$.

Every countable elementary substructure $\bar{\mathcal{M}}$ of $\mathcal{M}$ in V is also a countable elementary substructure of $\mathcal{M}$ in $\mathrm{Ult}(V, G)$, so it has a $j(\omega_1)$-iteration strategy in $j(\mathrm{Lp}(\mathbb{R})|\alpha)$, namely the pullback of such a strategy for $\mathcal{M}$. Therefore by the elementarity of j the premouse $\bar{\mathcal{M}}$ has an ω_1-iteration strategy in $\mathrm{Lp}(\mathbb{R})|\alpha$. Given any set of reals $B \in \mathrm{Lp}(\mathbb{R})|\alpha$—for example, the set coding this iteration strategy—there is a pair of ω-absolutely complementing trees $(T, S) \in \mathrm{Lp}(\mathbb{R})|\alpha$ with $p[T] = B$. The trees $j(S)$ and $j(T)$ are in V by pseudo-homogeneity, and the pair $(j(S), j(T))$ is $\mathbb{R}$-absolutely complementing: it is complementing in $V[h]$ where $h \subset \mathrm{Col}(\omega, \mathbb{R}^V)$ is $V[G]$-generic, because $\mathrm{Ult}(V, G)[h]$ is a Cohen generic extension of the model $\mathrm{Ult}(V, G)$, where the pair $(j(S), j(T))$ is ω-absolutely complementing. Therefore $\bar{\mathcal{M}}$ has a $\mathfrak{c}^+$-iteration strategy. This shows that $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$, so $D \in \mathrm{Lp}(\mathbb{R})$.

Now assume toward a contradiction that this inclusion is proper; that is, $\underset{\sim}{\mathbf{Env}}(\Gamma) \subsetneq \mathrm{Lp}(\mathbb{R}) \cap \wp(\mathbb{R})$. Let β be least such that $\mathrm{Lp}(\mathbb{R})|(\beta + 1) \cap \wp(\mathbb{R}) \not\subset \underset{\sim}{\mathbf{Env}}(\Gamma)$. Then because $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is projectively closed we have

$$\underset{\sim}{\mathbf{Env}}(\Gamma) = \mathrm{Lp}(\mathbb{R})|\beta \cap \wp(\mathbb{R}).$$

We have $\alpha \leq \beta$ because $\underset{\sim}{\mathbf{\Delta}}_\Gamma \subset \underset{\sim}{\mathbf{Env}}(\Gamma)$.[4] Because $\mathrm{Lp}(\mathbb{R})|\beta$ projects to $\mathbb{R}$ every countable sequence from $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is in $\mathrm{Lp}(\mathbb{R})|(\beta + 1)$. Let $\vec{A} = (A_n : n < \omega)$ be a self-justifing system sealing $\underset{\sim}{\mathbf{Env}}(\Gamma)$ as given by Proposition 4.4.1, that is, A_0 is a universal Γ set and each A_n is in $\underset{\sim}{\mathbf{Env}}(\Gamma)$. Some subsequence $(A_{n_i} : i < \omega)$ is a scale on a universal $\check{\Gamma}$ set U. So the set U has a scale in $\mathrm{Lp}(\mathbb{R})|(\beta + 1)$, but it cannot have a scale already in $\mathrm{Lp}(\mathbb{R})|\beta$ because the envelope cannot contain a uniformization of U. Therefore β is a critical ordinal in $\mathrm{Lp}(\mathbb{R})$.

We want to relate β to the Σ_1-gap structure of $\mathrm{Lp}(\mathbb{R})$. For any ordinal β^*, if $\mathrm{Lp}(\mathbb{R})|\alpha \prec_1^{\mathbb{R}} \mathrm{Lp}(\mathbb{R})|\beta^*$ then $\mathrm{Lp}(\mathbb{R})|\beta^* \cap \wp(\mathbb{R}) \subset \underset{\sim}{\mathbf{Env}}(\Gamma)$ by the proof of Proposition 3.3.3. Therefore we can take $\beta^* \leq \beta$ such that $[\alpha, \beta^*]$ is a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$. We either have $\beta^* = \beta$ or $\beta^* = \beta - 1$.[5] We cannot have $\beta^* < \beta - 1$; otherwise the pointclass $\Sigma_1^{\mathrm{Lp}(\mathbb{R})|(\beta^*+1)}$ has the scale property by Steel's theorem 2.6.3, but the envelope cannot contain any scaled pointclass beyond Γ. In any case the ordinal $\beta + 1$ begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$.

Now taking a real z such that each set A_i is in $\mathrm{Env}(z)$ we can get a sequence $(M_n^{\vec{A},\sharp} : n < \omega)$ of $\vec{A}$-mouse operators on H_{ω_1} with parameter z from repeated applications of Theorem 2.4.4 just as in our proof of projective determinacy, but starting with the model operator F_J

[4] In fact $\alpha < \beta$ because $\underset{\sim}{\mathbf{\Gamma}} \subset \underset{\sim}{\mathbf{Env}}(\Gamma)$.

[5] Using the construction of scales at the end of a weak gap from [**41**], one can show that these cases are the weak gap case and strong gap case respectively, but the distinction does not matter for us.

coding the $\vec{A}$-mouse operator given by $J(\mathcal{M}) = \mathcal{M} \oplus \vec{A}$. These $\vec{A}$-mouse operators are all projective in $\vec{A}$ and are cofinal in the projective-like hierarchy containing $\vec{A}$, or equivalently in the Levy hierarchy of sets of reals definable from parameters over $\mathrm{Lp}(\mathbb{R})|\beta$. Together they can be used to establish the coarse mouse witness condition $W^*_{\beta+1}$ in $\mathrm{Lp}(\mathbb{R})$. Therefore $\beta < \alpha$ by the definition of α, which is a contradiction. □

Now it is a simple matter to get a model of AD.

THEOREM 4.5.2. *If there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$ then* AD *holds in $L(\mathbb{R})$.*

PROOF. We have $L(\mathbb{R}) \cap \wp(\mathbb{R}) \subset \mathrm{Lp}(\mathbb{R}) \cap \wp(\mathbb{R}) = \underset{\sim}{\mathbf{Env}}(\Gamma)$ by Lemma 4.5.1, or else there is a model of $\mathsf{AD} + \theta_0 < \Theta$. The pointclass $\underset{\sim}{\mathbf{Env}}(\Gamma)$ is determined, so in either case AD holds in $L(\mathbb{R})$. □

At this point we can apply Σ_1 reflection in the model $L(\mathrm{Lp}(\mathbb{R}))$ combined with the fact that the pointclass $\underset{\sim}{\mathbf{\Delta}}_\Gamma$, which is equal to $\mathrm{Lp}(\mathbb{R})|\alpha \cap \wp(\mathbb{R})$, is the class of Suslin co-Suslin sets of this model, to see that

$$\Gamma = \Sigma_1^{\mathrm{Lp}(\mathbb{R})}.$$

The reason that the Σ_1-reflection principle holds in the model $L(\mathrm{Lp}(\mathbb{R}))$ is that this model satisfies $\mathsf{ZF} + \mathsf{AD}$ and has a largest Suslin cardinal, namely α. This observation that $\Gamma = \Sigma_1^{\mathrm{Lp}(\mathbb{R})}$ is not necessary for the following argument and we will use it only to simplify the notation by omitting mention of α in several places.

4.6. $\mathsf{AD} + \theta_0 < \Theta$ from a strong pseudo-homogeneous ideal

Let Γ be the pointclass $\Sigma_1^{\mathrm{Lp}(\mathbb{R})}$ and let $\vec{A} = (A_n : n < \omega)$ be a self-justifing system sealing $\underset{\sim}{\mathbf{Env}}(\Gamma)$ as given by Proposition 4.4.1. That is, A_0 is a universal Γ set and each A_n is in $\underset{\sim}{\mathbf{Env}}(\Gamma)$. We suppose that there is no inner model of $\mathsf{AD} + \theta_0 < \Theta$ containing all the reals and ordinals, because otherwise we are done. Under this smallness assumption we have $\underset{\sim}{\mathbf{Env}}(\Gamma) = \mathrm{Lp}(\mathbb{R}) \cap \wp(\mathbb{R})$ by Lemma 4.5.1, so each A_n is in $\mathrm{Lp}(\mathbb{R})$, but $\vec{A}$ itself cannot be in $\mathrm{Lp}(\mathbb{R})$ because $\underset{\sim}{\mathbf{Env}}(\Gamma)$ cannot contain a scale on a universal $\check{\Gamma}$ set.

