University of California
Los Angeles

Generating function zeros of Markov processes and their applications

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

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2011

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2011

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2011

Table of Contents

Acknowledgments

Without the support and dedication of my advisor, Thomas M. Liggett, this dissertation would never have been written. In particular, I am very grateful for his sustained encouragement, even before I was his student, to read and pursue many directions by myself. Although I often became lost in the mathematical wilderness, Tom's confidence in me, along with his many good questions, allowed me to find my own way.

I would also like to thank Marek Biskup for his support, several motivating conversations, and his careful reading of this dissertation.

Chapter 4 is a version of:

T. M. Liggett, A. Vandenberg-Rodes, *Stability theory on* $\{0,1,2,\dots\}^S$*: birth-death chains and particle systems*, to appear in the volume "Notions of Positivity and the Geometry of Polynomials", Trends in Mathematics, Birkhauser, 2011.

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(with Thomas M. Liggett) *Stability theory on* $\{0,1,2,\dots\}^S$*: birth-death chains and particle systems*, to appear in the volume "Notions of Positivity and the Geometry of Polynomials", Trends in Mathematics, Birkhauser, 2011. http://arxiv.org/abs/1009.4899

Abstract of the Dissertation

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University of California, Los Angeles, 2011

Professor Thomas M. Liggett, Chair

Through the correspondence between probability distributions on $\{0,1\}^n$ and multivariate generating polynomials, restricting the locations of the zeros of the latter to certain subsets of $\mathbb{C}^n$ turns out to have probabilistic relevance. For example, one kind of restriction on the generating function zeros gives rise to the class of *strong Rayleigh* measures, as constructed in [9]. The strong Rayleigh measures individually satisfy several notions of negative dependence, yet the class as a whole has the advantage of being closed under many operations such as projection onto subsets of coordinates, conditioning under external fields, and weak convergence.

This work is primarily concerned with a similar restriction of generating function zeros, but for measures on product spaces where each coordinate is a non-negative integer. The negative dependence results for the binary-valued case extend nicely to this more general situation, which gives a new negative association result for collections of independent Markov chains and reaction diffusion processes. Furthermore, measures in this class have a further useful property: the distribution of each coordinate decomposes into a sum of independent Bernoulli

and Poisson random variables. A version of this property was already used in [34] to obtain a central limit theorem for the current of particles in a particular symmetric exclusion process. We will generalize this result to symmetric exclusion processes with nearly arbitrary initial conditions and transition rates.

The last part of this dissertation is concerned with the single-coordinate case, that is, non-negative integer-valued measures satisfying the same restriction as above on their generating function zeros. In particular, we will classify the birth-death chains preserving this class.

CHAPTER 1

Introduction

Infinite stochastic systems of interacting particles have been studied intensively since F. Spitzer's founding paper [48]. Such systems act on a fixed (countable) collection of sites S, with a configuration of the system being just an arrangement of particle on the sites. The configurations then update in continuous time according to a specified interaction rule. They have been used as models for several natural processes; some typical examples are: the spread of infection through a population (the contact process), predator-prey dynamics (the voter model), and the movement of cars along a highway (the totally asymmetric exclusion process).

A common assumption is that the collection of sites S is the d-dimensional lattice $\mathbb{Z}^d$, and so the state space is just $\{0,1\}^{\mathbb{Z}^d}$. Within this framework, we will say that a state η has a particle at the site $x \in S$ if $\eta(x) = 1$, otherwise, the site is empty. Our primary example — considered in chapter 3 — will be the *symmetric exclusion process* (SEP). This has the following structure:

1. Particles move around the lattice, subject to the condition that only one particle at a time may occupy a site.
2. For each pair of sites $\{x, y\}$ let $p(x, y) = p(y, x)$ denote the jump rate between the two sites.
3. At rate $p(x, y)$ a particle at x attempts to jump to site y. If site y is occupied the jump fails, and the particle at x remains where it is.

The resulting Markov process will be denoted by $\{\eta_t : t \geq 0\}$ (we will give one construction in Chapter 3).

For conservative particle systems such as symmetric exclusion, a natural quantity to consider is the net flow, or current, of particles across some bond. The prototypical example is the one-dimensional lattice $\mathbb{Z}$, with the following step-initial condition: particles are placed at each site $x \leq 0$, and there no particles to the right of the origin. Then the current across the bond $\{0, 1\}$ — after some fixed time — is just the number of particles with final position in the right half $\{x > 0\}$. It had been expected (for at least a decade [39]) that the fluctuations of the current are Gaussian. Quite recently [34], this was finally shown in the case of translation-invariant jump distributions satisfying a second moment condition. In Chapter 3 we will generalize Liggett's result to (nearly) arbitrary initial conditions and transition rates. Here is one immediate application:

Proposition 1.0.1. *Suppose η_t is the symmetric exclusion process with translation-invariant transition rates in the domain of a symmetric stable law of index $\alpha > 1$. That is, $\sum_{y \geq x} p(0, y) = |x|^{-\alpha} L(x)$, with L a slowly-varying function. Assuming the step-initial condition above, the current $W_t = \sum_{x>0} \eta_t(x)$ satisfies the central limit theorem*

$$\frac{W_t - \mathbb{E}W_t}{\sqrt{\operatorname{Var} W_t}} \Rightarrow \mathcal{N}(0, 1). \tag{1.0.1}$$

This problem — with nearest neighbor instead of long-range jumps — originally served as a motivating application for the conjectured strong negative dependence properties of symmetric exclusion. It has long been known that symmetric exclusion has pairwise negative correlations, when started from a deterministic distribution [1]:

$$\operatorname{Cov}(\eta_t(x), \eta_t(y)) \leq 0, \text{ for all } t \text{ and distinct sites } x, y. \tag{1.0.2}$$

When again started from a deterministic distribution, it was conjectured that SEP satisfies the much stronger *negative association* (NA) property:

$$\mathrm{Cov}(f(\eta_t), g(\eta_t)) \leq 0, \tag{1.0.3}$$

for all increasing functions f, g depending on disjoint finite sets of coordinates. As Pemantle pointed out, there are good comparison inequalities between the characteristic functions of negatively associated random variables and their independent counterparts (due to C. Newman [37]), which should give central limit theorems for linear functionals of the NA variables such as the current. (However, one should note that the specific applications in [37] are to stationary sequences.)

The negative association conjecture for SEP was proved quite recently through two surprising connections to the study of linear transformations on polynomials that preserve certain zero-free regions.Recall that a probability measure μ on the state space $\{0,1\}^n$ is uniquely associated with its generating polynomial

$$f_\mu(z_1, \ldots, z_n) := \sum_{\eta \in \{0,1\}^n} \mu(\eta) z_1^{\eta(1)} \cdots z_n^{\eta(n)}. \tag{1.0.4}$$

If $\{\eta_t; t \geq 0\}$ is a stochastic process with the state $\eta_t \in \{0,1\}^n$ at each fixed time t (e.g. the symmetric exclusion process), we can consider the evolution of η_t as inducing a collection of linear transformations $\{T_t; t \geq 0\}$ on multiaffine polynomials as follows:

$$T_t \left(\sum_{\eta \in \{0,1\}^n} a_\eta z_1^{\eta(1)} \cdots z_n^{\eta(n)} \right) := \sum_{\eta \in \{0,1\}^n} a_\eta \mathbb{E}^\eta \left[z_1^{\eta_t(1)} \cdots z_n^{\eta_t(n)} \right] \tag{1.0.5}$$

For example, if μ is the initial distribution of the process, and μ_t the distribution at time t, then T_t relates their generating polynomials (1.0.4) in the obvious way:

$$T_t(f_\mu) = f_{\mu_t}. \tag{1.0.6}$$

Thus, if one is interested in certain classes of measures preserved by the original stochastic process, it is natural to consider instead classes of polynomials which are preserved by the induced linear transformations T_t.

Recall that we are interested in the negative association property. This is not easily expressed in terms of the generating polynomial. However, there is a stronger condition than negative association, termed *stability*, that has a simple formulation in terms of zeros of the generating polynomial. More specifically, a multivariate polynomial in n variables is called *stable* if it has no zeros in the region $\mathbb{H} \times \cdots \times \mathbb{H} \subset \mathbb{C}^n$ ($\mathbb{H}$ is the upper half plane), and by an abuse of terminology a probability measure with stable generating polynomial will also be called stable.

This remarkable result of Borcea, Brändén, and Liggett [9] — that stable probability measures on $\{0, 1\}^n$ are negatively associated — allowed them to construct a strong and robust negative dependence theory. Indeed, the stability condition implies a wide variety of negative dependence conditions beyond negative association, for example: ultra-log-concave rank sequence, conditional negative association under external fields, the negative lattice condition, and others.

In Chapter 4 we will generalize this negative dependence theory to measures on $\{0, 1, \ldots, N\}^n$, and in the limit $N \to \infty$, obtaining the following result:

Theorem 1.0.2. *Suppose μ is a measure on $\{0, 1, 2, \ldots\}^n$, with generating function a (coefficientwise) limit of stable polynomials. Then μ is negatively associated.*

The problem of determining linear transformations preserving the set of polynomials with particular zero-free regions goes back to the work of Pólya and Schur [42], in which they classify the multiplier transforms ($T(x^j) = \gamma_j x^j$) preserving the set of univariate polynomials with real zeros. However, the first explicit statement of this general problem was much more recent [12]:

Problem 1.0.3. *For $U \subset \mathbb{C}$, where U is a set of interest, we let $\pi_n(U)$ denote the set of polynomials of degree at most n with no zeros in U. Determine all linear transformations $T : \pi_n(U) \longrightarrow \pi_n(U)$.*

In [6–8], Borcea and Brändén completely solved this problem for $U = \mathbb{H}$, the upper half plane; furthermore, their general approach results in a classification of both the linear transformations preserving the set of multi-affine, and general *multivariate* stable polynomials (where U is a product of upper-half-planes). From this result follows the surprising fact that the linear transformations T_t induced by the evolution of the symmetric exclusion process naturally preserve the stability condition. (There are more elementary proofs of this latter fact, such as the one given in [34].) In Chapter 4 we will give examples of Markov processes on the more general state space $\{0, 1, 2, \dots\}^n$ ($n \leq \infty$), which preserve the generalized class of stable measures considered in Theorem 1.0.2.

Other non-trivial cases of Problem 1.1 remain unsolved, including the special case $U = \mathbb{C} \setminus \{x < 0\}$. A univariate polynomial f of degree at most n is in $\pi_n(U)$ if and only if its zeros are real and non-positive, i.e., after multiplication by a constant f is a generating polynomial for a stable probability measure on $\{0, 1, \dots, n\}$. This latter set has its own probabilistic relevance, as evidenced by the following result from Chapter 4.

Theorem 1.0.4. *A probability measure on the non-negative integers has the same distribution as a (possibly infinite) sum of independent Bernoulli and Poisson random variables, if and only if its generating function is a (coefficient-wise) limit of stable polynomials.*

See also Pitman [41] for more combinatorial and probabilistic properties of generating functions with only real and non-positive zeros, and the connection with Pólya frequency sequences.

Although we lack a general solution to Problem 1.1 in this case, we will produce non-trivial examples of such stability preservers by considering birth-death chains on the non-negative integers. Generating functions that are limits of stable polynomials (and the corresponding probability measures, as in Theorem 1.0.4) will be referred to as *t-stable*.

Theorem 1.0.5. *The birth-death chain* $\{X_t; t \geq 0\}$ *on* $\{0, 1, 2, \dots\}$ *preserves the class of t-stable measures if and only if the birth rates are constant and the death rates satisfy* $\delta_k = d_1 k + d_2 k^2$ *for some constants* d_1, d_2.

Theorem 1.0.6. *The birth-death chain* $\{X_t; t \geq 0\}$ *on* $\{0, 1, \dots, N\}$, $N \geq 3$, *preserves the class of stable measures on* $\{0, 1, \dots, N\}$ *if and only if there are constants* b_1, b_2, d_1, d_2 *such that the birth and death rates satisfy*

$$\beta_k = b_1(N-k) + b_2(N-k)^2, \quad \delta_k = d_1 k + d_2 k^2. \tag{1.0.7}$$

1.1 Notation

Our state spaces will usually be of the form $\{0, 1, \dots\}^n$, for some integer n. Elements of such product spaces will be denoted by η, with $\eta(j)$ the j'th coordinate of η. The letters μ and ν will refer to probability measures on $\{0, 1, \dots\}^n$, with P^μ and $\mathbb{E}^\mu$ being the probability and mathematical expectation under μ. Furthermore, for probability measures μ we will always let

$$f_\mu(\mathbf{z}) = \sum_{\eta \in \{0,1,\dots\}^n} \mu(\eta) z_1^{\eta(1)} \cdots z_n^{\eta(n)} \tag{1.1.1}$$

denote its generating function.

One particular note of caution: polynomials will be written from a more algebraic perspective, i.e., the rings of polynomials $\mathbb{C}[x]$ and $\mathbb{C}[y]$ will be considered as distinct, despite the obvious isomorphism. Thus, if T is a linear operator on polynomials in the x variable, and $h(x, y) = \sum c_{ij} x^i y^j$, then

$$Tf(x, y) = \sum_{i,j} c_{ij} T(x^i) y^j. \tag{1.1.2}$$

Single variables will be written in the normal fonts x, y, z, while boldface variables are always multivariate: $\mathbf{x}$ generally refers to the collection $x_1, x_2, \ldots, x_n$, for some fixed n. With $[n]$ denoting the set of integers $\{1, 2, \ldots, n\}$, and $S \subset [n]$, we will use the compact notation $\mathbf{x}^S$ to denote the monomial $\prod_{j \in S} x_j$.

CHAPTER 2

Stable polynomials

Our goal in this chapter is to give a mostly self-contained exposition of the theory of stable polynomials, as developed by J. Borcea and P. Brändén [6–8], along with their application to negative association [9] and Lee-Yang problems in statistical physics [5].

Section 2.1 briefly describes the classical theory of interlacing univariate polynomials, Section 2.2 gives the multivariate generalization, and Sections 2.3–2.5 are concerned with the linear transformations on polynomials preserving certain properties of the roots. This is all part of the domain of classical analysis, but it will be useful to keep in mind the examples of generating polynomials of probability measures, as defined above in (1.0.4).

We begin with the needed definition:

Definition 2.0.1. *A polynomial $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ is called **stable** if $f \neq 0$ on the set*

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \operatorname{Im}(x_j) > 0 \ \forall j\}. \tag{2.0.1}$$

Let $\mathfrak{G}[\mathbf{x}]$ be the set of all stable polynomials in the variables $\mathbf{x}$.

*If f has only real coefficients, it is also called **real** stable. The corresponding set of real stable polynomials is denoted $\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$.*

Note that a univariate real stable polynomial can only have real zeros.

One key fact from complex analysis is the (multivariate) Hurwitz's theorem on zeros of analytic functions: (e.g. footnote 3 in [11])

Theorem 2.0.2. *Let Ω be a connected open subset of $\mathbb{C}^n$. Suppose the analytic functions $\{f_k\}$ converge uniformly on compact subsets of Ω (normal convergence in the vocabulary of complex analysis). If each f_k has no zeros in Ω then their limit f is either identically zero, or has no zeros in Ω. In particular, a normal limit of stable polynomials with bounded degree is either stable or* 0.

Definition 2.0.3. *An operation on stable polynomials (adding, multiplying, taking limits, derivatives, etc...) is said to **preserve stability** if the result is either a stable polynomial or zero.*

Some examples are: taking limits (Hurwitz's theorem), permuting variables, and dilations by positive constants ($f(\mathbf{x}) \mapsto f(a_1x_1, \dots, a_nx_n)$).

Here are a few others.

Proposition 2.0.4. *The following operations also preserve stability:*

1. *Restriction: for $a \in \overline{\mathbb{H}}$, $f(\mathbf{x}) \mapsto f(a, x_2, \dots, x_n)$.*
2. *Inversion: if f has degree d in x_1, $f(\mathbf{x}) \mapsto x_1^d f(-x_1^{-1}, x_2, \dots, x_n)$.*
3. *Differentiation: for any $1 \le j \le n$, $f(\mathbf{x}) \mapsto \partial_j f(\mathbf{x})$.*

Proof. Restriction is clear for $a \in \mathbb{H}$. For $a \in \mathbb{R}$, take

$$f_k(\mathbf{x}) = f(a + \frac{i}{k}, x_2, \dots, x_n), \tag{2.0.2}$$

which is stable for each $k > 0$. Taking $k \to \infty$, we conclude the proof by Hurwitz's theorem.

For (2), note that $x \in \mathbb{H} \Rightarrow -x^{-1} \in \mathbb{H}$.

For (3), we fix $x_2, \ldots, x_n$ in $\mathbb{H}$, and let $g(x) = f(x, x_2, \ldots, x_n)$. Because f is stable, the zeros $z_1, \ldots, z_k$ of g satisfy $\mathrm{Im}(z_j) \leq 0$. Then

$$\frac{\partial_1 f(\mathbf{x})}{f(\mathbf{x})} = \frac{g'(x_1)}{g(x_1)} = \frac{d}{dx_1} \log \left\{ c \prod_{j=1}^{k} (x_1 - z_j) \right\} = \sum_{j=1}^{k} \frac{1}{x_1 - z_j}, \tag{2.0.3}$$

which cannot be zero for $x_1 \in \mathbb{H}$, because each term in the latter sum has strictly negative imaginary part. Since $x_2, \ldots, x_n$ are arbitrary values in $\mathbb{H}$, $\partial_1 f$ is stable. □

The work of Borcea and Brändén results in a successful classification of the linear transformations preserving stability. The goal of sections 2.1–2.5 below is to quickly obtain the *sufficient* conditions (showing necessity is often much harder), which will be all we need for our applications in sections 2.6. Indeed, these first few sections can be read almost independently from the rest of this dissertation, if classical analysis is not to the reader's taste.

Credit is due to the excellent survey of D. G. Wagner [54], which was the inspiration for this chapter. Although the structure of this chapter is taken from there, a key goal here is to make the circle of ideas involving interlacing roots, the Wronskian, and $\mathrm{Im}(f/g)$ more concrete. For more on polynomials with interlacing roots, see the book by Fisk on that subject [19]. Although almost all of this chapter is written as an introduction, Proposition 2.6.5 in its full generality is due to the author [53].

2.1 Univariate stable polynomials

Let us begin with $f, g \in \mathfrak{G}_{\mathbb{R}}[x]$ (that is, real stable univariate polynomials). Let $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ be the (ordered) zeros for f and g respectively, counted with their multiplicities. We say that the zeros of f and g are *interlaced* if either

$$\beta_1 \le \alpha_1 \le \beta_2 \le \alpha_2 \le \cdots , \tag{2.1.1}$$

or

$$\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \cdots . \tag{2.1.2}$$

Notice this ensures that $|\deg g - \deg f| \le 1$. In particular, if fg has only simple zeros, then f and g are interlaced if and only if there is a zero of f located between any two zeros of g.

We assume for now that $\deg f \le \deg g = n$, with $n \ge 1$. Writing

$$\hat{g}_j(x) = \frac{g(x)}{x - \beta_j}, \tag{2.1.3}$$

by linear independence of the set $\{\hat{g}_1 \ldots, \hat{g}_n, g\}$ in the vector space of polynomials of degree $n = \deg g$, we see that there are unique $b_1, \ldots, b_n, a \in \mathbb{R}$ such that

$$f = ag + \sum_{j=1}^{n} b_j \hat{g}_j. \tag{2.1.4}$$

The Wronskian of f and g is defined as $W[f,g] = f'g - g'f$.

Proposition 2.1.1. *Suppose $f, g \in \mathfrak{G}_{\mathbb{R}}[x]$, $\deg f \le \deg g = n$, $n \ge 1$, and fg has only simple zeros. Then the following are equivalent:*

(a) The zeros of f and g are interlaced.

(b) The values $f(\beta_j)$ alternate sign.

(c) With $f = ag + \sum_j b_j \hat{g}_j$, all the b_j are non-zero and have the same sign.

(d) Either $\operatorname{Im}(f/g) < 0$ on $\mathbb{H}$, or $\operatorname{Im}(f/g) > 0$ on $\mathbb{H}$.

(e) Either $W[f,g](x) < 0$ for all $x \in \mathbb{R}$, or $W[f,g] > 0$ for all $x \in \mathbb{R}$.

Proof. (a) $\Rightarrow$ (b). Since fg has only simple zeros, we have that $\beta_1 < \alpha_1 < \beta_2 < \cdots$ (or with β and α switched). So if $f(\beta_j)$ and $f(\beta_{j+1})$ both have the same sign then the zero at α_j must be a local min or max, hence $f'(\alpha_j) = 0$, contradicting the assumption that f has only simple zeros.

(b) $\Rightarrow$ (c). In equation (2.1.4), $\hat{g}_j(\beta_j)$ must alternate sign, so all the b_j are of the same sign.

(c) $\Rightarrow$ (a). Plugging in the zeros of g into (2.1.4) shows that the sequence $\{f(\beta_j), 1 \leq j \leq n\}$ alternates sign, so by continuity of f the zeros interlace.

(c) $\Rightarrow$ (d). Notice that

$$\operatorname{Im}(f/g) = \sum_j b_j \operatorname{Im}\left(\frac{1}{x - \beta_j}\right). \tag{2.1.5}$$

(d) $\Rightarrow$ (c). From (2.1.4), taking $x = \beta_j + i\epsilon$ for small ϵ shows that each b_j is either of the same sign or 0. But if $b_j = 0$ then the rational function f/g does not have a pole at β_j, which contradicts fg having simple zeros.

(c) $\Leftrightarrow$ (e). Immediate from the calculation

$$\frac{W[f,g]}{g^2} = (f/g)' = \left(a + \sum_j \frac{b_j}{x - \beta_j}\right)' = \sum_j b_j \frac{-1}{(x-\beta_j)^2}. \tag{2.1.6}$$

□

The above relationship between interlacing roots, the Wronskian, and $\operatorname{Im}(f/g)$ still holds when fg does not have simple roots:

Proposition 2.1.2. *Let $f, g \in \mathfrak{G}_{\mathbb{R}}[x]$. The following are equivalent:*

(1) The zeros of f and g are interlaced.

(2) Either $\operatorname{Im}(f/g)(z) \leq 0$ for all $z \in \mathbb{H}$, or $\operatorname{Im}(f/g)(z) \geq 0$ for all $z \in \mathbb{H}$,

(3) Either $W[f,g](x) \leq 0$ for all $x \in \mathbb{R}$, or $W[f,g](x) \geq 0$ for all $x \in \mathbb{R}$.

Proof. We show the equivalence of (1) and (2) only, since the equivalence of (1) and (3) is obtained in exactly the same way.

(1) $\Rightarrow$ (2). First interchange f and g so that $\deg(f) \leq \deg(g)$. Now suppose that the zeros interlace. By perturbing the roots of f and g slightly so that fg has simple roots, but with the same interlacing of the roots, we see by the above proposition that either $\operatorname{Im}(f/g)(z) < 0$ for all $z \in \mathbb{H}$, or $\operatorname{Im}(f/g)(z) > 0$ for all $z \in \mathbb{H}$. In the limit as the perturbations go to zero, we obtain (2).

(2) $\Rightarrow$ (1). If f/g has only simple zeros and poles then we are done, because we can divide out the common zeros from f and g to be left with f_1 and g_1 such that $f_1/g_1 = f/g$ and $f_1 g_1$ has only simple zeros. The above proposition then implies that f and g have interlacing roots.

Now suppose instead that f/g has a zero or a pole of degree $k \geq 2$ at α. Then for very small $\epsilon > 0$

$$\arg(f/g)(\alpha + \epsilon e^{i\theta}) = \pm ki\theta \cdot (C_\alpha + O(\epsilon)), \tag{2.1.7}$$

so by varying $\theta \in (0, \pi)$ we can flip the sign of $\operatorname{Im}(f/g)$, contradicting our assumption of (2). □

Here are some ways to characterize when the zeros of f and g interlace in a particular direction. The equivalence of (a) and (c) is known as the Hermite-Biehler Theorem.

Theorem 2.1.3. *Suppose $f, g \in \mathbb{R}[x] \setminus \{0\}$. Then the following are equivalent:*

(a) The complex polynomial $g + if$ is stable.

(b) $f, g \in \mathfrak{G}_{\mathbb{R}}[x]$ and $\operatorname{Im}(f/g) \leq 0$ on $\mathbb{H}$.

(c) $f, g \in \mathfrak{G}_{\mathbb{R}}[x]$, and $W[f, g](x) \leq 0$ for all $x \in \mathbb{R}$.

(d) The real bi-variate polynomial $g(x) + yf(x) \in \mathfrak{G}_{\mathbb{R}}[x, y]$.

Proof. (b) $\Rightarrow$ (a). If $\mathrm{Im}(f/g) \leq 0$ on $\mathbb{H}$ then $\Re(1 + if/g) \geq 1$ on $\mathbb{H}$, so $g + if$ is complex stable.

(a) $\Rightarrow$ (d). Let $\{\theta_j; 1 \leq j \leq \deg(g + if)\}$ be the zeros of $h = g + if$. As $h(z) = c\prod(z - \theta_j)$ is complex stable, $\mathrm{Im}(\theta_j) \leq 0$ for all j, so for $z \in \mathbb{H}$ we have $|z - \theta_j| \geq |\bar{z} - \theta_j|$, and hence

$$|h(\bar{z})| = |g(\bar{z}) + if(\bar{z})| = |g(z) - if(z)| \leq |g(z) + if(z)|. \tag{2.1.8}$$

Dividing by $g(z)$ we obtain $|1 - if/g| \leq |1 + if/g|$, hence $\mathrm{Im}(f/g) \leq 0$ on $\mathbb{H}$, except where $g = 0$ (we do not know that g and f are stable, so we cannot yet conclude (b)). Notice that f and g have no common zeros in $\mathbb{H}$, so if $x \in \mathbb{H}$ and $f(x) = 0$, then $h(x, y) = g(x) \neq 0$. If $x \in \mathbb{H}$ and $f(x) \neq 0$, then

$$\mathrm{Im}[(g/f)(x) + y] \geq \mathrm{Im}\, y > 0 \text{ for } y \in \mathbb{H}, \tag{2.1.9}$$

which shows (d).

(d) $\Rightarrow$ (b). Assuming $g(x)+yf(x)$ is stable, by restriction to $y = 0$ we see that g is stable. Furthermore, for each $y \in \mathbb{R}$ we have $y^{-1}g + f \in \mathfrak{G}_{\mathbb{R}}[x]$, so by taking $y \to \infty$ and Hurwitz' theorem we see that f is stable also. Since $(g/f)(x) + y$ has no zeros for $x, y \in \mathbb{H}$, we must have $\mathrm{Im}(g/f) \geq 0$ on $\mathbb{H}$, which implies (b).

(b) $\Leftrightarrow$ (c). Assuming either statement, by Proposition 2.1.2 the zeros of f and g are interlaced. Perturbing the roots (while keeping the same interlacing order) so that fg has only simple zeros, we see from Proposition 2.1.1 that the signs of $\mathrm{Im}(f/g)(z)$ and $W[f, g](x)$ are the same for all $z \in \mathbb{H}$ and $x \in \mathbb{R}$. This equivalence of (b) and (c) continues to hold as we take the perturbations to zero. □

Definition 2.1.4. *We will say that f and g are* ***in proper position****, denoted $f \ll g$, if one of the equivalent conditions in Theorem 2.1.3 hold.*

Borcea and Brändén initially consider (c) as their definition of proper position, but later — for multivariate purposes — consider (a) to be the right definition.

Indeed, (a) does not need the assumption that f, g are both real stable. However, (b) will be an easier condition to use, especially as we will often suppose f and g to be stable.

The Obreschkoff theorem states that all linear combinations of two stable polynomials are stable if and only if their zeros are interlaced:

Theorem 2.1.5. *Suppose $f, g \in \mathfrak{G}_{\mathbb{R}}[x]$. Then $af + bg \in \mathfrak{G}_{\mathbb{R}}[x] \cup \{0\}$ for all $a, b \in \mathbb{R}$ if and only if $f \ll g$ or $g \ll f$.*

Proof. We can assume that $\deg f \leq \deg g$.

If f/g is constant the result is clear, so we assume otherwise. Perturbing the roots of f and g as in the proof of Proposition 2.1.2, we can also assume that fg has simple roots.

If $f \ll g$, then for any $a \neq 0$, $\operatorname{Im}[a(f/g) + b] \neq 0$ everywhere on $\mathbb{H}$ by Proposition 2.1.1. Hence $af + bg$ has no zeros in $\mathbb{H}$. The same holds if $g \ll f$.

Conversely, suppose $af + bg$ has no roots in $\mathbb{H}$ for all $a, b \in \mathbb{R}$ not both zero. Then if $\operatorname{Im}(f/g)(x) = 0$ for some $x \in \mathbb{H}$, we can choose $a \neq 0, b \in \mathbb{R}$ such that $a(f/g) + b$ has a zero at x, a contradiction. Hence $\operatorname{Im}(f/g) \neq 0$ everywhere on $\mathbb{H}$, so by continuity $f \ll g$ or $g \ll f$. □

2.2 Multivariate stability

There is one trivial observation that allows us to describe multivariate stability in terms of univariate stability:

Proposition 2.2.1. *$f \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$ iff $f(\mathbf{a} + t\mathbf{b}) \in \mathfrak{G}_{\mathbb{R}}[t]$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \mathbf{b} > \mathbf{0}$.*

Proof. Notice that any $\mathbf{z} \in \mathbb{H}^n$ can be written as $\mathbf{z} = \mathbf{a} + t\mathbf{b}$, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{b} > \mathbf{0}$, and $t \in \mathbb{H}$. Conversely, any $\mathbf{a} + t\mathbf{b}$ of that form lies in $\mathbb{H}^n$. □

We want to extend the concept of proper position to multivariate polynomials. Fortunately, all of our equivalent conditions in Proposition 2.1.4 extend to the multivariate case, with just one change.

Define the *j-th Wronskian* to be

$$W_j[f,g] = \partial_j f \cdot g - f \cdot \partial_j g. \tag{2.2.1}$$

Proposition 2.2.2. *Suppose $f, g \in \mathbb{R}[\mathbf{x}]\backslash\{0\}$. Then the following are equivalent:*

(a) $g + if \in \mathfrak{G}[\mathbf{x}]$.

(b) $f, g \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$ and $\operatorname{Im}(f/g) \leq 0$ on $\mathbb{H}^n$.

(c) $f, g \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$ and $W_j[f,g](\mathbf{x}) \leq 0$ for all $j \in [n]$ and $\mathbf{x} \in \mathbb{R}^n$.

(d) $g(\mathbf{x}) + yf(\mathbf{x}) \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}, y]$.

Proof. (a) $\Leftrightarrow$ (b). Clear from Propositions 2.2.1 and 2.1.3.

(b) $\Rightarrow$ (d). Note that $\operatorname{Im}[(g/f)(\mathbf{x}) + y] \geq \operatorname{Im} y > 0$ for $x \in \mathbb{H}^n, y \in \mathbb{H}$, which implies (d).

(d) $\Rightarrow$ (b). Set $y = i$.

(b) $\Rightarrow$ (c). Let $\mathbf{e}_j$ be the vector $(0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with the 1 in the j'th coordinate. If (b) holds then

$$\operatorname{Im}\left(\frac{f(\mathbf{c} + \mathbf{e}_j t)}{g(\mathbf{c} + \mathbf{e}_j t)}\right) \leq 0 \tag{2.2.2}$$

for all $\mathbf{c} \in \mathbb{R}^n$ and $t \in \mathbb{H}$, hence by Proposition 2.1.2(b) we see that

$$W_j[f,g](\mathbf{c} + \mathbf{e}_j t) = W[f(\mathbf{c} + \mathbf{e}_j t), g(\mathbf{c} + \mathbf{e}_j t)] \leq 0, \text{ for all } t \in \mathbb{R}. \tag{2.2.3}$$

As $\mathbf{c} \in \mathbb{R}^n$ was arbitrary we obtain (c) by setting $t = 0$.

(c) $\Rightarrow$ (b). Notice by the chain rule that

$$W[f(\mathbf{c}+\mathbf{d}t), g(\mathbf{c}+\mathbf{d}t)] = \sum_{j=1}^{n} d_j W_j[f,g](\mathbf{c}+\mathbf{d}t) \leq 0 \text{ for all } \mathbf{c}, \mathbf{d} \in \mathbb{R}^n, \mathbf{d} > 0, t \in \mathbb{R}. \tag{2.2.4}$$

Then by the equivalence (b) $\Leftrightarrow$ (c) in Theorem 2.1.3 we see that $\mathrm{Im}(f(\mathbf{c}+\mathbf{d}t)/g(\mathbf{c}+\mathbf{d}t)) \leq 0$ for all $t \in \mathbb{H}$, and so $f \ll g$ by Proposition 2.2.1. $\square$

So we say again that $f \ll g$ if any of the above equivalent conditions are satisfied.

Extending the Obreschkoff theorem is almost as trivial now:

Theorem 2.2.3. *Suppose $f, g \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$. Then $af + bg \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}] \cup \{0\}$ for all $a, b \in \mathbb{R}$ if and only if $f \ll g$ or $g \ll f$.*

Proof. The claim is obvious if one of f, g is zero, so assume otherwise. Suppose $af + bg \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$. If $\mathrm{Im}(f/g(\mathbf{z})) = 0$ for some $\mathbf{z} \in \mathbb{H}^n$ then with $b = (f/g)(\mathbf{z})$ we have that $f - bg$ has a zero in $\mathbb{H}^n$, which by assumption can only happen if $f = bg$. Hence $\mathrm{Im}(f/g) \neq 0$ on $\mathbb{H}^n$, so by continuity either $f \ll g$ or $g \ll f$.

In the other direction, suppose that $f \ll g$. Then $f(\mathbf{c} + t\mathbf{d}) \ll g(\mathbf{c} + t\mathbf{d})$ for all $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ with $\mathbf{d} > 0$, and so by the univariate Obreshkoff theorem $af(\mathbf{c} + t\mathbf{d}) + bf(\mathbf{c} + t\mathbf{d}) \in \mathfrak{G}_{\mathbb{R}}[t]$ for all $a, b \in \mathbb{R}$. The above proposition then shows us that $af + bg \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$. $\square$

Here is a key lemma, essentially due to Lieb and Sokal [29].