Take a real z such that each set A_i is in $\mathrm{Env}(\Gamma(z))$. Notice that $\vec{A}$ is definable from a countable sequence of ordinals because each A_i is definable from z, an ordinal, and a universal Γ set that is itself definable. So we are still in the realm of sets where pseudo-homogeneity is a useful property.

DEFINITION 4.6.1 ($\mathrm{Lp}^{\vec{A}}$). For a model $\mathcal{P}$ over z, let $\mathrm{Lp}^{\vec{A}}(\mathcal{P})$ denote $\mathrm{Lp}^F(\mathcal{P})$ where F is the model operator F_J coding the $\vec{A}$-mouse operator given by $J(\mathcal{M}) = \mathcal{M} \oplus \vec{A}$. Let $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$ denote $\mathrm{Lp}^{\vec{A}}(\mathcal{P})$ where $\mathcal{P} = (V_{\omega+1}; \in, \emptyset)$.

The proof that $\mathrm{Lp}(\mathbb{R}) \models \mathsf{AD}$ can be relativized to $\vec{A}$ to show that $\mathrm{Lp}^{\vec{A}}(\mathbb{R}) \models \mathsf{AD}$. To do this, we begin by defining an ordinal that represents the progress of our core model induction through $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$. First we make the following definition.

DEFINITION 4.6.2 ($W_\gamma^{\vec{A},*}$). The *coarse mouse witness condition* $W_\gamma^{\vec{A},*}$ says that for any set $U \subset \mathbb{R}$ such that both U and its complement have scales in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\gamma$, for all $k < \omega$ and $x \in \mathbb{R}$ there is a coarse (k, U)-Woodin mouse containing x with an $(\omega_1 + 1)$-iteration strategy whose restriction to H_{ω_1} is in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\gamma$.

We say that γ is *critical in* $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$ if some set of reals has a scale in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|(\gamma+1)$ but not in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\gamma$, so that $W_{\gamma+1}^{\vec{A},*}$ does not follow trivially from $W_\gamma^{\vec{A},*}$.

Then we let $\alpha_{\vec{A}}$ be the strict supremum of the ordinals γ such that

- The coarse mouse witness condition $W_{\gamma+1}^{\vec{A},*}$ holds,
- γ is a critical ordinal in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$, and
- $\gamma + 1$ begins a Σ_1-gap in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$.

When dealing with Σ_1 formulas and Σ_1-gaps in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$, the following remark is important to keep in mind.

REMARK 4.6.3. A Σ_1 formula in the language of $\vec{A}$-premice is allowed to refer to the term relation, so in particular it can define $\vec{A}$ itself over $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$.

We have AD in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})\big|\alpha_{\vec{A}}$ because if γ is a critical ordinal in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$ and $W_{\gamma+1}^{\vec{A},*}$ holds then the coarse mice can be used to prove determinacy in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|(\gamma+1)$. We can relativize the proof of Proposition 2.6.2 to $\vec{A}$ to get

PROPOSITION 4.6.4 (ZFC). *Suppose that there is no model of $\mathsf{AD} + \theta_0 < \theta$ containing all the reals and ordinals. Let α be the strict supremum of the ordinals γ such that $W_{\gamma+1}^{\vec{A},*}$ holds, γ is critical in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$, and $\gamma+1$ begins a Σ_1-gap in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$. If there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$, then $\mathrm{Lp}^{\vec{A}}(\mathbb{R})\big|\alpha_{\vec{A}}$ is admissible.*

The very first step of the proof is to show that $W_1^{\vec{A},*}$ holds using the $\vec{A}$-mouse operators $M_n^{\vec{A},\sharp}$ for $n < \omega$, which implies determinacy for sets of reals projective in $\vec{A}$. These $\vec{A}$-mouse operators are obtained just as in the “gap in scales” case in the proof of $\mathrm{Lp}(\mathbb{R})$ in Lemma 4.5.1. The only difference is that here the gap does not end inside $\mathrm{Lp}(\mathbb{R})$ but instead ends just beyond it.

We omit the full proof of admissibility, remarking only that under our smallness assumption that there is no model of $\mathsf{AD} + \theta_0 < \theta$ containing all the reals and ordinals, for limit ordinals γ the coarse mouse witness condition $W_\gamma^{\vec{A},*}$ implies the $\vec{A}$-mouse witness condition $W_\gamma^{\vec{A}}$, which is the relativization of W_γ saying that Σ_1 facts true in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\gamma$ are witnessed by $\vec{A}$-mice with ω_1-iteration strategies in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\gamma$. The proof of W_γ from W_γ^* in [**44**] and [**36**] relativizes to show this by using $K^{c,F}$ constructions in the coarse mice in place of K^c constructions, where F is the model operator F_J corresponding to the $\vec{A}$-mouse operator given by $J(\mathcal{M}) = \mathcal{M} \oplus \vec{A}$.

As with W_γ^* the coarse mice that witness $W_\gamma^{\vec{A},*}$ will not be particularly coarse. Often they will just be $\vec{A}$-mice. However, to cross a gap in scales in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$ we will need to consider "two-step" term relation hybrid mice with term relations both for $\vec{A}$ and for some self-justifying system $\vec{B}$ that seals the gap in scales. We will call these $(\vec{A}, \vec{B})$-mice. The implication $W_\gamma^{\vec{A},*} \implies W_\gamma^{\vec{A}}$ for limit ordinals γ is needed to show that these self-justifying systems do not pile up as we go along, so we never need to consider $(\vec{A}, \vec{B}, \vec{C})$-mice—or, what would be more problematic, hybrid mice with uncountably many self-justifying systems.

Relativizing Steel's theorem on scales in $\mathrm{Lp}(\mathbb{R})$ in [**37**] to $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$, one can show that the pointclass

$$\Gamma_{\vec{A}} = \Sigma_1^{\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}}}$$

has the scale property. The condensation property of $\vec{A}$-premice plays a crucial role in relativizing the proof. Combined with the admissibility of $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}}$ this scale property shows that the pointclass $\Gamma_{\vec{A}}$ is inductive-like. Admissibility also implies that, defining the ambiguous part $\Delta_{\vec{A}} = \Gamma_{\vec{A}} \cap \check{\Gamma}_{\vec{A}}$, the corresponding boldface pointclass $\utilde{\mathbf{\Delta}}_{\Gamma_{\vec{A}}}$ is equal to $(\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}}) \cap \wp(\mathbb{R})$. Therefore $\utilde{\mathbf{\Delta}}_{\Gamma_{\vec{A}}}$ is determined, which implies that the envelope $\utilde{\mathbf{Env}}(\Gamma_{\vec{A}})$ is determined. The next lemma is just the relativization of Lemma 4.5.1 to $\vec{A}$.

Lemma 4.6.5. *Suppose that there is no model of* $\mathsf{AD} + \theta_0 < \theta$ *containing all the reals and ordinals. Let* α *be the strict supremum of the ordinals* γ *such that* $W_{\gamma+1}^{\vec{A},*}$ *holds,* γ *is critical in* $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$*, and* $\gamma+1$ *begins a* Σ_1*-gap in* $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$*. Define the pointclass* $\Gamma_{\vec{A}} = \Sigma_1^{\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}}}$*. If there is a strong pseudo-homogeneous ideal* $\mathcal{I}$ *on* $\wp_{\omega_1}(\mathbb{R})$ *then we have* $\utilde{\mathbf{Env}}(\Gamma_{\vec{A}}) = \mathrm{Lp}^{\vec{A}}(\mathbb{R}) \cap \wp(\mathbb{R})$.