Lemma 2.2.4. *If $g + yf$ is stable then $g - \partial_j f$ is either stable or zero.*

Proof. We can assume that $j = 1$. Notice that

$$g - \partial_1 f = f\left[\frac{g}{f} - \frac{\partial_1 f}{f}\right]. \tag{2.2.5}$$

If $\partial_1 f = 0$, we are clearly done. If not, for each $(x_2, \ldots, x_n)$ consider $\tilde{f}(x) := f(x, x_2, \ldots, x_n)$. Let the roots of $\tilde{f}$ be $\alpha_1, \ldots, \alpha_d$. Then

$$\mathrm{Im}\left(\frac{-\partial_1 \tilde{f}(x)}{\tilde{f}(x)}\right) = \mathrm{Im}\left(\sum_{k=1}^{d} \frac{-1}{x - \alpha_k}\right) > 0 \text{ for all } x \in \mathbb{H}. \tag{2.2.6}$$

(Recall that the roots must have non-positive imaginary part.) As this holds for all choices of $(x_2, \ldots, x_n)$, we see that $\mathrm{Im}(-\partial_1 f/f) > 0$ for all $\mathbf{x} \in \mathbb{H}^n$. Now by Theorem 2.2.2, $\mathrm{Im}(g/f) \geq 0$ on $\mathbb{H}^n$, and hence

$$\mathrm{Im}\left[\frac{g}{f} - \frac{\partial_1 f}{f}\right] > 0. \tag{2.2.7}$$

Multiplying by f we have that $g - \partial_1 f$ is stable. □

2.3 Multiaffine stable polynomials

We say that a multivariate polynomial is **multiaffine** if it has degree at most 1 in each variable. In particular, this includes the generating functions f_μ of measures on $\{0,1\}^n$ via equation (1.0.4).

Now, any multiaffine $f(\mathbf{x})$ can be written as $f = f^j + x_j f_j$, where we are using the very compact notation

$$f^j(\mathbf{x}) = f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n), \tag{2.3.1}$$

and

$$f_j(\mathbf{x}) = \frac{\partial}{\partial x_j} f(\mathbf{x}). \tag{2.3.2}$$

Now Proposition 2.2.2 shows us that stability of $f^j + x_j f_j$ is related to the quantity $W_k[f_j, f^j]$ for $k \neq j$. Expanding out

$$f = f^{jk} + x_k f^j_k + x_j f^k_j + x_j x_k f_{jk}, \tag{2.3.3}$$

we observe that

$$W_k[f_j, f^j] = f_{jk} f^j - f_j f^j_k = f_{jk} f^{jk} - f^k_j f^j_k. \tag{2.3.4}$$

Let

$$\Delta_{jk} f = \frac{\partial}{\partial x_j} f \cdot \frac{\partial}{\partial x_k} f - f \cdot \frac{\partial^2}{\partial x_j \partial x_k} f. \tag{2.3.5}$$

Expanding this out we see that $\Delta_{jk} = -W_k[f_j, f^j]$. The condition

$$\Delta_{jk} f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n, \text{ and all } 1 \leq j \neq k \leq n \tag{2.3.6}$$

is known as the *strong Rayleigh* condition. Notice what this says for probability measures μ with generating polynomial f_μ: since $(f_\mu)_j(\mathbf{1}) = \partial_j f_\mu(1, \ldots, 1) = \mu(\eta(j) = 1)$, the strong Rayleigh condition implies that

$$0 \leq \Delta_{jk} f_\mu(\mathbf{1}) = \mu(\eta(j) = 1)\mu(\eta(k) = 1) - \mu(\eta(j) = 1, \eta(k) = 1), \tag{2.3.7}$$

which is precisely the statement that μ is pairwise negatively correlated.

The strong Rayleigh condition also gives us an effective criterion for stability:

Theorem 2.3.1. *Suppose $f \in \mathbb{R}[\mathbf{x}]$ is multiaffine. Then $f \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$ if and only if $\Delta_{jk} f(\mathbf{x}) \geq 0$ for all $j \neq k$, $\mathbf{x} \in \mathbb{R}^n$.*

Proof. If $f = f^1 + x_1 f_1$ is stable, then by restriction to $x_1 = 0$ and $x_1 \to \infty$ we see that f^1 and f_1 are stable. Proposition 2.2.2 implies that $-W_k[f_j, f^j](\mathbf{x}) = \Delta_{jk} f(\mathbf{x}) \geq 0$ for all $j \neq k$ and $\mathbf{x} \in \mathbb{R}^n$.

For the converse, suppose $\Delta_{jk} f(\mathbf{x}) \geq 0$ for all $j \neq k$ and $\mathbf{x} \in \mathbb{R}^n$. We will show f is stable by induction on n. If $n = 1$ then f is automatically stable. For $n > 1$, we fix $a \in \mathbb{R}$ and set $g(\mathbf{x}) = f(a, \mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^{n-1}$. Now $\Delta_{jk} g(\mathbf{x}) = \Delta_{jk} f(a, \mathbf{x}) \geq 0$ for $j, k > 1$, $j \neq k$. Hence by induction $g = f^1 + a f_1$ is stable. Setting $a = 0$ and taking $a \to \infty$ we see that both f^1 and f_1 are stable. As $W_k[f_j, f^j] = -\Delta_{jk} f \leq 0$ on $\mathbb{R}^n$, Proposition 2.2.2 implies that $f = f^1 + x_1 f_1$ is stable. □

Our goal now is to find general conditions for a linear operator T to preserve stability. We have just been considering multiaffine polynomials. When is stability preserved in this case?

Theorem 2.3.2. *Suppose T is a linear operator on multi-affine polynomials in the n variables $x_1, \ldots, x_n$. Consider the polynomial $h(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n}(x_i + y_i)$ in $2n$ variables. We can apply T to h by first fixing $y_1, \ldots, y_n$ (see also equation (1.1.2)). Suppose the result*

$$T(h)(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \tag{2.3.8}$$

is stable. Then T preserves n-variable stability.

Proof. By inversion (see (b) of Proposition 2.0.4),

$$\begin{aligned} T\left(\prod_{i=1}^{n}(y_i + x_i)\right) &= (y_1 \cdots y_n) T\left(\prod (1 + x_i y_i^{-1})\right) \\ &\mapsto T\left(\prod (1 - x_i y_i)\right) = \sum_{S \subset [n]} T(\mathbf{x}^S)(-\mathbf{y})^S \end{aligned}$$

is also stable. Then for any stable f,

$$\sum_{S \subset [n]} T(\mathbf{x}^S)(-\mathbf{y})^S f(\mathbf{z}) \tag{2.3.9}$$

is $\mathbb{H}^{3n}$-stable. Now use the Lieb-Sokal lemma (Lemma 2.2.4) to replace term-by-term $-y_j$ with ∂z_j. Restricting to $\mathbf{z} = 0$ and recalling the compact notation seen in (2.3.3) for multiaffine f, this expression becomes exactly

$$\sum_{S \subset [n]} T(\mathbf{x}^S) f_S^{[n] \setminus S} = Tf(\mathbf{x}). \tag{2.3.10}$$

□

Remark 2.3.3. *As long as the image of T has dimension at least 2, the converse also holds.*

Here is an application of this theorem that will be very useful in both the next section and the following chapter. Define

$$T_{jk}^p f = (1 - p)f + p\tau_{jk} f, \tag{2.3.11}$$

where τ_{jk} exchanges the x_j and x_k coordinates.

In particular, if one exchanges the contents of the sites j, k with probability p, and f_μ is the initial generating polynomial for μ, then Tf_μ is the generating polynomial for μ after the random exchange. This transformation is thus the building block for the symmetric exclusion process, as we will see in Section 3.5.

Proposition 2.3.4. *The linear transformation T^p_{jk} preserves multiaffine stability.*

Proof. We can assume that $j = 1$, $k = 2$, and so then by Theorem 2.3.2 we need only show that

$$T^p_{12}((x_1 + y_1)(x_2 + y_2)) = x_1x_2 + [px_2 + (1 - p)x_1]y_1 + [px_1 + (1 - p)x_2]y_2 + y_1y_2 \tag{2.3.12}$$

is $\mathbb{H}^4$-stable. We do this by using the strong Rayleigh condition (Theorem 2.3.1). Renaming $y_1 = x_3$, $y_2 = x_4$, one can compute

$$\Delta_{jk}\left\{x_1x_2 + [px_2 + (1 - p)x_1]x_3 + [px_1 + (1 - p)x_2]x_4 + x_3x_4\right\} \tag{2.3.13}$$

for $j \neq k$ to see that it is always non-negative. For example (after canceling and combining terms), $\Delta_{12} = p(1 - p)(x_3 - x_4)^2$. □

2.4 General stable polynomials

We now consider stability preservers of non-multiaffine polynomials. In order to use Lemma 2.2.4 like we did above, we need to represent general polynomials in a multiaffine way. Let $f(x)$ be a polynomial of degree d. Now define the *j'th elementary symmetric function* of $\mathbf{x}$ to be

$$e_j(\mathbf{x}) = \sum_{S\subset[d]:|S|=j} \mathbf{x}^S = \sum_{1\leq i_1<i_2<\cdots<i_j\leq d} x_{i_1}x_{i_2}\cdots x_{i_j}. \tag{2.4.1}$$

For $f(x) = \sum_{j=0}^{d} a_j x^j$, define the *d'th polarization* of f to be

$$\mathrm{Pol}_d\, f(\mathbf{x}) = \sum_{j=0}^{d} a_j \binom{d}{j}^{-1} e_j(\mathbf{x}). \tag{2.4.2}$$

The d-th polarization of a multivariate polynomial is then defined to be the composition of polarizations in each variable.

Notice that going from $\mathrm{Pol}_d\, f$ to f is straightforward: $\mathrm{Pol}_d\, f(x, x, \ldots, x) = f(x)$. In particular, if $\mathrm{Pol}_d\, f$ is $\mathbb{H}^n$-stable then f is $\mathbb{H}$-stable. What about the converse? This is one form of the Grace-Walsh-Szegö theorem (for some other forms, see chapter 5 of [46]).

Theorem 2.4.1. *Suppose that $f(x)$ is a univariate, stable polynomial of degree d. Then $\mathrm{Pol}_d\, f(\mathbf{x})$ is a $\mathbb{H}^d$-stable polynomial.*

Proof. (Sketch). Recall from Proposition 2.3.4 that the linear operator

$$T_{j,k} f = \frac{1}{2}[f + \tau_{j,k} f] \tag{2.4.3}$$

preserves stability, where $\tau_{j,k}$ interchanges x_j, x_k. Now if $f(x) = b \prod_{j=1}^{d} (x - \alpha_j)$, where $\mathrm{Im}(\alpha_j) \leq 0$ for each j, so that f is stable, then

$$g(\mathbf{x}) = b \prod_{j=1}^{d} (x_j - \alpha_j) \tag{2.4.4}$$

is $\mathbb{H}^d$-stable. We will show that by applying to g the operators $T_{j,k}$ infinitely many times, we end up with $\mathrm{Pol}_d\, f$. Indeed, we claim that by applying $\prod_{j<k} T_{j,k}$ repeatedly, any $x_{j_1} \cdots x_{j_r}$ will converge to $\binom{d}{r}^{-1} e_r(\mathbf{x})$. If this is true, then by expanding out $g(\mathbf{x})$ and using the linearity of $T_{j,k}$, we see that

$$\left(\prod_{j<k} T_{j,k} \right)^{m} g(\mathbf{x}) \longrightarrow \mathrm{Pol}_d\, f(\mathbf{x}), \text{ as } m \to \infty, \tag{2.4.5}$$

so by preservation of stability under the $T_{j,k}$ and Hurwitz' theorem we can conclude that $\mathrm{Pol}_d\, f(\mathbf{x})$ is stable. (The convergence here and below is in the sense of the coefficients, which implies uniform convergence on compact sets.)

To show the claim, consider the following discrete-time Markov chain on the state space $\Omega = \{\eta \in \{0,1\}^d; \sum \eta(j) = r\}$ - the space of configurations of r particles on $\{1,\ldots,d\}$ subject to at most one particle per site. Let $\mathcal{T}$ be the ordered set of all transpositions $(j\ k)$ of $[d]$, $j < k$. At each time step we consider all the transpositions $(j\ k) \in \mathcal{T}$ in order, flipping a fair coin to determine whether or not to exchange the contents of sites j and k. This chain is irreducible, as the transpositions generate S_d, and is aperiodic, because any permutation is a composition of at most d transpositions. By the fundamental convergence theorem for finite Markov chains, from any initial configuration the distribution of the chain converges to the unique invariant measure, which is the uniform measure on Ω.

Now we note that $f_0(\mathbf{x}) = x_{j_1} \cdots x_{j_r}$ is the generating function for the initial condition with particles at sites $j_1, \ldots, j_r$ and that at the n'th time step the generating function is

$$f_n(\mathbf{x}) := \left(\prod_{j<k} T_{j,k}\right) f_{n-1}(\mathbf{x}). \tag{2.4.6}$$

Crucially, the generating function for the uniform measure on Ω is $\binom{d}{r}^{-1} e_r(\mathbf{x})$, proving the claim. $\square$

From this we can obtain a sufficient condition for a linear differential operator to preserve stability. First consider the following generalization of the Lieb-Sokal lemma. For the following statement to be well defined, we write each monomial of $f(\mathbf{x})$ with the x_i in increasing order. For example, if $f(x_1, x_2, x_3) = x_1x_2x_3$, then

$$f\Big(\frac{\partial}{\partial x_3}, x_2, x_3\Big) = x_2, \text{ while } f\Big(x_1, x_2, \frac{\partial}{\partial x_1}\Big) = 0. \tag{2.4.7}$$

Lemma 2.4.2. *Suppose that $f(x_1, \ldots, x_n) \in \mathbb{C}[\mathbf{x}]$ is stable. Then for any*

$$f\Big(x_1, \ldots, x_{j-1}, -\frac{\partial}{\partial x_k}, x_{j+1}, \ldots, x_n\Big) \tag{2.4.8}$$

is either stable or the zero polynomial.

Proof. This is of course only interesting if $j < k$, as the result would be zero otherwise. By fixing the variables x_i for $i \notin \{j, k\}$, we need only show that $f(x, y)$ is $\mathbb{H}^2$-stable implies that $f(-\frac{\partial}{\partial y}, y)$ is $\mathbb{H}$-stable. With d the degree of f in the first variable, we consider the polarization $\mathrm{Pol}_d\, f(x_1, \ldots, x_d, y)$, which is also stable by the above theorem. Then $\mathrm{Pol}_d\, f$ is multi-affine in the first d variables, so we can replace each x_j by $-\frac{\partial}{\partial y}$ by Lemma 2.2.4. But

$$\mathrm{Pol}_d\, f(-\frac{\partial}{\partial y}, \ldots, -\frac{\partial}{\partial y}, y) \tag{2.4.9}$$

collapses to $f(-\frac{\partial}{\partial y}, y)$, which is then stable or zero. □

Theorem 2.4.3 (Half of Theorem 1.2 of [8])**.** *Let*

$$T = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha,\beta} \mathbf{x}^\alpha \frac{\partial^\beta}{\partial \mathbf{x}^\beta} \tag{2.4.10}$$

be a differential operator of finite order. Let

$$F_T(\mathbf{x}, \mathbf{y}) = \sum_{\alpha,\beta} c_{\alpha,\beta} \mathbf{x}^\alpha \mathbf{y}^\beta \tag{2.4.11}$$

be the symbol of T. If $F_T(\mathbf{x}, -\mathbf{y}) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is stable then T preserves stability.

Proof. If $f(\mathbf{z})$ is stable, then so is

$$\sum_{\alpha,\beta} c_{\alpha,\beta} \mathbf{x}^\alpha (-\mathbf{y})^\beta f(\mathbf{z}). \tag{2.4.12}$$

Using the above lemma to replace the $-y_j$ with $\frac{\partial}{\partial z_j}$ and then setting $\mathbf{z} = \mathbf{x}$, we find that $T(f)$ is stable. □

2.5 Stability for general half-planes

So far, we have only considered stability to mean an absence of zeros when all coordinates are in the upper half plane $\mathbb{H}$. We can easily consider other half-planes, such as

$$\mathbb{H}_\theta := \{e^{-i\theta}z : z \in \mathbb{H}\}. \tag{2.5.1}$$

Chief among these is the open right half-plane, $\mathbb{H}_{\pi/2}$, which will have application to the Lee-Yang Circle Theorem in the next section. A $\mathbb{H}_{\pi/2}$-stable polynomial is also known as (weakly) Hurwitz stable. Luckily, much of the above theory carries over to $\mathbb{H}_\theta$-stable polynomials by merely noting that $f(\mathbf{z})$ is $\mathbb{H}_\theta$-stable if and only if $f(e^{-i\theta}\mathbf{z})$ is $\mathbb{H}_0$-stable. In particular, for $\mathbb{H}_{\pi/2}$ the Lieb-Sokal lemma has the following formulation:

Lemma 2.5.1. *If $g + yf$ is $\mathbb{H}_{\pi/2}$-stable then so is $g + \partial_j f$.*

The Grace-Walsh-Szego theorem carries over verbatim, and so by inspection we have the following theorem for Hurwitz stability preservers:

Theorem 2.5.2. *Let*

$$T = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha,\beta}\mathbf{x}^\alpha \frac{\partial^\beta}{\partial \mathbf{x}^\beta} \tag{2.5.2}$$

be a differential operator of finite order. Let

$$F_T(\mathbf{x}, \mathbf{y}) = \sum_{\alpha,\beta} c_{\alpha,\beta}\mathbf{x}^\alpha \mathbf{y}^\beta \tag{2.5.3}$$

be the symbol of T. If $F_T(\mathbf{x}, \mathbf{y}) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is $\mathbb{H}^{2n}_{\pi/2}$-stable then T preserves $\mathbb{H}^n_{\pi/2}$-stability.

2.6 Applications of stable polynomials

2.6.1 Lee-Yang Circle Theorem

Consider the Ising model on the finite lattice $[n]$ with complex fugacities (external magnetic field) $(h_1, \dots, h_n)$ and coupling constants J_{jk} — which encode the geometry of the lattice. Then the partition function is written as

$$Z(h_1, \dots, h_n) = \sum_{\eta \in \{-1,1\}^n} e^{\eta \cdot h + \sum_{j,k=1}^n J_{jk} \eta(j) \eta(k)}. \tag{2.6.1}$$

In the 50's, Lee and Yang [26] investigated the phase transitions of the Ising model by considering where the zeros of this partition function accumulated. The motivation here is that the free energy is the log of the partition function, so for non-analyticities to form in the thermodynamic limit at some h there needs to be a cluster of partition function zeros coming arbitrarily close to h.

Recall the definition of $\mathbb{H}^n_{\pi/2}$ from Section 2.5.

Theorem 2.6.1. *(Lee-Yang circle theorem.) If $J_{jk} \geq 0$ for all $1 \leq j, k \leq n$ (i.e. the Ising model is ferromagnetic), then*

(a) $Z(h_1, \dots, h_n)$ has no zeros in $\mathbb{H}^n_{\pi/2}$.

(b) All zeros of $Z(h, \dots, h)$ lie on the imaginary axis.

In particular, this shows that the Ising model has no phase transitions in the presence of an external magnetic field.

It is termed the "circle theorem" by considering the change of variables $w = e^h$ — then all zeros lie on the unit circle in the complex plane.

We present here the quick proof given in [5].

Proof. Notice that (b) follows from (a) by considering $\eta \mapsto -\eta$.

For (a), suppose first that $J_{jk} = 0$ for all j, k. Then

$$Z(h_1, \dots, h_n) = (e^{h_1} + e^{-h_1}) \cdots (e^{h_n} + e^{-h_n}), \tag{2.6.2}$$

which is the normal limit of the product of the $\mathbb{H}_{\pi/2}$-stable polynomials

$$(1 + h_j/k)^k + (1 - h_j/k)^k. \tag{2.6.3}$$

Now, suppose $J_{jk} = 0$ for some j, k, and that $Z(h_1, \dots, h_n)$ is the normal limit of $\mathbb{H}^n_{\pi/2}$-stable polynomials. Now change J_{jk} to be positive, and let $Z_{jk}(h_1, \dots, h_n)$ be the resulting partition function. The theorem follows if we can show that Z_{jk} is again a normal limit of $\mathbb{H}^n_{\pi/2}$-stable polynomials.

Now one can check directly from (2.6.1) that

$$Z_{jk}(h_1, \dots, h_n) = \frac{1}{2}\left[(e^{J_{jk}} + e^{-J_{jk}}) + (e^{J_{jk}} - e^{-J_{jk}})\frac{\partial^2}{\partial h_j \partial h_k}\right] Z(h_1, \dots, h_n). \tag{2.6.4}$$

The operator $T = a + b\partial_{jk}$ has symbol $F_T(\mathbf{x}, \mathbf{y}) = a + by_j y_k$, which is clearly a $\mathbb{H}^{2n}_{\pi/2}$-stable polynomial for $a, b > 0$. Theorem 2.5.2 shows that T preserves stability, and so as there are stable f_k converging to Z normally, $T(f_k) \to T(Z) = Z_{jk}$ normally. □

2.6.2 Negative association

In this section we let mu be a probability measure on $\{0,1\}^n$. Recall that if μ has stable generating function

$$f_\mu(x_1, \dots, x_n) = \mathbb{E}^\mu x_1^{\eta(1)} \cdots x_n^{\eta(n)}, \tag{2.6.5}$$

we say that μ is a *stable probability measure*. By Theorem 2.3.1, f_μ also satisfies the strong Rayleigh condition (2.3.6). Because of this connection, the terms "stable" and "strong Rayleigh" are used interchangeably.

In the setting $\{0,1\}^S$ for S infinite, we say that the measure μ is strong Rayleigh if every projection of μ onto finitely many coordinates is strong Rayleigh. One class of strong Rayleigh measures is given by the product probability measures on $\{0,1\}^S$. Indeed, if μ is a product measure with probabilities $\mu(\eta(j) = 1) = p_j$, then for any finite $T \subset S$ the generating function of the projection $\mu|_T$ can be expressed as

$$f_{\mu|_T}(\mathbf{x}) = \mathbb{E}^\mu \prod_{j\in T} x_j^{\eta(j)} = \prod_{j\in T} \mathbb{E}^\mu x_j^{\eta(j)} = \prod_{j\in T} [(1-p_j) + p_j x_j], \tag{2.6.6}$$

a product of stable linear factors.

There is already a well-developed theory of positive correlations, starting from the Harris inequality for product measures, and culminating with the celebrated FKG inequality. Recall that a measure μ is positively associated if for all monotone increasing functions f, g — assuming the natural partial ordering on $\{0,1\}^S$

$$\int fg d\mu \geq \int f d\mu \int g d\mu.$$

Theorem 2.6.2 (FKG Theorem)**.** *Suppose μ is a probability measure on $\{0,1\}^n$, satisfying the lattice condition*

$$\mu(\zeta \wedge \eta)\mu(\zeta \vee \eta) \geq \mu(\zeta)\mu(\eta), \text{ for all } \eta, \zeta \in \{0,1\}^n. \tag{2.6.7}$$

Of course, $\zeta \wedge \eta$ ($\zeta \vee \eta$) is the coordinate-wise minimum (maximum) of η and ζ. Then μ is positively associated.

Considering also a time component, as with interacting particle systems, it is known that "attractive" spin-flip dynamics — satisfied by the ferromagnetic Ising and Voter models, for example — preserve positive association. That is, assuming an initial distribution that is positively associated, the distribution of the process at later times is still positively associated. For this result of Harris, and the definition of "attractive", we refer the reader to chapters 2 and 3 of [30].

We consider the following analogue for negative correlations: μ is *negatively associated* if

$$\int fgd\mu \leq \int fd\mu \int gd\mu, \tag{2.6.8}$$

for all increasing functions f, g that depend on disjoint sets of coordinates. The latter condition is required by the fact that any random variable is positively correlated with itself.

Constructing a general theory of negative dependence has not been easy — see [39] for a survey of the many earlier attempts. Surprisingly, the theory of stable polynomials provides a very strong, yet analytically flexible notion of negative dependence.

At the most basic level, one notices that a probability measure μ satisfying the strong Rayleigh property with the vector $(1, 1, \ldots, 1)$ has pairwise negative correlations:

$$\mathbb{E}^{\mu}(\eta_j \eta_k) \leq \mathbb{E}^{\mu}\eta_j \mathbb{E}^{\mu}\eta_k \text{ for each } j \neq k. \tag{2.6.9}$$

In fact, stable measures satisfy much stronger negative dependence properties. From a review of the literature, Borcea, Brändén and Liggett assembled the following theorem:

Theorem 2.6.3 (Theorem 4.8 of [9])**.** *Let $\mathcal{P}$ be a set of probability measures satisfying:*

1. *Each $\mu \in \mathcal{P}$ is a measure on $\{0,1\}^E$, $E \subset \{1, 2, 3, \ldots\}$ some finite subset depending on μ.*
2. *$\mathcal{P}$ is closed under conditioning.*
3. *Each $\mu \in \mathcal{P}$ has pairwise negative correlations (see equation (2.6.9)).*

4. *Each $\mu \in \mathcal{P}$ has homogeneous generating polynomial (i.e, each monomial in f_μ has the same degree).*

Then every $\mu \in \mathcal{P}$ is negatively associated.

The class of homogeneous stable measures easily satisfies the above conditions. Furthermore, certain symmetrization and homogenizing procedures — via hyperbolic polynomials — preserve stability; hence one can show that non-homogeneous stable probability measures are also negatively associated. See §4.1 in [9] for details. We restate this in the following theorem:

Theorem 2.6.4 (Theorem 4.9 of [9])**.** *Suppose a probability measure μ on $\{0,1\}^S$, S countable, has a stable generating function. Then μ is negatively associated.*

In Chapter 3 we will prove a generalization of this theorem for measures on $\{0,1,2,\dots\}^S$.

2.6.3 Distribution of particle counts

Given a measure μ on $\{0,1\}^S$, we would like to gain some understanding of the distribution of the total number of 1's in certain subsets. This will be useful in the next chapter when we analyze the current flow of particles in the symmetric exclusion process. For S finite, the following theorem goes back to Lévy [28]. See also Pitman [41] for more combinatorial and probabilistic properties of generating function zeros, and the connection with Pólya frequency sequences.

Proposition 2.6.5. *Suppose μ is a stable measure on $\{0,1\}^S$. Then for each $T \subset S$ there exists a collection $\{\zeta_x; x \in T\}$ of independent Bernoulli random variables, such that $\sum_{x\in T}\eta(x)$ has the same distribution as $\sum_{x\in T}\zeta_x$. Note that each ζ_x depends on T.*

Combining the above proposition with standard conditions for convergence to the normal distribution yields a central limit theorem for strong Rayleigh random variables.

Proposition 2.6.6. *Suppose that for each n the collection of Bernoulli random variables $\{\eta_n(x); x \in S\}$ determines a stable probability measure on $\{0,1\}^S$. Furthermore, assume the variances $\operatorname{Var}(\sum_{x\in S}\eta_n(x))$ tend to infinity as $n \to \infty$. Then*

$$\frac{\sum_S \eta_n(x) - \mathbb{E}(\sum_S \eta_n(x))}{\sqrt{\operatorname{Var}(\sum_S \eta_n(x))}} \overset{d}{\Rightarrow} \mathcal{N}(0,1) \text{ as } n \to \infty. \tag{2.6.10}$$

These kinds of results were already known for determinantal processes — see [20,36] — although this is hardly a coincidence, as a large subset of determinantal measures are strong Rayleigh [9, Proposition 3.5].

We conclude this chapter by proving Proposition 2.6.5.

Proof. By a generalization of the Borel-Cantelli lemmas [16], applied to the negatively dependent events $\{\eta(x) = 1\}$,

$$\sum_{x\in T}\eta(x) = \infty,\ \mu\text{-a.s. if } \mathbb{E}^\mu \sum_{x\in T}\eta(x) = \infty. \tag{2.6.11}$$

In this case the proposition is trivially true, hence we may assume that

$$\mathbb{E}^\mu \sum_{x\in T}\eta(x) < \infty. \tag{2.6.12}$$

First take an increasing sequence of finite subsets $T_n \nearrow T$. Now define for $z \in \mathbb{C}$ the polynomials

$$Q_n(z) = \mathbb{E}^\mu z^{\sum_{x\in T_n}\eta(x)}. \tag{2.6.13}$$

The limit

$$Q(z) = \mathbb{E}^\mu z^{\sum_{x\in T}\eta(x)} \tag{2.6.14}$$

exists, and in fact $Q_n \to Q$ uniformly on compact sets. Indeed,

$$\begin{aligned} |Q_n(z) - Q(z)| &\le \mathbb{E}^\mu \left| z^{\sum_{x\in T_n} \eta(x)} \left[1 - z^{\sum_{x\in T\setminus T_n} \eta(x)} \right] \right| \\ &\le \mathbb{E}^\mu \left(\max \left\{ 1, |z|^{\sum_{x\in T} \eta(x)} \right\} 1_{\{\eta \not\equiv 0 \text{ on } T\setminus T_n\}} \right). \end{aligned}$$

Now for $|z| = r > 1$, r^t is an increasing function of t, so the negative dependence property of μ (2.6.8), implies that

$$\mathbb{E}^\mu r^{\sum_{x\in T} \eta(x)} \le \prod_{x\in T} \mathbb{E}^\mu r^{\eta(x)} = \prod_{x\in T} [1 + (r-1)\mathbb{E}^\mu \eta(x)] \le e^{r\mathbb{E}^\mu \sum_{x\in T} \eta(x)} < \infty. \tag{2.6.15}$$

(The last inequality uses the estimate $1 + x \le e^x$.) Dominated convergence then gives the normal convergence $Q_n \to Q$. In particular, Q is entire and has at most exponential growth.

Claim: The limit $Q(z)$ factors into

$$Q(z) = e^{\sigma(z-1)} \prod_{k=1}^{\infty} \frac{z - a_k}{1 - a_k}, \tag{2.6.16}$$

for some $\sigma \ge 0$, $a_k \le 0$ and $\sum (1 - a_k)^{-1} < \infty$.

Indeed, this is a trivial consequence of a classical theorem on entire functions [27, VIII, Theorem 1], which shows an equivalent factorization when the zeros lie in an arc of angle strictly less than π. In our case — the zeros lying on the negative real axis, and an exponential bound such as (2.6.15) — the proof is much simpler, and we will reproduce it now.

By stability, and since the coefficients are all non-negative, $Q_n(z)$ has only real non-positive zeros, say, $\{a_j^{(n)}; 1 \le j \le n\}$. Let $\{a_j\}$ be the set of zeros of Q. Since $\{Q_n; n > 0\}$ is a normal family, the zeros of Q_n converge to the zeros of Q, and all zeros of Q are accounted for. Furthermore, $Q_n(1) = 1$ for all n because μ

is a probability measure, so we can write

$$Q_n(z) = \prod_{j=1}^{n} \frac{z - a_j^{(n)}}{1 - a_j^{(n)}} = \prod_{j=1}^{n} \left[1 - \frac{1 - z}{1 - a_j^{(n)}} \right]. \tag{2.6.17}$$

Since $Q_n(0) \to Q(0)$, we see that $\sum_j (1 - a_j^{(n)})^{-1}$ stays bounded, so that

$$R(z) = \prod_j \frac{z - a_j}{1 - a_j} \tag{2.6.18}$$

is entire. Thus $Q(z)/R(z)$ is an entire function with no zeros, $Q(1)/R(1) = 1$, and the estimate (2.6.15) implies that $Q(z)/R(z)$ has order of growth at most 1. That is, $Q(z)/R(z) = e^{\sigma(z-1)}$, which proves the claim. (The fact that $\sigma \geq 0$ is obvious from the coefficients of Q being all non-negative.)

To conclude the proof it is enough to show that $\sigma = 0$, because then

$$Q(z) = \prod_j \frac{z - a_j}{1 - a_j} = \prod_j \left[p_j z + (1 - p_j) \right], \tag{2.6.19}$$

where $p_j = 1/(1 - a_j)$. But this last expression is just the generating function for the sum of independent Bernoulli r.v.'s having the parameters p_k.

To obtain $\sigma = 0$, we show that $|Q(z)| \leq e^{c|z|}$ for any $c > 0$ and $|z|$ large enough. It is clear that $|Q(z)| \leq Q(|z|)$, hence we consider only $z = r > 1$. Recall from (2.6.15) that

$$Q(r) \leq \prod_{x \in T} [1 + (r - 1)\mathbb{E}^{\mu}\eta(x)]. \tag{2.6.20}$$

Let

$$a_x = \mathbb{E}^{\mu}\eta(x). \tag{2.6.21}$$

With $a = \sum_{x \in T} a_x < \infty$, note that $\#\{x : a_x > r^{-1/2}\} \leq a r^{1/2}$. Then by a trivial bound we have

$$\begin{aligned} Q(r) &\le \left(\prod_{a_x > r^{-1/2}} r \right) \left(\prod_{a_x \le r^{-1/2}} e^{(r-1)a_x} \right) \\ &\le r^{ar^{1/2}} \exp\left((r-1) \sum_{a_x \le r^{-1/2}} a_x \right). \end{aligned}$$

As $r \to \infty$ the sum inside the exponential goes to zero, which concludes the proof. $\square$

CHAPTER 3

Current flow in the symmetric exclusion process

3.1 Introduction

The *exclusion process* on a countable set S is a continuous-time Markov process describing the motion of an interacting family of Markov chains on S, subject to the condition that each site can contain only one particle at a time. With the assumption that the jump rates from sites x to sites y satisfy $p(x, y) = p(y, x)$, the resulting process is termed the *symmetric* exclusion process (SEP).

In conservative particle systems such as the exclusion process and the zero-range process — systems where particles are neither created nor destroyed — one topic of study is the bulk flow or current of particles. By this we mean the net amount of particles that have flowed from one part of the system into the other. Finding the expected current in such systems is usually quite straightforward, however, given the interdependence of the particle motions, characterizing the current fluctuations is a harder problem. In the case of asymmetric exclusion on the integer lattice, the variance of the current as seen by a moving observer has been shown to have the curious order of $t^{2/3}$, with connections to random matrix theory [4, 18, 52].

For symmetric exclusion on $\mathbb{Z}$, when only nearest-neighbor jumps are allowed, the current flow is intimately tied to the classical problem of determining the mo-

The material in this chapter is based on [53].

tion of a tagged particle. This is especially clear when the process is started from the equilibrium measure ν_ρ — the homogeneous product measure on $\{0,1\}^{\mathbb{Z}}$ with density ρ. Since particles cannot jump over each other, and the spacings between subsequent particles are independent geometric-(ρ) random variables, the tagged particle's displacement is asymptotically proportional to the current across the origin. In this case Arratia [2] gave the first central limit theorem for a tagged particle, and Peligrad and Sethuraman [38] showed process-level convergence of the current (and hence of a tagged particle) to a fractional Brownian motion. Non-equilibrium results were obtained by Jara and Landim [24, 25] under a hydrodynamic rescaling of the process, even with non-translation invariant jump rates (quenched random bond disorder). The heat equation machinery used there requires the initial distributions to be smooth profiles, giving results only in an average sense.