Proof. Let $D \in \utilde{\mathbf{Env}}(\Gamma_{\vec{A}})$, say $D \in \mathrm{Env}(\Gamma_{\vec{A}}(x))$ where $x \in \mathbb{R}$. We will shows that $D \in \mathrm{Lp}^{\vec{A}}(\mathbb{R})$. Let $j : V \to \mathrm{Ult}(V, G) \subset V[G]$ be the elementary embedding associated to a V-generic filter $G \subset \mathcal{I}^+/\mathcal{I}$. Then the set D, which is equal to $j(D) \cap \mathbb{R}^V$, is in $C_{j(\Gamma_{\vec{A}})}(\mathbb{R}^V)$. By $j(\vec{A})$-mouse capturing[6] in $j(\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}})$ we have that $D \in \mathcal{M}$ for some ω-sound $j(\vec{A})$-premouse $\mathcal{M}$ over $\mathbb{R}^V$ projecting to $\mathbb{R}^V$ with a $j(\omega_1)$-iteration strategy in $j(\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}})$. The least such $\mathcal{M}$ is ordinal-definable from $\mathbb{R}^V$ and $j(\vec{A})$ in the generic ultrapower. Because $\vec{A}$ is definable from a countable sequence of ordinals in V, this gives $\mathcal{M} \in V$ by the pseudo-homogeneity of $\mathcal{I}$.

Every countable elementary substructure $\bar{\mathcal{M}}$ of $\mathcal{M}$ in V is also a countable elementary substructure of $\mathcal{M}$ in $\mathrm{Ult}(V, G)$. Therefore by the condensation lemma for $j(\vec{A})$-premice, $\bar{\mathcal{M}}$ is also a $j(\vec{A})$-premouse. Moreover $\bar{\mathcal{M}}$ has a $j(\omega_1)$-iteration strategy in $j(\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}})$, namely the pullback of such a strategy for $\mathcal{M}$. So by the elementarity of j the structure $\bar{\mathcal{M}}$ is an $\vec{A}$-premouse with an ω_1-iteration strategy in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}}$. As in the proof of Lemma 4.5.1 one can show that every set of reals in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\alpha_{\vec{A}}$ is $\mathfrak{c}^+$-universally Baire, so this ω_1-iteration strategy can be extended to a $\mathfrak{c}^+$-iteration strategy. This shows that $\mathcal{M} \lhd \mathrm{Lp}^{\vec{A}}(\mathbb{R})$, so $D \in \mathrm{Lp}^{\vec{A}}(\mathbb{R})$.

[6]That is, capturing of $j(C_{\Gamma_{\vec{A}}})$ by $j(\vec{A})$-mice with iteration strategies in $j(\utilde{\mathbf{\Delta}}_{\Gamma_{\vec{A}}})$, which follows from the coarse mouse witness condition $W_{j(\alpha_{\vec{A}})}^{j(\vec{A}),*}$ in the generic ultrapower under the smallness assumption that there is no model of $\mathsf{AD} + \theta_0 < \theta$ containing all the reals and ordinals.

Now assume toward a contradiction that this inclusion is proper; that is, $\underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}}) \subsetneq \mathrm{Lp}^{\vec{A}}(\mathbb{R}) \cap \wp(\mathbb{R})$. Let β be least such that $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|(\beta+1) \cap \wp(\mathbb{R}) \not\subset \underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}})$. Then because $\underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}})$ is projectively closed we have

$$\underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}}) = \mathrm{Lp}^{\vec{A}}(\mathbb{R})|\beta \cap \wp(\mathbb{R}).$$

We have $\alpha_{\vec{A}} \leq \beta$ because $\underset{\sim}{\mathbf{\Delta}}_{\Gamma_{\vec{A}}} \subset \underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}})$. Because $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\beta$ projects to $\mathbb{R}$ every countable sequence from $\underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}})$ is in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|(\beta+1)$. Let $\vec{B} = (B_n : n < \omega)$ be a self-justifing system sealing $\underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}})$ as given by Proposition 4.4.1, that is, B_0 is a universal $\Gamma_{\vec{A}}$ set and each B_n is in $\underset{\sim}{\mathbf{Env}}(\Gamma_{\vec{A}})$. Some subsequence $(B_{n_i} : i < \omega)$ is a scale on a universal $\check{\Gamma}_{\vec{A}}$ set U. So the set U has a scale in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|(\beta+1)$, but it cannot have a scale already in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\beta$ because the envelope cannot contain a uniformization of U. Therefore β is a critical ordinal. As in the proof of Lemma 4.5.1 one can show that $\beta+1$ begins a Σ_1-gap in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})$. (Recall that Σ_1 formulas are allowed to refer to the term relation for $\vec{A}$.)

Taking a real t such that each set B_i is in $\mathrm{Env}(\Gamma_{\vec{A}}(t))$ we can get a sequence of "two-step" hybrid $(\vec{A},\vec{B})$-mouse operators $(M_n^{\vec{A},\vec{B},\sharp} : n < \omega)$ on H_{ω_1} with parameter (z,t). If $\mathcal{P}$ is a countable model with parameter (z,t) then $M_n^{\vec{A},\vec{B},\sharp}(\mathcal{P})$ denotes the least term relation hybrid premouse over $\mathcal{P}$ that is active, ω-sound, projects to $\mathcal{P}$, has n Woodin cardinals, and has term relations for both $\vec{A}$ and $\vec{B}$. An $(\vec{A},\vec{B})$-premouse is like an $\vec{A}$-premouse, but in addition to adding the term relation for $\vec{A}$ at appropriate limit levels that are full with respect to ordinary mice, we also add the term relation for $\vec{B}$ at appropriate limit levels that are full with respect to $\vec{A}$-mice with ω_1-iteration strategies in $\mathrm{Lp}^{\vec{A}}(\mathbb{R})\big|\alpha_{\vec{A}}$. Because $C_{\Gamma_{\vec{A}}}$ is captured by such mice, this amount of fullness suffices to locally term-capture each set B_n.

We get these $(\vec{A},\vec{B})$-mouse operators from repeated applications of Theorem 2.4.4 just as in our proof of projective determinacy, but starting with the model operator F_J coding the $(\vec{A},\vec{B})$-mouse operator given by $J(\mathcal{M}) = \mathcal{M} \oplus \vec{A} \oplus \vec{B}$. These operators are all projective in $\vec{B}$ and are cofinal in the projective-like hierarchy containing $\vec{B}$, or equivalently in the Levy hierarchy of sets of reals definable from parameters over $\mathrm{Lp}^{\vec{A}}(\mathbb{R})|\beta$. Together they can be used to establish the coarse mouse witness condition $W_{\beta+1}^{\vec{A},*}$, so $\beta < \alpha_{\vec{A}}$ by the definition of $\alpha_{\vec{A}}$, which is a contradiction. □

Now we are ready to complete the proof of the main theorem.

Theorem 4.6.6 (Main Theorem, part 2). *If there is a strong pseudo-homogeneous ideal on $\wp_{\omega_1}(\mathbb{R})$, then there is a model of $\mathsf{AD} + \theta_0 < \Theta$ containing all the reals and ordinals.*

Proof. Assume not. We will obtain a contradiction by showing that the model $M = L(\mathrm{Lp}^{\vec{A}}(\mathbb{R}))$ is such a model, where $\vec{A}$ is a self-justifying system sealing $\mathrm{Lp}(\mathbb{R})$. Lemma 4.6.5 together with the determinacy of the envelope implies that $M \models \mathsf{AD}$. We complete the proof of the theorem by showing that $\vec{A}$ is not ordinal-definable from a real in M. (Here we do *not* equip M with the term relation for $\vec{A}$.)

Under our smallness assumption the existence of coarse mouse witnesses in M implies (ordinary) mouse capturing in M. That is, if σ is a countable transitive set and the subset

$a \subset \sigma$ is $\mathrm{OD}_{\sigma\cup\{\sigma\}}$ in M, then a is in a premouse over σ with an ω_1-iteration strategy in M. By Σ_1-reflection in M, we can take the iteration strategy to be Suslin and co-Suslin in M.