More recently, Derrida and Gerschenfeld [14] applied techniques used for the more difficult asymmetric exclusion [52] to SEP, obtaining asymptotics for the cumulants of the (non-normalized) current. Although their results are sharp, translation invariance of the jump rates and a step-initial condition seem to be required by that approach.

In this chapter we exploit the theory of stable polynomials to obtain a central limit theorem for the current throughout a wide range of transition rates and initial conditions.

3.2 The stirring process

In this section we give a brief sketch of how to construct Harris's stirring representation of the symmetric exclusion process. For simplicity, we consider the set of sites to be the integer lattice $\mathbb{Z}$, and assume the symmetric jump rates $p(x, y) = p(y, x)$ are uniformly bounded.

The stirring representation is a collection of randomly jumping labels. Initially (at time zero), we place at each site $x \in \mathbb{Z}$ the label x.

With each pair of sites $\{x, y\}$ we associate an independent Poisson process (clock) $N^{\{x,y\}}$ with the rate $p(x, y)$. When the clock $N^{\{x,y\}}$ rings, the labels at x and y switch places.

Let ξ_t^x denote the position at time t of the label x, and let $L_t(x)$ denote the label occupying site x at time t. Note that $\xi_0^x = x = L_0(x)$ for all x. In particular, if the clock $N^{\{x,y\}}$ rings at time t, we see that

$$L_t = L_{t-}^{x,y}, \text{ where } L^{x,y}(z) = \begin{cases} L(y) & \text{if } z = x \\ L(x) & \text{if } z = y \\ L(z) & \text{if } z \neq x, y \end{cases} \tag{3.2.1}$$

Given a realization of the Poisson clocks, there are two sources of ambiguity when defining ξ_t^x (equivalently, $L_t(x)$). One ambiguity occurs if two Poisson clocks ring at the very same time (which transposition should be applied first?), however, we can ensure that this can't happen by first restricting to a set of full probability.

Another ambiguity can arise if there is a sequence of clocks, ringing at times $t > t_1 > t_2 > \cdots > 0$, which carries a label along infinitely many jumps before this fixed time. In other words, we must show that the collection of bonds does not percolate, i.e., the graph of sites $\mathbb{Z}$, containing only those (undirected) edges $\{x, y\}$ for which the associated Poisson clocks have rung before time t, does not contain an connected infinite subgraph.

Suppose that the rates satisfy a finite-range condition: with N fixed,

$$p(x, y) = 0 \text{ for all } |x - y| > N. \tag{3.2.2}$$

Then the events

$$A_k := \{\text{none of the clocks } N^{\{x,y\}} \text{ with } x < 2kN \le y \text{ have rung by time } t\} \tag{3.2.3}$$

are independent of each other, and the uniform boundedness condition on the rates ensures that $P(A_k)$ is uniformly bounded away from zero. Thus the second Borel-Cantelli Lemma ensures for an a.s. realization of the clocks, and for any fixed time t, that $\mathbb{Z}$ can be partitioned into finite sets with no jumps attempted between them before t. As within each finite set there are a.s. only finitely many attempted jumps, the value of $L_t(x)$ is defined by applying a finite number of transpositions (3.2.1) to the initial state L_0 (where of course $L_0(x) = x$).

Using a percolation result of Schulman [45], one can still construct the process of labels $\{\xi_t; t \ge 0\}$ under the weaker

Assumption 3.2.1. *There is a function $\overline{p} : \mathbb{Z} \to \mathbb{R}$, satisfying a first moment condition*

$$\sum_{x \in \mathbb{Z}} |x| \overline{p}(x) < \infty, \tag{3.2.4}$$

such that $p(x, y) \le \overline{p}(y - x)$ for all $x, y \in \mathbb{Z}$.

3.3 Symmetric exclusion

Given an initial state $\eta \in \{0, 1\}^{\mathbb{Z}}$, we can now construct the symmetric exclusion process $\{\eta_t; t \ge 0\}$ by setting

$$\eta_t(x) = \eta(L_t(x)). \tag{3.3.1}$$

Notice that if the clock $N^{\{x,y\}}$ rings at time t, this produces an effect on the prior state η_{t-} if and only if there is one particle and one hole between sites x and y; in that case they switch locations.

The main references for symmetric exclusion, and interacting particle systems in general, are the books by Liggett [30, 31]. For convenience we reproduce some useful facts below.

Let P^η and $\mathbb{E}^\eta$ be the probability and expectation, respectively, given that the process has initial configuration η. As SEP is a Feller process, the associated Markov semigroup $\{S(t); t \geq 0\}$ on continuous functions $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ can be defined:

$$S(t)f(\eta) = \mathbb{E}^\eta f(\eta_t). \tag{3.3.2}$$

Given the process with an initial *distribution* μ on $\{0,1\}^{\mathbb{Z}}$, the distribution at a later time t will be denoted by $\mu_t = \mu S(t)$, where

$$\int f(\eta) d[\mu S(t)](\eta) := \int S(t)f(\eta) d\mu(\eta). \tag{3.3.3}$$

Once we have the Markov semigroup, recall that we can form the infinitesimal generator:

$$\mathcal{L}f = \lim_{t \downarrow 0} \frac{S(t)f - f}{t}, \tag{3.3.4}$$

acting on functions for which the limit exists under uniform convergence. In the case of SEP, the infinitesimal generator has the particularly simple form

$$\mathcal{L}f(\eta) = \sum_{x,y \in \mathbb{Z}} p(x,y)[f(\eta^{x,y}) - f(\eta)]. \tag{3.3.5}$$

(Recall that $\eta^{x,y}$ is defined as in equation (3.2.1).)

We recall that there is a one-to-one correspondence between Feller processes, Markov semigroups, and infinitesimal generators. In fact, the usual way to construct interacting particle systems is to first write down the form of the infinitesimal generator when applied to local functions, and then use this correspondence to form the stochastic process. As exhibited here, the infinitesimal generator is usually straightforward to write down from an intuitive description of the process; however, verifying the technical conditions a generator needs to satisfy is not easy. Again, [30] gives the general account of how this is done.

Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)f = \mathcal{L}S(t)f = S(t)\mathcal{L}f \tag{3.3.6}$$

for all f in the domain of $\mathcal{L}$, the following formula is straightforward:

Proposition 3.3.1 (Integration by parts formula)**.** *Suppose that $\mathcal{L}^{(1)}$, $\mathcal{L}^{(2)}$ are the infinitesimal generators associated with the semigroups $S^{(1)}(t)$, $S^{(2)}(t)$. Suppose D is a set of functions on $\{0,1\}^{\mathbb{Z}}$ with the following properties:*

(1) $S^{(j)}(t) : D \longrightarrow D$ for all $t \geq 0$, $j = 1, 2$.

(2) D is a subset of the domains of the generators $\mathcal{L}^{(j)}$, $j = 1, 2$.

Then for any $f \in D$,

$$[S^{(1)}(t) - S^{(2)}(t)]f = \int_0^t S^{(2)}(t-s)[\mathcal{L}^{(1)} - \mathcal{L}^{(2)}]S^{(1)}(s)f ds. \tag{3.3.7}$$

For particle systems such as exclusion and independent Markov chains, the construction in [30] gives a class of 'smooth' functions D satisfying (1) and (2) in the above. It will be enough for our purposes to know that this choice of D includes the functions only depending on a finite number of coordinates.

3.3.1 Duality

The notion of duality for interacting particle systems is extremely useful. Basically — when available — it allows one to compute probabilities for the infinite system in terms of a (often-simpler) "dual" system. The symmetric exclusion process in particular has a finite version of itself as its dual.

Let $(Y_1(t), Y_2(t), \dots, Y_n(t))$ be the Feller process on

$$\mathbb{Z}^n \setminus \{(x_1, \dots, x_n); x_i = x_j \text{ for some } i \neq j\} \tag{3.3.8}$$

with generator

$$V_n f(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{y \neq x_i} p(x_i, y)[f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)]. \tag{3.3.9}$$

This describes the symmetric exclusion process with exactly n particles. We can also think of n-particle symmetric exclusion as being a Markov chain on $\{A \subset \mathbb{Z}; |A| = n\}$ — just let $A_t = \{Y_1(t), Y_2(t), \dots, Y_n(t)\}$. We write P^A for the probability of the finite system when started from the set A.

Theorem 3.3.2 (Duality relation for SEP, Theorem 1.1, Chapter VIII of [30])**.** *Let $A \subset \mathbb{Z}$, with $|A| = n$. Then*

$$P^\eta(\eta_t = 1 \text{ on } A) = P^A(\eta = 1 \text{ on } A_t). \tag{3.3.10}$$

Let X_t be the one-particle Markov chain on $\mathbb{Z}$ with the transition rates $p(x, y)$. When $n = 1$, the duality relation implies that

$$\mathbb{E}^\eta \eta_t(x) = \mathbb{E}^x \eta(X_t). \tag{3.3.11}$$

For $n > 1$, the n-particle exclusion process does not have easily expressed probabilities, but luckily we can compare this process to a collection of n independent 1-particle Markov chains. This is captured by the following theorem. Recall that a bounded symmetric function $f(x, y)$ is said to be positive semi-definite if for all functions $\beta(x)$ with $\sum_x |\beta(x)| < \infty$ and $\sum_x \beta(x) = 0$ we have

$$\sum_{x,y} \beta(x)\beta(y)f(x, y) \geq 0. \tag{3.3.12}$$

An bounded symmetric function of n variables is called positive semi-definite if it is so in each pair of variables.

Proposition 3.3.3 (Proposition 1.7, Chapter VIII of [30]). *Suppose* $\{X_i(t); 1 \leq i \leq n\}$ *are independent copies of the 1-particle Markov chain with the transition rates* $p(x, y)$*, and* $\{Y_1(t), \ldots, Y_n(t)\}$ *is the* n*-particle symmetric exclusion process. Then for any* $\{x_1, \ldots, x_n\}$ *of cardinality* n*, and bounded symmetric positive semi-definite* f*, we have the comparison*

$$\mathbb{E}^{\{x_1,\ldots,x_n\}} f(Y_1(t), \ldots, Y_n(t)) \leq \mathbb{E}^{\{x_1,\ldots,x_n\}} f(X_1(t), \ldots, X_n(t)). \tag{3.3.13}$$

3.4 Particle current

The original problem as described by Pemantle [39] was proved and generalized to the following by Liggett [34]. Consider SEP on $\mathbb{Z}$ with translation invariant transition probabilities that describe a random increment with finite variance, i.e.,

$$\sum_{n>0} n^2 p(0, n) < \infty. \tag{3.4.1}$$

Start with particles initially occupying the whole half lattice $\{x \in \mathbb{Z}; x \leq 0\}$. Then the current of particles across the origin after time t,

$$W_t = \sum_{x>0} \eta_t(x), \tag{3.4.2}$$

satisfies the central limit theorem

$$\frac{W_t - \mathbb{E}W_t}{\sqrt{\mathrm{Var}(W_t)}} \Rightarrow \mathcal{N}(0, 1) \text{ in distribution.} \tag{3.4.3}$$

It was conjectured in [34] that this result would also hold in the case where the transition probabilities lie in the domain of a stable law of index $\alpha > 1$. We will show this in Section 5.

In this chapter we consider the following general setting: For a partition $\mathbb{Z} = A \cup B$, we think of the net current of particles from A to B as

$$W_t = W^+(t) - W^-(t), \tag{3.4.4}$$

where $W^+(t)$ is the number of particles that start in A and end up in B at time t, and $W^-(t)$ is the number of particles that start in B and end up in A. Although the usual construction of SEP does not distinguish particles, we can make this quantity rigorously defined through the above stirring representation, as used by De Masi and Ferrari [13].

Define the current from A to B as

$$W_t = \sum_{x \in A} \eta(x) 1_{\{\xi_t^x \in B\}} - \sum_{x \in B} \eta(x) 1_{\{\xi_t^x \in A\}}. \tag{3.4.5}$$

This is well defined for any initial condition η as long as

$$\mathbb{E}\left(\sum_{x \in A} 1_{\{\xi_t^x \in B\}} \right) < \infty. \tag{3.4.6}$$

When η contains only finitely many particles, the current can be written as just

$$W_t = \sum_{x \in B} \Big\{ \eta_t(x) - \eta(x) \Big\}. \tag{3.4.7}$$

This coincides with the definition given above, because

$$\begin{aligned} \sum_{x \in B} \Big\{ \eta(L_t(x)) - \eta(x) \Big\} &= \sum_{x \in B} \sum_{y \in \mathbb{Z}} \eta(y) 1_{\{\xi_t^y = x\}} - \sum_{y \in B} \eta(y) \\ &= \sum_{y \in \mathbb{Z}} \eta(y) 1_{\{\xi_t^y \in B\}} - \sum_{y \in B} \eta(y) \\ &= \sum_{y \in A} \eta(y) 1_{\{\xi_t^y \in B\}} - \sum_{y \in B} \eta(y)[1 - 1_{\{\xi_t^y \in B\}}], \end{aligned}$$

which is precisely the expression (3.4.5).

For instance, suppose $A = \{x \leq 0\}$ and $B = \{x > 0\}$. Each ξ_t^x has the same distribution as the one-particle Markov chain X_t started from the site x, though of course for different x the ξ_t^x are highly dependent. Under Assumption 3.2.1 we can compare (using a coupling argument) X_t to a translation-invariant random walk Z_t, having finite first moment, to show that $P^x(X_t \leq 0) \leq P^0(Z_t \geq x)$. Condition (3.4.6) then holds, because

$$\mathbb{E}\left(\sum_{x>0} 1_{\{\xi_t^x \leq 0\}} \right) \leq \sum_{x>0} P^0(Z_t \geq x) = \mathbb{E}(Z_t^+) < \infty. \tag{3.4.8}$$

We say that the partition $\mathbb{Z} = A \cup B$ is *balanced* if there is a $c > 0$, not depending on x, such that

$$c < \liminf_{t\to\infty} P^x(X_t \in A) \leq \limsup_{t\to\infty} P^x(X_t \in A) < 1 - c. \tag{3.4.9}$$

Recall that the Lévy distance $d(X,Y)$ between two random variables X and Y is

$$\inf\{\epsilon > 0 : P(X \leq x - \epsilon) - \epsilon \leq P(Y \leq x) \leq P(X \leq x + \epsilon) + \epsilon \text{ for all } x \in \mathbb{R}\}.$$

Here is our main theorem:

Theorem 3.4.1. *Let $A \cup B$ be any balanced partition of $\mathbb{Z}$, and $\eta \in \{0,1\}^{\mathbb{Z}}$ be a (deterministic) initial condition for η_t — the symmetric exclusion process. Suppose (3.4.6) holds at all times, and that*

$$\sup_{t\geq 0} \mathbb{E}^\eta \left(\sum_{\eta(x)=1} (1 - \eta_t(x)) \right) = \infty. \tag{3.4.10}$$

Then the current W_t^η of particles between A and B satisfies the central limit theorem

$$\overline{W_t^\eta} := \frac{W_t^\eta - \mathbb{E}W_t^\eta}{\sqrt{\operatorname{Var} W_t^\eta}} \Rightarrow^d \mathcal{N}(0,1). \tag{3.4.11}$$

Furthermore, we have the following rate of convergence in the Lévy metric:

$$d(\overline{W_t^\eta}, \mathcal{N}) \leq C(\operatorname{Var} W_t^\eta)^{-\frac{1}{2}}. \tag{3.4.12}$$

Condition (3.4.10) is a measure of how rigid the system is: by varying the time parameter, the expected number of initially occupied sites that are then empty needs to be unbounded.

Remark 3.4.2. *The above theorem is true for symmetric exclusion processes on more general sets of sites, as long as the stirring process is well-defined. For concreteness, and with a view to the examples, we have specialized here to the integer lattice.*

The reader can skip to the last section to see these conditions checked for a couple of examples.

3.5 Negative dependence and SEP

Because of the hard-core repulsion of particles, the Symmetric Exclusion process tends to spread out more than independent particles would. One example of this is the following correlation inequality of Andjel [1]: for disjoint sets A, B, and deterministic starting configuration η,

$$P^\eta(\eta_t \equiv 1 \text{ on } A \cup B) \leq P^\eta(\eta_t \equiv 1 \text{ on } A) P^\eta(\eta_t \equiv 1 \text{ on } B). \tag{3.5.1}$$

Unfortunately SEP does not preserve negative association [33]. But the subclass of stable measures is preserved:

Theorem 3.5.1 (Theorem 5.2 of [9])**.** *The symmetric exclusion process preserves the class of stable measures. That is, if its initial distribution μ is stable, then the distribution at any positive time t is stable. In particular (by Theorem 2.6.4), starting from any product measure the distribution at later times is negatively associated.*

This latter theorem is proved by first specializing to $S = \{1, 2, \ldots, n\}$, and then only allowing jumps between sites 1 and 2. If one then starts the process from the distribution μ, then the generating function at time t has the form

$$f_{\mu_t}(z_1, z_2, \ldots, z_n) = p(t) f_\mu(z_1, z_2, \ldots, z_n) + [1 - p(t)] f_\mu(z_2, z_1, z_3, \ldots, z_n), \tag{3.5.2}$$

where $p(t)$ is the probability of having an even number of exchanges between sites 1 and 2 by time t. In other words, SEP acts on the generating function via the linear transformation T^p_{jk} seen in (2.3.11), which preserves stability by Proposition 2.3.4. One can then use Trotter's product formula to allow jumps between other pairs of sites, while still preserving stability. Finally, one can extend to all of $\mathbb{Z}$ by relabeling, and noticing that weak convergence preserves the strong Rayleigh property.

3.6 A lower bound for the current variance

By the CLT for stable measures (Proposition 2.6.6), the proof of Theorem 3.4.1 hinges upon the following proposition:

Proposition 3.6.1. *Under the conditions of Theorem 3.4.1,*

$$\operatorname{Var} W_t^\eta \to \infty \ \textit{as}\ t \to \infty. \tag{3.6.1}$$

For example, suppose we start with $\eta(x) = 1$ for $x \leq 0$ and $\eta(x) = 0$ for $x > 0$. Then we may write the current variance as

$$\operatorname{Var}(W_t^\eta) = \operatorname{Var}\left(\sum_{x>0} \eta_t(x)\right) = \sum_{x>0} \operatorname{Var}(\eta_t(x)) + \sum_{\substack{x,y>0 \\ x\neq y}} \operatorname{Cov}(\eta_t(x), \eta_t(y)). \tag{3.6.2}$$

Because of the negative association property of symmetric exclusion, all the off-diagonal covariances ($x \neq y$) are negative, while of course the diagonal terms are positive. The approach in [34] for the Pemantle problem described above was to compute the exact asymptotics of the diagonal terms, and then estimate the negative (off-diagonal) terms to be at most some fixed percentage smaller. Getting tight-enough bounds on the negative terms was already tricky in that case - considering even slightly more general transition functions $p(x, y)$ seems to require quite delicate analysis to obtain bounds that even approached the positive terms' asymptotics. To get around this obstacle, we do a generator computation in order to rewrite the *positive* variances, and obtain term-by-term domination of the off-diagonal covariances.

There is one additional complication when extending the result to the more general initial conditions of Theorem 3.4.1: when η contains infinitely many particles in both A and B we cannot write W_t^η as a convergent sum of occupation variables. Instead, we approximate by considering initial conditions with only finitely many particles. Consider first the following lemma.

Lemma 3.6.2. *Suppose $S_n \nearrow \mathbb{Z}$ is an increasing sequence of finite subsets, and for $\eta \in \{0,1\}^{\mathbb{Z}}$ define $\eta^n(x) = 1_{x\in S_n}\eta(x)$. Then for fixed $t \geq 0$ such that (3.4.6) holds,*

$$W_t^{\eta^n} \to W_t^\eta \text{ in } L^2 \text{ as } n \to \infty. \tag{3.6.3}$$

Proof. First note that the stirring process has the following negative correlations:

$$\mathbb{E}(1_{\xi_t^x\in B}1_{\xi_t^y\in B}) \leq \mathbb{E}1_{\xi_t^x\in B}\mathbb{E}1_{\xi_t^y\in B}, \text{ for any } x \neq y \text{ and } B \subset \mathbb{Z}. \tag{3.6.4}$$

Indeed, let $\{Y_1(t), Y_2(t)\}$ be the 2-particle symmetric exclusion process with initial condition $\{x, y\}$, and notice that through the relation (3.3.1) the events $\{Y_1(t), Y_2(t) \in B\}$ and $\{\xi_t^x, \xi_t^y \in B\}$ are equal. So (3.6.4) follows immediately from the comparison inequality (Proposition 3.3.3).

Using the stirring representation (3.4.5) and the inequality $(a-b)^2 \leq a^2 + b^2$ for $a, b \geq 0$,

$$\begin{aligned}\mathbb{E}(W_t^\eta - W_t^{\eta^n})^2 \leq \mathbb{E}\Big(\sum_{x\in A} 1_{\{\xi_t^x\in B\}}(\eta(x) - \eta^n(x))\Big)^2 \\ +\mathbb{E}\Big(\sum_{x\in B} 1_{\{\xi_t^x\in A\}}(\eta(x) - \eta^n(x))\Big)^2.\end{aligned}$$

Both expectations are dealt with identically, so we consider here only the first one. Expanding gives

$$\begin{aligned}&\sum_{x,y\in A}\mathbb{E}1_{\{\xi_t^x\in B\}}1_{\{\xi_t^y\in B\}}(\eta(x)-\eta^n(x))(\eta(y)-\eta^n(y))\\ \le&\Big(\mathbb{E}\sum_{x\in A}1_{\{\xi_t^x\in B\}}(\eta(x)-\eta^n(x))\Big)^2+\mathbb{E}\sum_{x\in A}1_{\{\xi_t^x\in B\}}(\eta(x)-\eta^n(x)).\end{aligned}$$

The inequality here follows from (3.6.4), and the expectations appearing in the last line converge to zero by Dominated Convergence and (3.4.6). □

Proof of Theorem 3.4.1 from Proposition 3.6.1. For (η, η^n) as above we note by the triangle inequality that

$$d(\overline{W_t^\eta},\mathcal{N})\le d(\overline{W_t^\eta},\overline{W_t^{\eta^n}})+d(\overline{W_t^{\eta^n}},\mathcal{N}). \tag{3.6.5}$$

Now recall from Proposition 2.6.5 that

$$W_t^{\eta^n}+\sum_{x\in B}\eta^n(x)\overset{d}{=}\sum_{x\in S_n}\zeta_{t,x}^n \tag{3.6.6}$$

where the ζ are Bernoulli and independent in x for each n and t. Normalize both sides to see that

$$\overline{W_t^{\eta^n}}\overset{d}{=}\overline{\sum_{x\in S_n}\zeta_{t,x}^n}. \tag{3.6.7}$$

Hence by Esseen's inequality [40, V, Theorem 3],

$$\begin{aligned}d(\overline{W_t^{\eta^n}},\mathcal{N})&\le C\left[\sum_{x\in S_n}\mathrm{Var}(\zeta_{t,x}^n)\right]^{-\frac{3}{2}}\left[\sum_{x\in S_n}\mathbb{E}|\zeta_{t,x}^n-\mathbb{E}\zeta_{t,x}^n|^3\right]\\&\le C\left[\sum_{x\in S_n}\mathrm{Var}(\zeta_{t,x}^n)\right]^{-\frac{1}{2}},\end{aligned}$$

because ζ Bernoulli implies that $|\zeta-\mathbb{E}\zeta|\le 1$, and hence $\mathbb{E}|\zeta-\mathbb{E}\zeta|^3\le \mathrm{Var}(\zeta)$.

Notice that Lemma 3.6.2 implies $d(\overline{W_t^{\eta^n}},\overline{W_t^\eta})\to 0$ in the Lévy metric as $n\to\infty$. Hence, taking $n\to\infty$ above and in (3.6.5),

$$d(\overline{W_t^\eta},\mathcal{N})\le C[\mathrm{Var}(W_t^\eta)]^{-\frac{1}{2}}. \tag{3.6.8}$$

Applying Proposition 3.6.1 finishes the proof. □

Our proof of Proposition 3.6.1 relies upon the following representation for the on-diagonal covariances.

Lemma 3.6.3. *Let X_t be defined as above Theorem 3.4.1. Then for any $\eta \in \{0,1\}^{\mathbb{Z}}$ with finite support (i.e., $\eta(x) = 1$ for only finitely many x),*

$$\sum_{x\in\mathbb{Z}} \mathrm{Var}(\eta_t(x)) = \int_0^t \sum_{\substack{x\neq y\\ x,y\in\mathbb{Z}}} p(x,y)[\mathbb{E}^y\eta(X_s) - \mathbb{E}^x\eta(X_s)]^2 ds. \tag{3.6.9}$$

Proof. (Essentially a generator computation). From the duality relation (Theorem 3.3.2), we can write

$$\sum_{x\in\mathbb{Z}} \mathrm{Var}\,\eta_s(x) = \sum_{x\in\mathbb{Z}} \left\{\mathbb{E}^\eta\eta_s(x) - (\mathbb{E}^\eta\eta_s(x))^2\right\} = \sum_{x\in\mathbb{Z}} \left\{\mathbb{E}^x\eta(X_s) - [\mathbb{E}^x\eta(X_s)]^2\right\}. \tag{3.6.10}$$

Let U and $\{U(t); t \geq 0\}$ be the generator and semi-group for X_t. Changing into the language of semigroups, we have

$$\sum_{x\in\mathbb{Z}} \mathrm{Var}(\eta_s(x)) = \sum_{x\in\mathbb{Z}} \left\{U(s)\eta(x) - [U(s)\eta(x)]^2\right\}. \tag{3.6.11}$$

Now take the derivative w.r.t. s. For the second equality below, recall that

$$Uf(x) = \sum_{y\in\mathbb{Z}:y\neq x} p(x,y)[f(y) - f(x)], \tag{3.6.12}$$

for bounded f.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\sum_{x\in\mathbb{Z}}\Big\{U(s)\eta(x)-[U(s)\eta(x)]^2\Big\} &= \sum_{x\in\mathbb{Z}} U[U(s)\eta](x)[1-2U(s)\eta(x)] \\ &= \sum_{\substack{x\neq y\\ x,y\in\mathbb{Z}}} p(x,y)[\mathbb{E}^y\eta(X_s)-\mathbb{E}^x\eta(X_s)][1-2\mathbb{E}^x\eta(X_s)] \\ &= \sum_{x\neq y} p(x,y)[\mathbb{E}^y\eta(X_s)-\mathbb{E}^x\eta(X_s)]^2 \\ &\qquad + \sum_{x\neq y} p(x,y)[\mathbb{E}^y\eta(X_s)-\mathbb{E}^x\eta(X_s)][1-\mathbb{E}^y\eta(X_s)-\mathbb{E}^x\eta(X_s)],\end{aligned} \tag{3.6.13}$$

where all sums converge absolutely because η has finite support. Since $p(x,y) = p(y,x)$, exchanging x and y in the latter sum in (3.6.13) shows it to be its own negative, hence zero. We thus conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\sum_{x\in\mathbb{Z}}\mathrm{Var}(\eta_s(x)) = \sum_{\substack{x\neq y\\ x,y\in\mathbb{Z}}} p(x,y)[\mathbb{E}^y\eta(X_s)-\mathbb{E}^x\eta(X_s)]^2. \tag{3.6.14}$$

Integrating from 0 to t finishes the proof. □

Proof of Proposition 3.6.1. Considering only initial conditions $\eta \in \{0,1\}^S$ containing finitely many particles, the net current from A to B can be written as

$$W_t^\eta = \sum_{x\in B}\Big\{\eta_t(x)-\eta(x)\Big\} = \sum_{x\in A}\Big\{\eta(x)-\eta_t(x)\Big\}. \tag{3.6.15}$$

We symmetrize this expression:

$$2W_t^\eta = \sum_{x\in\mathbb{Z}}[H(x)\eta_t(x)-H(x)\eta(x)], \text{ where } H(x) = \begin{cases} 1 & \text{if } x\in B \\ -1 & \text{if } x\in A,\end{cases} \tag{3.6.16}$$

and consider the variance:

$$4\,\mathrm{Var}(W_t^\eta) = \sum_{x\in\mathbb{Z}}\mathrm{Var}(\eta_t(x)) + \sum_{\substack{x\neq y\\ x,y\in\mathbb{Z}}} H(x)H(y)\,\mathrm{Cov}(\eta_t(x),\eta_t(y)).$$

We first deal with the covariances above, proceeding almost identically to [34]. Let $\{U_2(t); t \geq 0\}$ be the semigroup for two identical, independent Markov chains with symmetric kernel $p(x, y)$, and let U_2 be its infinitesimal generator. Specifically,

$$U_2 f(x, y) = \sum_{z \in \mathbb{Z}} \Big\{ p(x, z)[f(z, y) - f(x, y)] + p(y, z)[f(x, z) - f(x, y)] \Big\}. \quad (3.6.17)$$

Let $\{V_2(t); t \geq 0\}$ and V_2 be the semigroup and generator for the process with the exclusion interaction — the latter is defined in equation (3.3.9) above. By a slight abuse of notation, define

$$\eta(x, y) = \eta(x)\eta(y) \;\; \text{and} \; H(x, y) = H(x)H(y). \quad (3.6.18)$$

By duality and the integration by parts formula (Proposition 3.3.1),

$$\begin{aligned} -\sum_{x \neq y} H(x, y) \operatorname{Cov}(\eta_t(x), \eta_t(y)) &= \sum_{x \neq y} H(x, y)[U_2(t) - V_2(t)]\eta(x, y) \\ &= \int_0^t \sum_{x \neq y} H(x, y) V_2(t - s)[U_2 - V_2] U_2(s) \eta(x, y) ds. \quad (3.6.19) \end{aligned}$$

Subtracting V from U, we obtain

$$\begin{aligned} [U_2 - V_2] U_2(s) \eta(x, y) &= p(x, y) \Big\{ U_2(s)\eta(x, x) + U_2(s)\eta(y, y) - 2U_2(s)\eta(x, y) \Big\} \\ &= p(x, y)[\mathbb{E}^y \eta(X_s) - \mathbb{E}^x \eta(X_s)]^2, \end{aligned}$$

which we substitute into (3.6.19), also using the fact that $V_2(t-s)$ is a symmetric linear operator on the space of functions on $\{(x, y) \in S^2; x \neq y\}$:

$$\begin{aligned} &-\sum_{x \neq y} H(x)H(y) \operatorname{Cov}(\eta_t(x), \eta_t(y)) \\ &\qquad = \int_0^t \sum_{x \neq y} p(x, y)[\mathbb{E}^y \eta(X_s) - \mathbb{E}^x \eta(X_s)]^2 V_2(t - s) H(x, y) ds \\ &\qquad \leq \int_0^t \sum_{x \neq y} p(x, y)[\mathbb{E}^y \eta(X_s) - \mathbb{E}^x \eta(X_s)]^2 U_2(t - s) H(x, y) ds, \end{aligned}$$

by the comparison inequality between interacting and non-interacting particles (Proposition 3.3.3).

Combining Lemma 3.6.3 and the above estimate we obtain:

$$4\,\mathrm{Var}(W_t^\eta) \geq \int_0^t \sum_{x\neq y} p(x,y)[\mathbb{E}^y\eta(X_s) - \mathbb{E}^x\eta(X_s)]^2 q_{t-s}(x,y)ds, \tag{3.6.20}$$

where

$$q_s(x,y) = 1 - U_2(s)H(x,y) = 1 - [1 - 2P^x(X_s \in A)][1 - 2P^y(X_s \in A)]. \tag{3.6.21}$$

Notice that there is a constant $c' > 0$, depending only on the c in (3.4.9), such that $q_s(x,y) > c'$ for each $x, y \in \mathbb{Z}$ and then s large enough. So for fixed $T > 0$, applying Fatou's Lemma twice, then using Lemma 3.6.3 again,

$$\begin{aligned}
4 \liminf_{t\to\infty} \mathrm{Var}(W_t^\eta) &\geq \int_0^T \sum_{x\neq y} p(x,y)[\mathbb{E}^y\eta(X_s) - \mathbb{E}^x\eta(X_s)]^2 \liminf_{t\to\infty} q_{t-s}(x,y)ds \\
&\geq \int_0^T \sum_{x\neq y} p(x,y)[\mathbb{E}^y\eta(X_s) - \mathbb{E}^x\eta(X_s)]^2 c' ds \\
&= c' \sum_{x\in\mathbb{Z}} \mathrm{Var}(\eta_T(x)).
\end{aligned}$$

Now for any finite $S' \subset \mathbb{Z}$, using duality,

$$\begin{aligned}
\sum_{x\in S'} \mathrm{Var}(\eta_T(x)) &= \sum_{x\in S'}\sum_{y\in\mathbb{Z}} \eta(y)p_T(x,y)\left[1 - \sum_{z\in\mathbb{Z}} \eta(z)p_T(x,z)\right] \\
&= \sum_{\eta(y)=1}\sum_{\eta(z)=0}\sum_{x\in S'} p_T(y,x)p_T(x,z) \to \sum_{\eta(y)=1}\sum_{\eta(z)=0} p_{2T}(z,y),
\end{aligned}$$

as $S' \nearrow \mathbb{Z}$, by monotone convergence. By duality again, this is precisely

$$\sum_{\eta(y)=1} \mathbb{E}^\eta(1 - \eta_{2T}(y)). \tag{3.6.22}$$

Hence by (3.4.10),

$$\liminf_{t\to\infty} \mathrm{Var}(W_t^\eta) = \infty, \tag{3.6.23}$$

as desired. □

3.7 Examples

1. Consider the partition of $\mathbb{Z}$ into $A = \{x \leq 0\}$ and $B = \{x > 0\}$, with translation invariant rates $p(0,x)$ in the domain of a symmetric stable law with index $\alpha > 1$. That is,

$$\sum_{y \geq x} p(0,y) \sim L(x)x^{-\alpha}, \; x > 0, \tag{3.7.1}$$

for a slowly varying function L. Consider the step initial condition η with particles at all $x \leq 0$. Condition (3.4.6) is satisfied by the remarks before (3.4.8). The balance condition holds by the central limit theorem for random variables in the domain of attraction of a stable law. By duality and translation invariance,

$$\sum_{x \leq 0} P^{\eta}(\eta_t(x) = 0) = \sum_{x \leq 0} \sum_{y > 0} P^{x}(X_t = y) = \sum_{n > 0} nP^{0}(X_t = n) = \mathbb{E}^0 X_t^+ \to \infty. \tag{3.7.2}$$

(In fact, it grows at rate $t^{1/\alpha}$). This shows (3.4.10), and the above expression is the same as $\mathbb{E}^{\eta} W_t$, so all conditions of Theorem 3.4.1 have been verified.