If $\vec{A}$ is OD_x in the model M for some real x, then $\vec{A} = j(\vec{A}) \cap \mathbb{R}^V$ is $\mathrm{OD}_{\{\mathbb{R}^V, x\}}$ in the model $j(M)$ and so $\vec{A}$ is in an ω-sound premouse $\mathcal{M}$ over $\mathbb{R}^V$ projecting to $\mathbb{R}^V$ with a $j(\omega_1)$-iteration strategy in $j(M)$. (We could have used a more general theorem of Steel in [**36**] on capturing via $\mathbb{R}$-mice, but the presence of a pseudo-homogeneous ideal allows this easier argument.) The least such $\mathcal{M}$ is ordinal-definable from $\mathbb{R}^V$ and $j(\vec{A})$ in the generic ultrapower, so we have $\mathcal{M} \in V$ by the pseudo-homogeneity of $\mathcal{I}$.

Every countable elementary substructure $\bar{\mathcal{M}}$ of $\mathcal{M}$ in V is also a countable elementary substructure of $\mathcal{M}$ in $\mathrm{Ult}(V, G)$, so it has a $j(\omega_1)$-iteration strategy that is Suslin and co-Suslin in $j(M)$, namely the pullback of such a strategy for $\mathcal{M}$. Therefore by the elementarity of j the premouse $\bar{\mathcal{M}}$ has an ω_1-iteration strategy in that is Suslin and co-Suslin in M. As in the proof of Lemma 4.6.5 one can show that every Suslin co-Suslin set of M is $\mathfrak{c}^+$-universally Baire, so this iteration strategy can be extended to a $\mathfrak{c}^+$-iteration strategy. This shows that $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$, so $\vec{A} \in \mathrm{Lp}(\mathbb{R})$. This is a contradiction, because $\vec{A}$ computes a scale on a universal $\Pi_1^{\mathrm{Lp}(\mathbb{R})}$ set, so it cannot be in $\mathrm{Lp}(\mathbb{R})$. Therefore the Wadge rank of $\vec{A}$ in M must be at least θ_0. (In fact one can easily show that it is exactly θ_0 and that M satisfies $\Theta = \theta_1$.) $\square$

CHAPTER 5

Sealing the envelope: an alternative method

The goal of this chapter is to discuss a method for constructing a countable hull $\overline{\mathcal{M}_\infty}$ of the direct limit premosue $\mathcal{M}_\infty$ (see Definition 5.1.5) and an iteration strategy for $\overline{\mathcal{M}_\infty}$ that is fullness-preserving and has branch condensation. The method is essentially the same as Ketchersid's thesis [**18**] (see also [**32**]) but in sections 5.2 and 5.3 we prove some intermediate results in a more general context. We also make two simplifications to the arguments, namely we do not use the fact that $\mathcal{M}_\infty$ "computes HOD" in a determinacy model, and we do not require that assumption that the envelope has countable cofinality (this can be deduced at the end instead.)

We also work more locally, defining a direct limit premouse $\mathcal{M}^\Gamma_\infty$ that is usually only considered when Γ is the pointclass Σ^2_1 of a model of $\mathsf{ZF}+\mathsf{AD}+\mathsf{MC}$, for example, $\Gamma = (\Sigma^2_1)^{L(\mathbb{R})}$. Our generalization pertains to pointclasses with fewer closure properties, such as $\Gamma = \mathrm{IND}$. The idea is to unify all "gap in scales" cases of the core model induction into a single case. The main inner-model-theoretic tool we will need for this generalization is only a conjecture at this point (Conjecture 5.1.2,) so for the Main Theorem we must rely on the descriptive set-theoretic approach to the "gap in scales" case that is given in Chapter 4.

5.1. Quasi-iterable pre-mice and $\mathcal{M}_\infty$

Definition 5.1.1. Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined and Γ-MC holds. A countable premouse $\mathcal{P}$ is Γ-*suitable* if it has a unique Woodin cardinal $\delta^{\mathcal{P}}$, and $\mathcal{P} = \mathrm{Lp}^\Gamma_\omega(\mathcal{P}|\delta^{\mathcal{P}})$, but no ordinal $\eta < \delta^{\mathcal{P}}$ is a Woodin cardinal in $\mathrm{Lp}^\Gamma(\mathcal{P}|\eta)$.

We refer the reader to [**32**, Ch. 7] for the definitions of the ordinals $\gamma^{\mathcal{P}}_A$ and hulls $H^{\mathcal{P}}_A$ for $A \in \mathrm{Env}(\Gamma)$, the notions of Lp-guided tree, short tree, and maximal tree, suitable sequence, A-good interval, A-guided partial quasi-iteration map $\pi^A_{\mathcal{P},\mathcal{Q}}$, quasi-limit of a suitable sequence, A-quasi-iterable, and strongly A-quasi-iterable.

Conjecture 5.1.2 (Quasi-iterability conjecture). Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined and Γ-MC holds. Then for every $A \in \mathrm{Env}(\Gamma)$ then there is a strongly A-quasi-iterable premouse.

If $\Gamma = (\Sigma^2_1)^M$ where $M \models \mathsf{ZF}+\mathsf{AD}+\mathsf{MC}$ and $\mathbb{R} \subset M$ and $\mathrm{Env}(\Gamma) \subset M$ then we have $\mathrm{Env}(\Gamma) = \mathrm{OD}^M \cap \wp(\mathbb{R})$ and the conjecture holds by the following theorem:

Theorem 5.1.3 (Woodin, unpublished). *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{MC}$. *Then for every* OD *set of reals A, there is a strongly A-quasi-iterable premouse.*

If $\mathcal{P}$ is A-quasi-iterable and $\mathcal{Q}$ is strongly B-quasi-iterable then comparing $\mathcal{P}$ and $\mathcal{Q}$ gives a common quasi-iterate $\mathcal{R}$ that is strongly $(A \oplus B)$-quasi-iterable where $\oplus$ denotes the operation of recursive join. This leads us to define the following directed system.

Definition 5.1.4. $\mathcal{F}_\infty^\Gamma$ is the directed system whose indices are pairs $(\mathcal{P}, A)$ where $A \in \mathrm{Env}(\Gamma)$ and $\mathcal{P}$ is a Γ-suitable, strongly A-quasi-iterable premouse, ordered by $(\mathcal{P}, A) \leq_{\mathcal{F}_\infty} (\mathcal{Q}, B)$ if $\mathcal{Q}$ is a quasi-iterate of $\mathcal{P}$ and A is obtained from B by a recursive substitution, and whose maps are the quasi-iteration maps $\pi^A_{\mathcal{P},\mathcal{Q}} : H^{\mathcal{P}}_A \to H^{\mathcal{Q}}_A$.

Our definition is a bit different than the standard one, which has finite tuples $\vec{A} \in \mathrm{Env}(\Gamma)^{<\omega}$ rather than single sets, and the ordering says that $\mathrm{ran}(\vec{A}) \subset \mathrm{ran}(\vec{B})$ rather than saying that A is obtained from B by a recursive substitution. The difference is purely cosmetic.

Definition 5.1.5. We define the premouse

$$\mathcal{M}_\infty^\Gamma = \varinjlim \mathcal{F}_\infty^\Gamma,$$

to be the direct limit of $\mathcal{F}_\infty^\Gamma$ under the partial quasi-iteration maps $\pi^A_{\mathcal{P},\mathcal{Q}}$.

The direct limit is well-founded because for any countable sequence $((\mathcal{P}_i, A) : i < \omega)$ from $\mathcal{F}_\infty^\Gamma$ we can quasi-compare all of the $\mathcal{P}_i$'s simultaneously to get a premouse $\mathcal{P}$. Then the partial quasi-iteration maps $\pi^{A_i}_{\mathcal{P}_i,\infty}$ all factor through the wellfounded model $\mathcal{P}$, so their range cannot contain an infinite decreasing sequence of ordinals.

When the pointclass Γ is clear from context we may omit it from the notation and refer to $\mathcal{F}_\infty^\Gamma$ and $\mathcal{M}_\infty^\Gamma$ as $\mathcal{F}_\infty$ and $\mathcal{M}_\infty$ respectively.

5.2. Full hulls

Definition 5.2.1. Let $j : N \to H_{(2^\mathfrak{c})^+}$ be an elementary embedding where N is countable and transitive, and $\Gamma \in \mathrm{ran}(j)$. Let $j(\bar{\Gamma}) = \Gamma$ and $j(\overline{\mathrm{Env}(\Gamma)}) = \mathrm{Env}(\Gamma)$. We say that N is Γ-*full* if $A \cap N \in \overline{\mathrm{Env}(\Gamma)}$ for every for every $A \in \mathrm{Env}(\Gamma)$.

The definition of Γ-fullness is quite robust, and it is straightforward to verify the following equivalences.

- N is Γ-full if and only if $A \cap N \in N$ for every $A \in \mathrm{Env}(\Gamma)$,
- N is Γ-full if and only if $A \cap N \in N$ for every $A \in \mathrm{OD}^{<\Gamma}$,
- N is Γ-full if and only if $A \cap N \in \overline{\mathrm{Env}(\Gamma)}$ for every $A \in \mathrm{Env}(\Gamma)$, and
- N is Γ-full if and only if $A \cap N \in \overline{\mathrm{Env}(\Gamma)}$ for every $A \in \mathrm{OD}^{<\Gamma}$.

Note that for a countable premouse $\mathcal{P} \in N$ and a set $A \in \mathrm{Env}(\Gamma)$, if $A \cap N \in N$ then we have $\tau^{\mathcal{P}}_A = (\tau^{\mathcal{P}}_{A\cap N})^N$.

The following proposition establishes the properties of Γ-full hulls that will be useful to us in the core model induction.

Proposition 5.2.2. *Let $j : N \to H_{(2^{\mathfrak{c}})^+}$ be an elementary embedding where N is countable and transitive, and $\Gamma \in \operatorname{ran}(j)$. Let $j(\bar{\Gamma}) = \Gamma$ and $j(\overline{\operatorname{Env}(\Gamma)}) = \operatorname{Env}(\Gamma)$. Let $j(\overline{\mathcal{F}_\infty}) = \mathcal{F}_\infty$ and $j(\overline{\mathcal{M}_\infty}) = \mathcal{M}_\infty$. Suppose that N is Γ-full. Then we have*

(1) the direct limit premouse $\overline{\mathcal{M}_\infty}$ is Γ-suitable and strongly $j(\bar{A})$-quasi-iterable for every $\bar{A} \in \overline{\operatorname{Env}(\Gamma)}$, and

(2) the restriction $j \upharpoonright \overline{\mathcal{M}_\infty} : \overline{\mathcal{M}_\infty} \to \mathcal{M}_\infty$ is the union of the maps $\pi^{j(\bar{A})}_{\overline{\mathcal{M}_\infty},\infty}$ of the system $\mathcal{F}_\infty$ for $\bar{A} \in \overline{\operatorname{Env}(\Gamma)}$.

Proof. Choose a cofinal sequence $((\mathcal{P}_i, \bar{A}_i) : i < \omega)$ in the directed system $\overline{\mathcal{F}_\infty}$ of N. Then $\overline{\mathcal{M}_\infty}$ is equal to the direct limit of the system

(C) $\qquad \{(\mathcal{P}_i, \bar{A}_i) : i < \omega\}$

under the partial quasi-iteration maps $\bar{\pi}^{\bar{A}_i}_{\mathcal{P}_i,\mathcal{P}_j}$ where $\bar{\pi}$ is used to denote partial quasi-iteration maps computed in N. The sequence $(\mathcal{P}_i : i < \omega)$ is a Γ-suitable sequence, so it has a Γ-suitable quasi-limit $\mathcal{Q}$, which is by definition the limit of the directed system

(Q) $\qquad \{(\mathcal{P}_i, A) : i < \omega \ \ \& \ \ A \in \operatorname{Env}(\Gamma)\}$

under whichever partial quasi-iteration maps $\pi^{A}_{\mathcal{P}_i,\mathcal{P}_j}$ happen to be defined. Let $A \in \operatorname{Env}(\Gamma)$. Because N is Γ-full the terms $(\tau^{\mathcal{P}_i}_{A\cap N})^N$ and $\tau^{\mathcal{P}_i}_{A}$ are equal for all $i < \omega$, so for all $i, j < \omega$ with $i \leq j$ the quasi-iteration maps $\bar{\pi}^{A\cap N}_{\mathcal{P}_i,\mathcal{P}_j}$ and $\pi^{A}_{\mathcal{P}_i,\mathcal{P}_j}$ are equal or are both undefined. Therefore the quasi-limit premouse $\mathcal{Q}$ is also equal to the limit of the directed system

($\bar{\text{Q}}$) $\qquad \{(\mathcal{P}_i, \bar{A}) : i < \omega \ \ \& \ \ \bar{A} \in \overline{\operatorname{Env}(\Gamma)}\}$

under whichever partial quasi-iteration maps $\bar{\pi}^{\bar{A}}_{\mathcal{P}_i,\mathcal{P}_j}$ of N happen to be defined. The directed system (C) is cofinal in ($\bar{\text{Q}}$) as well as in $\overline{\mathcal{F}_\infty}$. Therefore the direct limits of all four systems $\overline{\mathcal{F}_\infty}$, (C), (Q), and ($\bar{\text{Q}}$) are equal. This shows that $\overline{\mathcal{M}_\infty} = \mathcal{Q}$, so it is Γ-suitable. For every $i < \omega$ the premouse $\mathcal{P}_i$ is strongly $\bar{A}_i$-quasi-iterable in N, so it is strongly $j(\bar{A}_i)$-quasi-iterable in V. The direct limit $\overline{\mathcal{M}_\infty}$ is equal to $\mathcal{Q}$, so it is a quasi-iterate of $\mathcal{P}_i$ and is therefore strongly $j(\bar{A}_i)$-quasi-iterable as well. The $\bar{A}_i$'s are cofinal, so this shows that $\overline{\mathcal{M}_\infty}$ is strongly $j(\bar{A})$-quasi-iterable for every $\bar{A} \in \overline{\operatorname{Env}(\Gamma)}$, thereby establishing part (1).

The fact that the premouse $\mathcal{P}_i$ is strongly $j(\bar{A}_i)$-quasi-iterable also implies that the partial quasi-iteration map $\pi^{j(\bar{A}_i)}_{\mathcal{P}_i,\overline{\mathcal{M}_\infty}}$ is defined and is equal to the direct limit map $\bar{\pi}^{\bar{A}_i}_{\mathcal{P}_i,\infty}$ of the system $\overline{\mathcal{F}_\infty}$, or equivalently of (C) or ($\bar{\text{Q}}$). To show that the map $j \upharpoonright \overline{\mathcal{M}_\infty}$ is equal to the union of the maps $\pi^{j(\bar{A})}_{\overline{\mathcal{M}_\infty},\infty}$, because the domain is a direct limit of the (C) system it suffices to show that their compositions with the maps $\bar{\pi}^{\bar{A}_i}_{\mathcal{P}_i,\infty}$ are equal for all $i < \omega$. Indeed, we have

$$j \circ \bar{\pi}^{\bar{A}_i}_{\mathcal{P}_i,\infty} = j(\bar{\pi}^{\bar{A}_i}_{\mathcal{P}_i,\infty}) = \pi^{j(\bar{A}_i)}_{\mathcal{P}_i,\infty} = \pi^{j(\bar{A}_i)}_{\overline{\mathcal{M}_\infty},\infty} \circ \pi^{j(\bar{A}_i)}_{\mathcal{P}_i,\overline{\mathcal{M}_\infty}} = \pi^{j(\bar{A}_i)}_{\overline{\mathcal{M}_\infty},\infty} \circ \bar{\pi}^{\bar{A}_i}_{\mathcal{P}_i,\infty},$$

establishing part (2). $\square$

Definition 5.2.3. Let $\mathcal{S} \subset \mathrm{Env}(\Gamma)$ be a countable set closed under recursive join. Let $\mathcal{P}$ be a suitable premouse that is strongly A-quasi-iterable for each $A \in \mathcal{S}$. For a quasi-iterate $\mathcal{N}$ of $\mathcal{P}$,

- The $\mathcal{S}$-*hull* $H^{\mathcal{P}}_{\mathcal{S}}$ of $\mathcal{P}$ is the union of the hulls $H^{\mathcal{P}}_{A}$ for $A \in \mathcal{S}$.
- The $\mathcal{S}$-*guided map* $\pi^{\mathcal{S}}_{\mathcal{P},\mathcal{N}}$ is the union of the maps $\pi^{A}_{\mathcal{P},\mathcal{N}}$ for $A \in \mathcal{S}$, for any quasi-iterate $\mathcal{N}$ of $\mathcal{P}$.
- The $\mathcal{S}$-*guided map* $\pi^{\mathcal{S}}_{\mathcal{P},\infty}$ is the union of the maps $\pi^{A}_{\mathcal{P},\infty}$ for $A \in \mathcal{S}$.