2. Now consider the one-dimensional symmetric exclusion process in an random environment, with the same partition as in the previous example. The random environment is described by $\{\omega_i\}$, an iid family of random variables with $\omega \in (0,1]$ almost surely and $E\frac{1}{\omega_i} < \infty$. For each realization of the environment, we consider the exclusion process with the rates $p(i, i+1) = p(i+1, i) = \omega_i$. By the result of Kawazu and Kesten [22], we know that for a.e. $\{\omega_i\}$ the process $\{X_{n^2 t}/n\}$ converges weakly to a scaled Brownian motion, from which follows the balance condition. Again by remarks before (3.4.8) we know that the current has finite expectation for any initial placement of particles. Let us consider the case where we pick a realization η of the homogeneous product measure ν_ρ. Then almost surely-ν_ρ there are infinitely many $x \in \mathbb{Z}$ such that $\eta(x) = 1$ and

$\eta(x+1)=0$, so from the ergodicity of the environment,

$$\infty = \sum_{\substack{\eta(x)=1 \\ \eta(x+1)=0}} P^x(X_t = x+1) \leq \sum_{\eta(x)=1} P^x(\eta(X_t)=0), \tag{3.7.3}$$

which shows (3.4.10) by duality. Thus Theorem 3.4.1 gives a quenched CLT for the current.

CHAPTER 4

Birth-death chains and reaction-diffusion processes.

4.1 Introduction

Using the framework of Chapter 2, we will generalize the negative dependence result of Theorem 2.6.4 to measures on $\{0, 1, 2, \dots\}^S$ — S countable — with application to independent Markov chains and reaction-diffusion processes. The one-coordinate case is also independently interesting; more specifically, the probability measures under consideration can be decomposed into a sum of independent Bernoulli and Poisson random variables.

Call such measures on $\{0, 1, 2, \dots\}$ *t-stable.* (The formal definition is given in Section 3.) In the last section we characterize the birth-and-death chains preserving such measures:

Theorem 4.1.1. *The birth-death chain $\{X_t; t \geq 0\}$ on $\{0, 1, 2, \dots\}$ preserves the class of t-stable measures if and only if the birth rates are constant and the death rates satisfy $\delta_k = d_1 k + d_2 k^2$ for some constants d_1, d_2.*

Theorem 4.1.2. *The birth-death chain $\{X_t; t \geq 0\}$ on $\{0, 1, \dots, N\}$, $N \geq 3$, preserves the class of stable measures on $\{0, 1, \dots, N\}$ if and only if there are*

The material in this chapter is based on [35].

constants b_1, b_2, d_1, d_2 such that the birth and death rates satisfy

$$\beta_k = b_1(N-k) + b_2(N-k)^2, \quad \delta_k = d_1 k + d_2 k^2. \tag{4.1.1}$$

One example is the pure death chain with rates $\delta_k = k(k-1)/2$, which expresses the number of ancestral genealogies in Kingman's coalescent — a well-studied model in mathematical biology [21, 23, 51]. In particular, by taking the initial number of particles to infinity, we obtain that the number of ancestors at any fixed time has the distribution of a sum of independent Bernoulli and Poisson random variables.

4.2 General Stable Measures

Suppose μ is a measure on $\{0, 1, 2, \dots\}^n$. The generating function of μ is now the formal power series

$$f_\mu(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n = 0}^{\infty} \mu(i_1, \dots, i_n) x_1^{i_1} \cdots x_n^{i_n}. \tag{4.2.1}$$

If μ has finite support, then f_μ is a polynomial.

Definition 4.2.1. *We say that a function $f(\mathbf{x})$ defined on $\mathbb{C}^n$ is transcendental stable, or t-stable, if there exist stable polynomials $\{f_m(\mathbf{x})\}$ such that $f_m \to f$ uniformly on all compact subsets of $\mathbb{C}^n$ (f can then be expressed as an absolutely convergent power series on $\mathbb{C}^n$). Let $\overline{\mathfrak{S}[\mathbf{x}]}$ be the set of all t-stable functions — this is also known as the Laguerre-Pólya class* [27]*. Let $\overline{\mathfrak{S}_{\mathbb{R}}[\mathbf{x}]}$ be the set of all real t-stable functions.*

We will again abuse notation and say that a measure μ on $\mathbb{N}^n$ is *transcendental stable* (or *t-stable*) if its generating function lies in $\overline{\mathfrak{S}_{\mathbb{R}}[\mathbf{x}]}$. Similarly, if μ has finite support and its generating polynomial is stable then we say that μ is stable. Of course, a stable measure is automatically t-stable.

The papers by Borcea and Brändén [6, 8] characterized the linear transformations preserving stable polynomials by establishing a bijection between linear transformations preserving n-variable stability and t-stable powers series in $2n$ variables — see Theorems 2.3.2 and 2.4.3 above. We will not require their full result here; however, the following characterization of t-stable powers series — the technical cornerstone upon which the above bijection rests — will be most useful.

Recall the standard partial order on $\mathbb{N}^n$: $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq n$. Then for $\alpha, \beta \in \mathbb{N}^n$ and letting $\beta! = \beta_1! \cdots \beta_n!$, define

$$(\beta)_\alpha = \frac{\beta!}{(\beta - \alpha)!} \text{ if } \alpha \leq \beta, \quad (\beta)_\alpha = 0 \text{ otherwise.} \tag{4.2.2}$$

Theorem 4.2.2 (Theorem 6.1 of [6]). *Let $f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$ be a formal power series in $\mathbf{x}$ with coefficients in $\mathbb{R}$. Set $\beta_m = (m, m, \ldots, m) \in \mathbb{N}^n$. Then $f(\mathbf{x}) \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$ (thus f is entire) if and only if*

$$f_m(\mathbf{x}) := \sum_{\alpha \leq \beta_m} (\beta_m)_\alpha c_\alpha \left(\frac{\mathbf{x}}{m}\right)^\alpha \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}] \cup \{0\}, \tag{4.2.3}$$

for all $m \in \mathbb{N}$. In this case, the polynomials $f_m(\mathbf{x}) \to f(\mathbf{x})$ uniformly on compact sets.

This classification also has the following immediate consequence:

Corollary 4.2.3. *The class $\overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$ is closed under convergence of coefficients. In particular, the set of t-stable probability measures on $\mathbb{N}^n$ is closed under weak convergence.*

Proof. Suppose that for each n,

$$f^{(n)}(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha^{(n)} \mathbf{x}^\alpha \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}, \tag{4.2.4}$$

with $c_\alpha^{(n)} \to c_\alpha$ for each α. Then for each m the stable polynomials $f_m^{(n)}(\mathbf{x})$, defined in (4.2.3) above, converge normally to the polynomial f_m likewise obtained from f. Hurwitz's Theorem implies that each f_m is stable, and applying Theorem 4.2.2 again we conclude that $f \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$. $\square$

By the following proposition, we can say that a measure on $\{0, 1, 2, \ldots\}^S$ is t-stable if every projection onto a finite subset of coordinates is a t-stable measure.

Proposition 4.2.4. *The class of t-stable measures is closed under projections onto subsets of coordinates.*

Proof. It suffices to consider projections of n coordinates onto $n-1$ coordinates. Suppose that μ is a t-stable measure on $\mathbb{N}^n$. If $f(x_1, \ldots, x_n)$ is its generating function, notice that the generating function of the projection of μ onto $\mathbb{N}^{n-1}$ is $f(x_1, \ldots, x_{n-1}, 1)$. By Theorem 4.2.2, it suffices to show that the approximating polynomials $f_m(x_1, \ldots, x_{n-1}, 1)$ are stable. But this follows by considering the (complex) stable polynomials $f_m(x_1, \ldots, x_{n-1}, 1 + i/k)$ and applying Hurwitz's theorem (Theorem 2.0.2) as $k \to \infty$. $\square$

We can now give an extension of Theorem 2.6.4.

Theorem 4.2.5. *Suppose μ is a t-stable probability measure on $\{0, 1, 2, \ldots\}^S$. Then μ is negatively associated.*

Proof. By the previous proposition and a limiting argument it is sufficient to show the result for measures on $\mathbb{N}^n$. Let $f(\mathbf{x})$ be the generating function of μ. By definition, $f \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$. Let $\{f_N(\mathbf{x})\}$ be the stable polynomials converging to f as in Theorem 4.2.2, which we can assume are normalized so that $f_N(1) = 1$. Let μ_N be the respective probability measures on $\{0, 1, 2, \ldots, N\}^n$. Hence by the

Grace-Walsh-Szegö theorem (Theorem 2.4.1), $\mathrm{Pol}_N f_N$ is the generating function for a stable measure $\tilde{\mu}_N$ on $\{0,1\}^{nN}$. Let

$$\{\zeta_{ij}; 1 \le i \le n, 0 \le j \le N\} \tag{4.2.5}$$

be the coordinates of $\tilde{\mu}_N$, such that $\eta_i = \sum_j \zeta_{ij}$ is the i-th coordinate of μ_N. Hence for bounded increasing functions F and G on $\{0,1,2,\dots\}^n$, depending on disjoint sets of coordinates, we have

$$\begin{aligned}
&\mathbb{E}^{\mu_N}[F(\eta_1,\dots,\eta_n)G(\eta_1,\dots,\eta_n)] \\
&= \mathbb{E}^{\mu_N}\left[F\Big(\sum_j \zeta_{1j},\dots,\sum_j \zeta_{nj}\Big)G\Big(\sum_j \zeta_{1j},\dots,\sum_j \zeta_{nj}\Big)\right] \\
&\le \mathbb{E}^{\mu_N}\left[F\Big(\sum_j \zeta_{1j},\dots,\sum_j \zeta_{nj}\Big)\mathbb{E}^{\mu_N}\left[G\Big(\sum_j \zeta_{1j},\dots,\sum_j \zeta_{nj}\Big)\right.\right. \\
&= \mathbb{E}^{\mu_N}F(\eta_1,\dots,\eta_n)\mathbb{E}^{\mu_N}G(\eta_1,\dots,\eta_n)
\end{aligned}$$

The inequality above follows because $F(x_{11}+\cdots+x_{1N},\dots,x_{n1}+\cdots+x_{nN})$ is an increasing function in the nN variables (similarly with G), and the ζ_{ij} are all negatively associated by Theorem 2.6.4. The normal convergence of $f_N \to f$ implies the weak convergence $\mu_N \to \mu$, concluding the proof. □

We can also characterize all t-stable measures on one coordinate. The following lemma is an almost trivial version of a Szász principle [49] — the observation being that a polynomial with no zeros in the upper half plane has its growth determined by the first few coefficients.

Lemma 4.2.6. *Suppose*

$$f(z) = \sum_{j=1}^{n} c_j z^j = c_0 \prod_{j=1}^{n} \left[1 - z/a_j\right], \tag{4.2.6}$$

where $c_0 > 0$ and all zeros a_j of f are strictly negative. Then $|f(z)| \le c_0 e^{|z|c_1}$.

Proof. Notice that $\sum_j |\frac{1}{a_j}| = \frac{c_1}{c_0}$. Hence

$$|f(z)| = c_0 \prod |1 - z/a_j| \le c_0 \prod e^{|z/a_j|} = c_0 e^{|z| c_1/c_0}. \tag{4.2.7}$$

□

Proposition 4.2.7. *A probability measure on $\{0, 1, 2, \dots\}$ is transcendental stable if and only if it has the same distribution as a (possibly infinite) sum of independent Bernoulli random variables and a Poisson random variable.*

Proof. Suppose f is a t-stable generating function for a non-negative, integer valued random variable. By Theorem 4.2.2, f is a normal limit of univariate polynomials f_n with all zeros on the negative real axis. As in the proof of Proposition 2.6.5, we first show that f can be expressed as the following infinite product:

$$f(z) = e^{\sigma(z-1)} \prod_{k=1}^{\infty} \frac{z - a_k}{1 - a_k}, \tag{4.2.8}$$

for some $\sigma \le 0$, $a_k \le 0$ and $\sum (1 - a_k)^{-1} < \infty$. Indeed, recalling the argument beneath equation (2.6.16), all we need to show (4.2.8) is an exponential bound of the form

$$|f(z)| \le C e^{c|z|}. \tag{4.2.9}$$

Since the coefficients of the f_n converge to the coefficients of f, and without loss of generality we can assume $f(0) \ne 0$, the exponential bound is provided by the (baby) Szász principle above (Lemma 4.2.6).

Thus

$$f(z) = e^{\sigma(z-1)} \prod_{k=1}^{\infty} \frac{z - a_k}{1 - a_k} = e^{\sigma(z-1)} \prod_{k=1}^{\infty} [(1 - p_k) + z p_k], \tag{4.2.10}$$

where $p_k = 1/(1 - a_k)$. This we recognize as the generating function for the sum of independent Poisson($-\sigma$) and Bernoulli(p_k) random variables. Conversely, any generating function of this form with $\sum p_k < \infty$ is automatically t-stable, as e^x is the normal limit of the polynomials $(1 + x/n)^n$. Note that σ can be strictly positive, in contrast to Proposition 2.6.5, because here we are allowing arbitrary normal limits of polynomials, instead of the sequence of polynomials generated from the occupation variables of the first n coordinates. □

By projecting onto finite subsets of coordinates (taking limits if need be) and setting all variables in the resulting generating function to be equal, we obtain the following extension of Theorem 2.6.5:

Corollary 4.2.8. *Suppose μ is a t-stable measure on $\mathbb{N}^S$. Then for any $T \subset S$ the number of particles located in T — according to μ — has the distribution of a sum of independent Bernoulli and Poisson random variables.*

4.2.1 Markov processes and stability

Suppose $\{\eta_t; t \geq 0\}$ is a Markov process on $\mathbb{N}^n$. We define the associated linear operator T_t on power series with bounded coefficients by letting $T_t(\mathbf{x}^\alpha)$ be the generating function of $\{\eta_t | \eta_0 = \alpha\}$ for each $\alpha \in \mathbb{N}^n$, and extending by linearity. This is well-defined because $\sum_{k \geq 0} P(\eta_t = k) = 1$.

Definition 4.2.9 (Preservation of stability)**.** *We say that a Markov process $\{\eta_t; t \geq 0\}$ on $\mathbb{N}^n$ preserves stability if for any stable initial distribution, the distribution at any later time is t-stable. That is, the associated linear operator T_t maps the set of stable polynomials with non-negative coefficients into the set of t-stable power series.*

The process η_t preserves t-stability if for any t-stable initial distribution, the distribution at a later time is again t-stable. That is, T_t maps the set of t-stable power series with non-negative coefficients into itself.

In fact, these two definitions are equivalent.

Proposition 4.2.10. *A Markov process preserves t-stability if and only if it preserves stability.*

Proof. Only one direction needs proof. Assume the process preserves stability. Let T_t be the associated linear operator, and $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ be the generating function of a t-stable distribution; hence $c_\alpha \geq 0$ for all α and $\sum_\alpha c_\alpha = 1$. By Theorem 4.2.2 there are stable polynomials $f_n = \sum_\alpha c_\alpha^{(n)} \mathbf{x}^\alpha$ with $c_\alpha^{(n)} \to c_\alpha$, all $c_\alpha^{(n)} \geq 0$, and with $\sum_\alpha c_\alpha^{(n)} \leq 1$. Suppose that

$$T_t(\mathbf{x}^\alpha) = \sum_\beta d_{\alpha,\beta} \mathbf{x}^\beta. \tag{4.2.11}$$

Since probability is conserved, $\sum_\beta d_{\alpha,\beta} = 1$, and hence by dominated convergence,

$$\sum_\alpha c_\alpha^{(n)} d_{\alpha,\beta} \longrightarrow \sum_\alpha c_\alpha d_{\alpha,\beta} \quad \text{as } n \to \infty. \tag{4.2.12}$$

In other words, the coefficients of $T_t f_n$ (which is t-stable by assumption), converge to the coefficients of $T_t f$, which is then t-stable by Corollary 4.2.3. □

We now give a couple examples.

4.2.2 Independent Markov chains

Suppose $\{X_t(1), X_t(2), \dots\}$ is a collection of independent Markov chains on S with identical jump rates. Set

$$\eta_t(x) = \sum_{i \geq 1} 1_{\{X_t(i)=x\}}, \tag{4.2.13}$$

so that the resulting process is a collection of particles on S jumping independently with the same rates. This is well defined as long as $\eta_t(x) < \infty$ for all $x \in S,\ t \geq 0$ — one possibility is to restrict initial configurations to the space E_0 defined below for reaction-diffusion processes.

Proposition 4.2.11. *The process $\{\eta_t; t \geq 0\}$ preserves t-stability. Hence, assuming that the initial distribution is t-stable, the distribution at any time is negatively associated by Theorem 4.2.5.*

Proof. Let μ_t be the distribution of η_t, with μ_0 t-stable. We need to show that for each finite $T \subset S$, the projection $\mu_t|_T$ is t-stable. Taking finite $T \subset S_1 \subset S_2 \subset \cdots$ with each S_n finite and $S_n \nearrow S$, we can approximate $\mu_t|_T$ by the sequence $\mu_t^{(n)}|_T$, with each $\mu_t^{(n)}$ the distribution of the independent Markov chain process on S_n given initial distribution $\mu_0|_{S_n}$ and jumps restricted to staying inside S_n. Hence by Corollary 4.2.3 we can assume finite S.