Notice that $\pi^{\mathcal{S}}_{\mathcal{P},\infty}$ is only a total map on $\mathcal{P}$ if $\mathcal{P} = H^{\mathcal{P}}_{\mathcal{S}}$. For a Γ-full hull $j : N \to H_{(2^{\mathfrak{c}})^+}$ in the situation of Proposition 5.2.2 where $j(\overline{\mathrm{Env}(\Gamma)}) = \mathrm{Env}(\Gamma)$ and $j(\overline{\mathcal{M}_\infty}) = \mathcal{M}_\infty$, letting $\mathcal{S} = j``\overline{\mathrm{Env}(\Gamma)}$ we have that $\pi^{\mathcal{S}}_{\overline{\mathcal{M}_\infty},\infty}$ is a total map $\overline{\mathcal{M}_\infty} \to \mathcal{M}_\infty$ and is equal to $j \restriction \mathcal{M}_\infty$.

5.3. The full factors property

Proposition 5.2.2 gave us a way to construct countable hulls $\overline{\mathcal{M}_\infty}$ of $\mathcal{M}_\infty$ that are Γ-suitable, and in particular are Lp^Γ-full. Next we consider the Lp^Γ-fullness of premice that are intermediate between $\overline{\mathcal{M}_\infty}$ and $\mathcal{M}_\infty$. In a more general context, we make the following definition.

Definition 5.3.1. Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined and Γ-MC holds. Let $j : \mathcal{M} \to \mathcal{R}$ be an elementary embedding where $\mathcal{M}$ and $\mathcal{R}$ are premice and $\mathcal{M}$ is Γ-suitable. We say that the map j has the Lp^Γ-*full factors property*, or just the *full factors property* when Γ is clear from context, if whenever $\mathcal{P}$ is a countable premouse and there are elementary embeddings

$$\mathcal{M} \xrightarrow{i} \mathcal{P} \xrightarrow{k} \mathcal{R}$$

such that $k \circ i = j$, then $\mathcal{P}$ is Lp^Γ-full.

A sufficient condition for the full factors property is that we can "carry a tree along on top" of j in the following sense.

Lemma 5.3.2. *Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined and Γ-MC holds. Let $j : \mathcal{M} \to \mathcal{R}$ be an elementary embedding where $\mathcal{M}$ and $\mathcal{R}$ are premice and $\mathcal{M}$ is Γ-suitable. Let T be the tree of a Γ-scale on a universal Γ set and let E_j be the extender of length $\mathrm{Ord}^{\mathcal{R}}$ derived from j. If the ultrapower $\mathrm{Ult}(L[T,\mathcal{M}], E_j)$ is well-founded, then the map j has the full factors property.*

Proof. First note that because $\mathcal{M}$ is Lp^Γ-full it is a cardinal initial segment of the model $L[T,\mathcal{M}]$ by Γ-MC, so we really can apply the extender E_j to this model. Now suppose j factors as $k \circ i$ where we have

$$\mathcal{M} \xrightarrow{i} \mathcal{P} \xrightarrow{k} \mathcal{R}$$

for some countable premouse $\mathcal{P}$. We want to show that $\mathcal{P}$ is Lp^Γ-full. Letting E_i be the extender of length $\mathrm{Ord}^{\mathcal{P}}$ derived from i, we can define the ultrapower map $i^* : L[T,\mathcal{M}] \to$

$\mathrm{Ult}(L[T,\mathcal{M}],E_i)$. The ultrapower is wellfounded because it embeds into the larger ultrapower $\mathrm{Ult}(L[T,\mathcal{M}],E_j)$ via the map k^* defined by

$$k^*(i^*(f)(s)) = j(f)(k(s)).$$

The ordinal $\mathrm{Ord}^{\mathcal{M}}$ has cofinality ω in $L[T,\mathcal{M}]$ by the definition of a suitable premouse, so it is a continuity point of i^* and we have

$$i^* : L[T,\mathcal{M}] \to L[i^*(T),\mathcal{P}] \quad \text{and} \quad i^*(\mathcal{M}) = \mathcal{P}.$$

Therefore the premouse $\mathcal{P}$ is a rank initial segment of $L[i^*(T),\mathcal{P}]$ by the elementarity of i_E. Consider every real x as coding a structure $\mathcal{P}_x$. Define the set

$$U = \{x : \mathcal{P}_x \text{ is not a } \mathrm{Lp}^\Gamma\text{-full premouse}\}.$$

Then we have $U \in \Gamma$. By Becker–Kechris [**2**] if we let S be the tree of a Γ-scale on U we have $S \in L[T]$. Because $\mathcal{M}$ is Lp^Γ-full, if we take an $L[T,\mathcal{M}]$-generic filter $g \subset \mathrm{Col}(\omega,\mathcal{M})$ then we have

$$L[T,\mathcal{M}][g] \models \dot{x} \notin p[S]$$

where $\dot{x}$ denotes the term for the generic real code of $\mathcal{M}$. So by the definability of forcing and the elementarity of i^*, if we take an $L[i^*(T),\mathcal{P}]$-generic filter $h \subset \mathrm{Col}(\omega,\mathrm{Ord}^{\mathcal{P}})$ then we have

$$L[i^*(T),\mathcal{P}][h] \models \dot{x} \notin p[i^*(S)]$$

where $\dot{x}$ denotes the term for the generic real code of $\mathcal{P}$. Let $x = \dot{x}_h$ be the generic real code of $\mathcal{P}$ corresponding to h. The tree $i^*(S)_x$ is wellfounded in the wellfounded model $L[i^*(T),\mathcal{P}][h]$, so it is really wellfounded. The tree S_x embeds into it by i^*, so S_x is also wellfounded. Therefore $x \notin U$ and the premouse $\mathcal{P} = \mathcal{P}_x$ is Lp^Γ-full. □

Given a countable hull $\mathcal{M}$ of $\mathcal{M}_\infty$ as in Proposition 5.2.2, if we additionally assume that the map $\mathcal{M} \to \mathcal{M}_\infty$ has the full factors property, then we can get an iteration strategy for $\mathcal{M}_\infty$. The first step is the following preservation lemma.

LEMMA 5.3.3. *Let $\mathcal{S} \subset \mathrm{Env}(\Gamma)$ be a countable set closed under recursive join. Let $\mathcal{M}$ be a suitable premouse that is strongly A-quasi-iterable for every $A \in \mathcal{S}$. Suppose that $H^{\mathcal{M}}_{\mathcal{S}} = \mathcal{M}$ and the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\infty} : \mathcal{M} \to \mathcal{M}_\infty$ has the full factors property. Let $\mathcal{N}$ be a quasi-iterate $\mathcal{N}$ of $\mathcal{M}$. Then $H^{\mathcal{N}}_{\mathcal{S}} = \mathcal{N}$ and the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{N},\infty} : \mathcal{N} \to \mathcal{M}_\infty$ has the full factors property.*

PROOF. First we show that the hull $H^{\mathcal{N}}_{\mathcal{S}}$ of $\mathcal{N}$, defined as the union of the hulls $H^{\mathcal{N}}_A$ for $A \in \mathcal{S}$, is equal to $\mathcal{N}$ itself. The $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$ factors through $H^{\mathcal{N}}_{\mathcal{S}}$, so it also factors through the transitive collapse $\mathcal{P}$ of $H^{\mathcal{N}}_{\mathcal{S}}$. Therefore $\mathcal{P}$ is Lp^Γ-full by the full factors property of $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$. Each hull $H^{\mathcal{N}}_A$ is transitive below $\delta^{\mathcal{N}}$ by definition, so $H^{\mathcal{N}}_{\mathcal{S}} \cap \delta^{\mathcal{N}}$ is an ordinal $\eta \leq \delta^{\mathcal{N}}$. This ordinal η is Woodin in the transitive collapse $\mathcal{P}$, which is full, so we must have $\eta = \delta^{\mathcal{N}}$ by the suitability of $\mathcal{N}$. Now from the fact that $\mathcal{P}$ is full and $\mathcal{N}$ is suitable it follows that $\mathcal{P}$

is all of $\mathcal{N}$, and moreover that the uncollapsed hull $H^{\mathcal{N}}_{\mathcal{S}}$ is all of $\mathcal{N}$. Therefore the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{N},\infty}$ is a total map $\mathcal{N} \to \mathcal{M}_\infty$. It has the full factors property because it factors into a map with the full factors property: $\pi^{\mathcal{S}}_{\mathcal{M},\infty} = \pi^{\mathcal{S}}_{\mathcal{N},\infty} \circ \pi^{\mathcal{S}}_{\mathcal{M},\mathcal{N}}$. □

The following lemma shows that if $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and has the full factors property then it guides an ω_1-iteration strategy. We will later improve this to an (ω_1, ω_1)-iteration strategy.

LEMMA 5.3.4. *If $\mathcal{M}$ is a suitable, quasi-iterable premouse and $\mathcal{S} \subset \mathrm{Env}(\Gamma)$ is a countable set, closed under recursive join, such that the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and has the full factors property, then for any quasi-iterate $\mathcal{P}$ of $\mathcal{M}$ by a maximal iteration tree $\mathcal{T}$, there is a unique non-dropping cofinal branch b of $\mathcal{T}$ with the property that $\mathcal{M}^{\mathcal{T}}_b = \mathcal{P}$ and $i^{\mathcal{T}}_b = \pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}$.*

PROOF. Let $(A_i : i < \omega)$ be a sequence of sets of reals that is cofinal in $\mathcal{S}$ under recursive substitution. Let b_i be an A_i-good branch of $\mathcal{T}$. Then by definition the partial quasi-iteration map $\pi^{A_i}_{\mathcal{M},\mathcal{P}}$ is equal to $i^{\mathcal{T}}_{b_i} \restriction H^{A_i}_{\mathcal{M}}$.

If $i < j < \omega$ then for every $\alpha < \mathrm{lh}\,\mathcal{T}$ with $\mathrm{lh}\,E^{\mathcal{T}}_\alpha < \gamma^{\mathcal{P}}_i$ (or even with $\mathrm{crit}\,E^{\mathcal{T}}_\alpha < \gamma^{\mathcal{P}}_i$) we have $\alpha \in b_i \iff \alpha \in b_j$: If b_i and b_j were to disagree at some α with $\mathrm{crit}\,E^{\mathcal{T}}_\alpha < \gamma^{\mathcal{P}}_i$ then $\mathrm{ran}\, i^{\mathcal{T}}_{b_i} \cap \mathrm{ran}\, i^{\mathcal{T}}_{b_j} \subset \mathrm{crit}\,E^{\mathcal{T}}_\alpha$ by the "zipper argument", but because both branches are A_i-good their ranges both contain the set of points definable from $\tau^{\mathcal{P}}_{A_i}$, which is cofinal in $\gamma^{\mathcal{P}}_{A_i}$, a contradiction.

We claim that the limit branch

$$\begin{aligned} b &= \{\alpha < \mathrm{lh}\,\mathcal{T} : (\exists i < \omega)\,(\mathrm{lh}\,E^{\mathcal{T}}_\alpha < \gamma^{\mathcal{P}}_{A_i} \;\&\; \alpha \in b_i)\} \\ &= \{\alpha < \mathrm{lh}\,\mathcal{T} : (\forall i < \omega)\,(\mathrm{lh}\,E^{\mathcal{T}}_\alpha < \gamma^{\mathcal{P}}_{A_i} \implies \alpha \in b_i)\} \end{aligned}$$

is a non-dropping cofinal branch of $\mathcal{T}$. The limit branch b cannot drop because none of the branches b_i drops. Suppose toward a contradiction that b is not cofinal, and let $\eta = \sup b$. By Lemma 5.3.3 we have that $\mathcal{P} = \bigcup \{H^{\mathcal{P}}_{A_i} : i < \omega\}$, so we can take $i < \omega$ such that $\gamma^{\mathcal{P}}_{A_i} > \mathrm{lh}\,E^{\mathcal{T}}_\eta$. Branches are closed, so $\eta \in b_i$. Let $E^{\mathcal{T}}_\xi$ be the extender applied to $\mathcal{M}^{\mathcal{T}}_\eta$ in b_i. We have $\mathrm{crit}\,E^{\mathcal{T}}_\xi < \mathrm{lh}\,E^{\mathcal{T}}_\eta < \gamma^{\mathcal{P}}_{A_i}$. We also have $\mathrm{lh}\,E^{\mathcal{T}}_\xi \geq \gamma^{\mathcal{P}}_i$; otherwise ξ would be in b. But then $\mathrm{ran}\, b_i$ cannot be cofinal in $\gamma^{\mathcal{P}}_{A_i}$, a contradiction.

We can define an elementary embedding

$$\begin{aligned} &k : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{P} \\ &i^{\mathcal{T}}_b(f)(s) \mapsto \pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}(f)(s), \qquad \text{for } f \in \mathcal{M} \text{ and } s \in \delta(\mathcal{T})^{<\omega}. \end{aligned}$$

To see that k is well-defined, if $i^{\mathcal{T}}_b(f)(s) = i^{\mathcal{T}}_b(g)(t)$ then take $\alpha < \mathrm{lh}(\mathcal{T})$ with $\mathrm{lh}\,E^{\mathcal{T}}_\alpha > \max(s \cup t)$, and take $i < \omega$ such that $\gamma^{\mathcal{P}}_{A_i} > \mathrm{lh}\,E^{\mathcal{T}}_\alpha$ and $f, g \in H^{\mathcal{M}}_{A_i}$. Then we have

$$\pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}(f)(s) = i^{\mathcal{T}}_{b_i}(f)(s) = i^{\mathcal{T}}_{\alpha,b_i}(i^{\mathcal{T}}_{0,\alpha}(f)(s)) = i^{\mathcal{T}}_{\alpha,b_i}(i^{\mathcal{T}}_{0,\alpha}(g)(t)) = i^{\mathcal{T}}_{b_i}(g)(t) = \pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}(g)(t)$$

We have $H^{\mathcal{P}}_{A_i} \subset \mathrm{ran}\, k$ for each $i < \omega$, so because $\mathcal{P} = \bigcup \{H^{\mathcal{P}}_{A_i} : i < \omega\}$ we have that $\mathrm{ran}\, k = \mathcal{P}$ and k is the identity map. This shows that $\mathcal{M}^{\mathcal{T}}_b = \mathcal{P}$ and $i^{\mathcal{T}}_b = \pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}$.

The branch b is unique with the property that $\mathcal{M}_b^{\mathcal{T}} = \mathcal{P}$ and $i_b^{\mathcal{T}} = \pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}$ by the zipper argument because $\operatorname{ran}\pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}$ is cofinal in $\delta^{\mathcal{P}}$. $\square$

PROPOSITION 5.3.5. *If $\mathcal{M}$ is a suitable, quasi-iterable premouse and $\mathcal{S} \subset \mathrm{Env}(\Gamma)$ is a countable set, closed under recursive join, such that the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$ is total on $\mathcal{M}$ and has the full factors property, then there is a unique (ω_1, ω_1)-iteration strategy Λ for $\mathcal{M}$ such that for every quasi-iterate $\mathcal{P}$ of $\mathcal{M}$*

- *$\mathcal{P}$ is a non-dropping Λ-iterate of $\mathcal{M}$, and*
- *the Λ-iteration map $i : \mathcal{M} \to \mathcal{P}$ equals the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}$.*

Moreover, the strategy Λ has the property that whenever $i : \mathcal{M} \to \mathcal{P}$ is a Λ-iteration map factors as a composition of maps $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$, then $\mathcal{N}$ is Lp^{Γ}-full. In particular $\mathcal{P}$ itself is full.

PROOF. Take a suitable sequence $(\mathcal{M}_\alpha : \alpha \leq \gamma)$ with $\gamma < \omega_1$ such that $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_\gamma = \mathcal{P}$. Let $\vec{\mathcal{T}} = (\mathcal{T}_\alpha : \alpha < \gamma)$ be the trees used in this sequence. The proof is by induction on γ. Assume the claim holds with respect to the sequences $(\mathcal{M}_\alpha : \alpha \leq \beta)$ for $\beta < \gamma$.

If $\gamma = \beta + 1$ for some β, then the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M}_\beta,\infty}$ (and therefore also the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M}_\beta,\mathcal{M}_{\beta+1}}$, which factors into it) is total on $\mathcal{M}_\beta$ and has the full factors property by Lemma 5.3.3. If $\mathcal{T}_\beta$ is short, then the quasi-iterate $\mathcal{M}_{\beta+1}$ is defined as $\mathcal{M}_b^{\mathcal{T}_\beta}$ where b is the unique branch satisfying $Q(M(\mathcal{T}))^{\mathcal{M}_b^{\mathcal{T}}} = Q(M(\mathcal{T}))$. For every $A \in \mathcal{S}$ this branch is A-correct because $\mathcal{M}$ is A-quasi-iterable. Therefore we can and must define $\Lambda(\vec{\mathcal{T}}) = b$. If $\mathcal{T}_\beta$ is maximal, then we can and must define $\Lambda(\vec{\mathcal{T}})$ to be the unique branch given by Lemma 5.3.4.

If γ is a limit ordinal then the branch $\Lambda(\vec{\mathcal{T}})$ can and must be defined as the concatenation of the branches $\Lambda(\vec{\mathcal{T}}_\alpha)$, $\alpha < \gamma$. The Λ-iteration map $\mathcal{M}_0 \to \mathcal{M}_\gamma$ is the direct limit of the Λ-iteration maps $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ for $\alpha < \beta < \gamma$, and the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M}_0,\mathcal{M}_\gamma}$ is the direct limit of the $\mathcal{S}$-guided maps $\pi^{\mathcal{S}}_{\mathcal{M}_\alpha,\mathcal{M}_\beta}$ for $\alpha < \beta < \gamma$, so the claim follows from the induction hypothesis.

The "moreover" follows from the full factors property of the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$, because the Λ-iteration map $\mathcal{M} \to \mathcal{P}$ equals the $\mathcal{S}$-guided map $\pi^{\mathcal{S}}_{\mathcal{M},\mathcal{P}}$, which factors into $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$. So every map $\mathcal{M} \to \mathcal{N}$ that factors into the Λ-iteration map $\mathcal{M} \to \mathcal{P}$ also factors into $\pi^{\mathcal{S}}_{\mathcal{M},\infty}$. $\square$

We summarize the conclusion of Proposition 5.3.5 by saying that $\mathcal{S}$ *guides an iteration strategy for $\mathcal{M}$ with the full factors property.* Note that in particular such a strategy Λ has the property of being *fullness-preserving*: the Λ-iterates of $\mathcal{M}$ are themselves Lp^{Γ}-full. The full factors property for iteration strategies was called "weak condensation" in [**18**] and [**32**], but here we will use a more descriptive name.

5.4. Application to strong pseudo-homogeneous ideals

THEOREM 5.4.1 (ZFC). *Assume that there is a strong pseudo-homogeneous ideal $\mathcal{I}$ on $\wp_{\omega_1}(\mathbb{R})$. Let Γ be an inductive-like pointclass such that $\underset{\sim}{\boldsymbol{\Delta}}_\Gamma$ is determined, Γ-MC holds, and every set in Γ is definable from a countable sequence of ordinals. Then there is a Γ-suitable* *premouse $\mathcal{P}$ and an (ω_1, ω_1)-iteration strategy for $\mathcal{P}$ that is* Γ*-fullness preserving, has branch condensation, and is definable from a countable sequence of ordinals.*

PROOF. Let $j : V \to \mathrm{Ult}(V, H) \subset V[H]$ be the generic embedding associated to a V-generic filter $H \subset \mathcal{I}^+ \setminus \mathcal{I}$. Let T be the tree of a Γ-scale on a universal Γ set. The tree T is definable from a countable sequence of ordinals, so $j(T) \in V$ by pseudo-homogeneity.

The pointclass $\underset{\sim}{\mathbf{Env}}(\Gamma)$ has size $\leq \mathfrak{c}$ by Lemma 4.4.2 because we have a strong ideal on $\wp_{\omega_1}(\mathbb{R})$. Take an elementary embedding $\sigma : N \to H_{(2^{\mathfrak{c}})^+}$ where N is a transtive set of size $\mathfrak{c}$ with $\underset{\sim}{\mathbf{Env}}(\Gamma) \subset N$. Then N is in $\mathrm{Ult}(V, H)$ and is countable there. Moreover the map $j(\sigma) \circ (j \restriction N)$ is in $\mathrm{Ult}(V, H)$, and is an elementary embedding $N \to j(H_{(2^{\mathfrak{c}})^+})$.

Note that N is $j(\Gamma)$-full: for any set $A \in j(\mathrm{Env}(\Gamma))$ we have $A \cap \mathbb{R}^V \in C_{j(\Gamma)}(\mathbb{R}^V)$, so $A \in L(j(T), \mathbb{R}^V)$. Because $j(T) \in V$, we have $A \in V$, so $A \in N$. Therefore by Proposition 5.2.2, the direct limit premouse $\mathcal{M}_\infty$ is $j(\Gamma)$-suitable and strongly $j(A)$-quasi-iterable for every set $A \in \mathrm{Env}(\Gamma)$, and $j \restriction \mathcal{M}_\infty$ is the union of the partial quasi-iteration limit maps $\pi^{j(A)}_{\mathcal{M}_\infty,\infty}$ as defined in the generic ultrapower.

First we will get an iteration strategy with the full factors property. Let E_j be the extender on $\mathcal{M}_\infty$ of length $j(\mathrm{Ord}^{\mathcal{M}_\infty})$ derived from j. The ultrapower of $L[j(T), \mathcal{M}_\infty]$ by E_j factors into the map $j \restriction L[j(T), \mathcal{M}_\infty]$, so it is wellfounded. Therefore the map $j \restriction \mathcal{M}_\infty$ has the $j(\Gamma)$-full factors property by Lemma 5.3.2. Now from Proposition 5.3.5 we see that there is a unique iteration strategy for $\mathcal{M}$ in the generic ultrapower that is guided by the collection of sets $j``\mathrm{Env}(\Gamma)$, and it has the Γ-full factors property.

Pulling this statement back to V with the elementarity of j, we see that there is a suitable premouse $\mathcal{M}$ and a countable collection of sets $\mathcal{S} \subset \mathrm{Env}(\Gamma)$ that guides an iteration strategy Λ for $\mathcal{M}$ with the full factors property. This strategy is definable from $\mathcal{M}$ and $\mathcal{S}$ and therefore from a countable sequence of ordinals.

Now we use the fact that, for any countable sequence $\pi \in \mathrm{Ord}^\omega$ and any OD set $B \subset \mathrm{Ord}$ there is an inner model containing π and B in which ω_1^V is measurable. This is a trivial consequence of pseudo-homogeneity. As in [**18**] (see also [**32**]) this fact can be used to get a tail of Λ with branch condensation, letting B code $\mathcal{M}_\infty$ and letting π code the $\mathcal{S}$-guided map $\mathcal{M} \to \mathcal{M}_\infty$ (which is equal to the direct limit of all Λ-iteration maps.) $\square$

We conclude this chapter by remarking that the author does not know whether getting branch condensation from the full factors property actually requires any special hypothesis (such as the aforementioned inner model where ω_1^V is measurable.) A general method for getting fullness-preserving strategies with branch condensation remains a key problem in the area.

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