Suppose now that $S = [n]$, and only jumps from site 1 to site 2 are allowed. In this case, assuming a jump rate $q(1,2)$, each particle at x independently has probability $p := 1 - e^{-tq(1,2)}$ of moving to y. Hence the associated linear operator T_t takes

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mapsto (px_2 + (1-p)x_1)^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \tag{4.2.14}$$

that is,

$$T_t f(x_1, \ldots, x_n) = f(px_2 + (1-p)x_1, x_2, \ldots, x_n), \tag{4.2.15}$$

which preserves the class of stable polynomials. By permuting variables, this argument also holds for general $i, j \in [n]$.

Recalling the Banach space $C_0(\mathbb{N}^n)$ of functions that vanish at infinity, consider the strongly continuous contraction semi-groups $S^{i,j}(t)$ on $C_0(\mathbb{N}^n)$ defined by

$$S^{i,j}(t) f(\eta) = \mathbb{E}^\eta f(\eta_t^{i,j}), \tag{4.2.16}$$

where $\{\eta_t^{i,j}; t \geq 0\}$ is the (Feller) process of independent Markov chains which only allow jumps from site i to site j. Then for a t-stable intial distribution μ, we just showed that $\mu S^{i,j}(t)$ — the distribution of $\eta_t^{i,j}$ assuming initial distribution

μ — is again t-stable. By Trotter's product theorem [17, p. 33], the process allowing the jumps $\{i \mapsto j\}, \{k \mapsto l\}$ has semigroup

$$S(t) = \lim_{n\to\infty} \left[S^{i,j}\Big(\frac{t}{n}\Big) S^{k,l}\Big(\frac{t}{n}\Big)\right]^n. \tag{4.2.17}$$

Hence by Corollary 4.2.3, $\mu S(t)$ is again t-stable. Including all the possible jumps one-by-one, we conclude that the whole process $\{\eta_t; t \geq 0\}$ on $\mathbb{N}^n$ preserves t-stability. □

Remark 4.2.12. *Let $\mathcal{M}$ be the set of probability measures on $\mathbb{N}^S$ described by random configurations η with coordinates*

$$\eta(k) = \sum_i 1_{\{Y_i = k\}}, \tag{4.2.18}$$

where $Y_1, Y_2, \dots$ are independent random variables with values in $S \cup \{\infty\}$. In [33] *it was shown that $\mathcal{M}$ is preserved by the process of independent Markov chains, and that measures in $\mathcal{M}$ are negatively associated. Proposition 4.2.11 is a generalization of this result, since it is easily checked that the class $\mathcal{M}$ is contained in the class of t-stable measures. Indeed, if $\mu \in \mathcal{M}$ and $S = [n]$, then μ has a generating function of the form*

$$\begin{aligned} &\mathbb{E}^\mu \left[x_1^{\sum_i 1_{\{Y_i=1\}}} \cdots x_n^{\sum_i 1_{\{Y_i=n\}}} \right] \\ &= \prod_i [\mu(Y_i = \infty) + \mu(Y_i = 1)x_1 + \cdots + \mu(Y_i = n)x_n], \end{aligned}$$

by the independence of the Y_i's. Furthermore, (non-constant) product measures for which each coordinate is a sum of independent Bernoulli and Poisson measures are t-stable, but are not contained in the class $\mathcal{M}$.

4.2.3 Reaction-diffusion processes

In addition to having the motion of particles following independent Markov chains, we can also allow particles to undergo a reaction at each site. Let $p(i, j)$

be transition probabilities for a Markov chain on S. Given a state $\eta \in \mathbb{N}^S$, we consider the following evolution:

1. at rate $\beta^i_{\eta(i)}$ a particle is created at site i,
2. at rate $\delta^i_{\eta(i)}$ a particle at site i dies.
3. at rate $\eta(i)p(i,j)$ a particle at site i jumps to site j.

The most common example is the *polynomial model* of order m, where the birth-death rates for each site are

$$\beta_k = \sum_{j=0}^{m-1} b_j k(k-1)\cdots(k-j+1), \quad \delta_k = \sum_{j=1}^{m} d_j k(k-1)\ldots(k-j+1). \tag{4.2.19}$$

Reaction-diffusion processes originated as a model for chemical reactions [44], and subsequent work by probabilists has focused on the ergodic properties [3, 10, 15].

To construct the process, we require a strictly positive sequence k_i on the index set S, and a positive constant M such that

$$\sum_j p(i,j)k_j \le Mk_i, \quad i \in S. \tag{4.2.20}$$

Furthermore, the birth rates must satisfy

$$\sum_i \beta^i_0 k_i < \infty. \tag{4.2.21}$$

Then we take the state space of the process to be

$$E_0 = \{\eta \in \mathbb{N}^S : \sum_i \eta(i)k_i < \infty\}. \tag{4.2.22}$$

See [10, chapter 13.2] for the details of the construction.

In a very simple case we have preservation of t-stability:

Proposition 4.2.13. *Suppose the reaction-diffusion process is well-constructed with* $\beta_k^i = b^i$ *and* $\delta_k^i = d^i k$ *— this is the polynomial model of order* 1 *and site-varying reaction rates. Then the process preserves t-stability, and hence — assuming a t-stable initial configuration — its distribution at any time is negatively associated.*

Proof. The strategy here is the same as with Proposition 4.2.11. To reduce to a reaction-diffusion process on a finite number of sites, we approximate using the construction in [10, Theorem 13.8]. Furthermore, on the locally compact space $\mathbb{N}^n$, the reaction-diffusion process with at most constant birth and linear death rates is now a Feller process, as can be seen from [17, Theorem 3.1, Ch. 8]. Hence by Trotter's product formula and Corollary 4.2.3 we only need to show that the following processes preserve stability on $\mathbb{N}^n$:

1. constant birth rate b^i at a single site i.
2. linear death rates at a single site i ($\delta_k^i = d^i k$),
3. jumps from site i to site j at rate $\eta(i)p(i,j)$

For (1), we note that with a constant birth rate, at time t a Poisson($b^i t$) number of particles has been added to the system — i.e. the original generating function is multiplied by $e^{b^i t(x_i - 1)}$, preserving t-stability.

(2) can be thought of as the process in which each particle at site i dies independently at rate d^i. Hence the associated linear transform is defined by

$$T_t(x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n}) = x_1^{\alpha_1} \cdots [1 - e^{-d^i t} + e^{-d^i t} x_i]^{\alpha_i} \cdots x_n^{\alpha_n}. \qquad (4.2.23)$$

As the affine transformation $x \mapsto ax + (1-a)$, ($a > 0$) maps the upper half plane onto itself, T_t preserves preserves the class of stable polynomials.

Finally, we note that (3) was already seen to preserve stability from the proof of Proposition 4.2.11 for independent Markov chains. □

Other reaction-diffusion processes do not preserve stability in general. On one coordinate, a reaction-diffusion process is just a birth-death chain, so by Theorem 4.1.1 the only possible generalization would be to quadratic death rates. In this case, unfortunately, the associated linear transformation T_t will *not* preserve stable polynomials with positive roots; indeed, assuming death rates $\delta_k = k(k-1)$, quadratic polynomials with a double root inside the interval $(0, 1)$ will not be mapped to stable polynomials under T_t. Furthermore, the following example shows that quadratic death rates on multiple sites does not always preserve the class of stable probability measures:

Example 4.2.14. *Consider the following generating function:*

$$f(x, y) = \frac{1}{4}(x^2 + 2xy + y^2). \tag{4.2.24}$$

This is obviously stable, and corresponds to an initial distribution of two particles on two sites. Now run a pure death chain only on the site associated with the variable y, with rates $\delta_k = k(k-1)$. After any fixed time $t > 0$, the generating function will be of the form

$$f_t(x, y) = \frac{1}{4}(x^2 + (2x + p_t)y + (1 - p_t)y^2 \text{ for some } 0 < p_t < 1. \tag{4.2.25}$$

As a quadratic equation in y, the discriminant is $(p_t^2 + 4xp_t(x + 1))/16$, hence $f_t(x, y) = 0$ for $x = -1/2$ and some $y \in \mathbb{H}$. By restriction (Proposition 2.0.4), f_t cannot be stable.

4.3 Birth-Death Chains

Our goal in this section is to prove Theorems 4.1.1 and 4.1.2. As just noted above, in the case of quadratic death rates the associated linear transformation does not preserve all polynomials with real zeros, and hence we cannot rely on general classifications such as Theorem 2.4.3. From the probabilistic point of view, it would be useful to have a similar theory for linear transformations on polynomials with positive coefficients, however, this is a significant open problem in complex function theory. Instead, we will make do with several perturbation arguments — the main idea being that a polynomial's roots move continuously under changes to its coefficients.

Recall that a (continuous-time) birth-death chain $\{X_t; t \geq 0\}$ is a Markov process on the non-negative integers with transitions

$$k \mapsto k+1 \quad \text{at rate } \beta_k, \quad k \mapsto k-1 \quad \text{at rate } \delta_k, \quad \text{with } \delta_0 = 0. \tag{4.3.1}$$

We only consider rates $\{\beta_k, \delta_k\}$ such that the process does not blow up in finite time — see Chapter 2 of [32] for the necessary and sufficient conditions, as well as for the construction of the process.

The generating function at time t is given by

$$\phi(t,z) = \sum_{k=0}^{\infty} P(X_t = k) z^k. \tag{4.3.2}$$

By Theorem 2.14 of [32], the transition probabilities $p_t(j,k) = P^j(X_t = k)$ are continuously differentiable in t and satisfy the Kolmogorov backward equations

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(j,k) = \beta_j p_t(j+1,k) + \delta_j p_t(j-1,k) - (\beta_j + \delta_j) p_t(j,k). \tag{4.3.3}$$

We will need the following lemma:

Lemma 4.3.1. *Suppose $\phi(t,z)$ is t-stable for all $t \geq 0$, and that $\phi(0,z)$ is a polynomial of degree $n \geq 3$ with roots z_i satisfying $-1 < z_1 = z_2 < z_3 \leq, \dots, z_n < 0$. Then*

$$\frac{\partial \phi}{\partial t}(0, z_1) = 0. \tag{4.3.4}$$

Proof. By Theorem 2.13 of [32], $|p_t(j,k) - p_s(j,k)| \leq 1 - p_{|t-s|}(j,j)$. It follows that

$$\frac{\partial}{\partial t}\phi(t,z) = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} P(X_t = k) z^k \tag{4.3.5}$$

for $|z| < 1$, since $X(0)$ is bounded. Iterating this argument, one sees that $\phi(t,z)$ is C^2 on $[0,\infty) \times \{z : |z| < 1\}$.

By assumption, the generating function $\phi(t,z)$ of X_t has only real roots for $t > 0$. If $z_1 = z_2 = w$ is a root of $\phi(0,z)$ of multiplicity exactly two, and ϵ is small enough that $|z_k - w| > \epsilon$ for $k \geq 3$, then Rouché's Theorem implies that for sufficiently small $t > 0$, $\phi(t,z)$ has exactly two roots in the disk $\{z : |z - w| < \epsilon\}$. Therefore, for small $t > 0$, there exist real $z(t)$ so that $\phi(t, z(t)) = 0$ and $\lim_{t\downarrow 0} z(t) = w$. By Taylor's Theorem, there exist $s(t) \in [0,t]$ and $y(t)$ between $z(t)$ and w so that

$$\begin{aligned} 0 =& \phi(t,z(t)) = t\frac{\partial \phi}{\partial t}(0,w) + \frac{1}{2}t^2 \frac{\partial^2 \phi}{\partial t^2}(s(t), y(t)) \\ &+ t(z(t) - w)\frac{\partial^2 \phi}{\partial t \partial z}(s(t), y(t)) + \frac{1}{2}(z(t) - w)^2 \frac{\partial^2 \phi}{\partial z^2}(s(t), y(t)). \end{aligned} \tag{4.3.6}$$

Dividing by t and letting $t \downarrow 0$ leads to

$$2\frac{\partial \phi}{\partial t}(0,w) \Big/ \frac{\partial^2 \phi}{\partial z^2}(0,w) = -\lim_{t\downarrow 0} \frac{(z(t) - w)^2}{t} \leq 0. \tag{4.3.7}$$

Noting that

$$\frac{\partial^2 \phi}{\partial z^2}(0,w) = 2c\prod_{k=3}^{n}(w - z_k), \tag{4.3.8}$$

we see that

$$\frac{\partial \phi}{\partial t}(0,w) \tag{4.3.9}$$

changes sign when z_3 crosses w, and hence is zero when $z_3 = w$. $\square$

Proposition 4.3.2. *Suppose the birth-death process* $\{X_t; t \geq 0\}$ *is a birth-death chain preserving stable polynomials of degree* $N \geq 3$*. Then there exist constants* b_1, b_2, d_1, d_2 *so that*

$$\beta_k = b_1(N-k) + b_2(N-k)^2 \quad \textit{and} \quad \delta_k = d_1 k + d_2 k^2. \tag{4.3.10}$$

Proof. Take $\phi(0, z)$ satisfying the conditions of the above lemma. We now compute (4.3.9). Recall the k^{th} elementary symmetric polynomials $e_k(x_1, \ldots, x_n)$ defined in (2.4.1). If μ is the distribution of X_0 and $X_0 \leq n$, then

$$\phi(0, z) = \sum_{k=0}^{n} \mu(k) z^k = c \prod_{k=1}^{n} (z - z_k) = c \sum_{k=0}^{n} (-1)^k e_k(z_1, ..., z_n) z^{n-k}, \tag{4.3.11}$$

so

$$\mu(k) = c(-1)^{n-k} e_{n-k}(z_1, ..., z_n). \tag{4.3.12}$$

Therefore for $|z| < 1$,

$$\frac{\partial \phi}{\partial t}(0, z) = \sum_{l=0}^{\infty} \sum_{k=0}^{n} \mu(k) q(k, l) z^l = c(1-z) \sum_{k=0}^{n} (-1)^{n-k} e_{n-k}(z_1, ..., z_n)[\delta_k z^{k-1} - \beta_k z^k], \tag{4.3.13}$$

where we have $\delta_0 = \beta_N = 0$. From Lemma 4.3.1 it follows that the expression on the right is zero if $z = z_1 = z_2 = z_3 = w$ for any values of $w, z_4, ..., z_n \in (-1, 0)$. In this case,

$$e_k(z_1, ..., z_n) = \sum_i w^i \binom{3}{i} e_{k-i}(z_4, ..., z_n), \tag{4.3.14}$$

and so

$$\sum_k \sum_i (-1)^{n-k} \binom{3}{i} e_{n-k-i}(z_4, ..., z_n)[\delta_k w^{k+i-1} - \beta_k w^{k+i}] \equiv 0, \tag{4.3.15}$$

where the boundary conditions are enforced by

$$\binom{3}{i} = 0, \text{ for } i < 0, \ i > 3, \text{ and } e_k(z_4, \ldots, z_n) = 0 \text{ for } k \leq 3, \ k > n. \tag{4.3.16}$$

Interchanging the order of summation and letting $k \mapsto k - i$, we see that the coefficient of each of the e_{n-k}'s is zero:

$$\sum_i \binom{3}{i} (-1)^i [\delta_{k-i} - \beta_{k-i} w] = 0, \tag{4.3.17}$$

or equivalently $\delta_k - 3\delta_{k+1} + 3\delta_{k+2} - \delta_{k+3} = 0$ and $\beta_k - 3\beta_{k+1} + 3\beta_{k+2} - \beta_{k+3} = 0$. Recalling that we set $\delta_k = 0 = \beta_N$, we obtain the restriction in (4.3.10). □

With the next proposition we resolve the "only if" part of Theorem 4.1.1.

Proposition 4.3.3. *The birth-death chain preserves t-stability only if the birth rate is constant and there are $d_1, d_2 \in \mathbb{R}$ such that $\delta_k = d_1 k + d_2 k^2$ for all k.*

Proof. The proof of Prop 4.3.2 holds, except that in the case of unbounded birth-death chains we don't have a boundary condition on β_N, hence we obtain the weaker condition that there exist $b_0, b_1, b_2 \in \mathbb{R}$ such that $\beta_k = b_0 + b_1 k + b_2 k^2$ for all k. The condition on δ_k remains unchanged. Hence we will now show that the birth rates β_k satisfy $\beta_k \geq \beta_{k+1}$ for each k, so that β_k must be constant.

By iterating the Kolmogorov backward equations, one can obtain the following approximations for small $t > 0$:

$$p_t(k, k+1) = t(\beta_k + o(1)), \quad p_t(k, k+2) = \frac{t^2}{2}(\beta_k \beta_{k+1} + o(1)). \tag{4.3.18}$$

Similarly,

$$p_t(k, k-j) = O(1) t^j, \tag{4.3.19}$$

where $O(1)$ denotes a uniformly bounded quantity, and $o(1) \to 0$, as $t \to 0$.

Suppose that we start the chain with k particles; the initial distribution has generating function $f(x) = x^k$. We also can assume that $\beta_k, \beta_{k+1} > 0$. Then the generating function for small $t > 0$ will be:

$$f_t(x) = \cdots + (1 + o(1)) x^k + (\beta_k + o(1)) t x^{k+1} + (\beta_k \beta_{k+1} + o(1)) \frac{t^2}{2} x^{k+2} + \cdots \tag{4.3.20}$$

Since $f_t(x)$ is t-stable, by Theorem 4.2.2 the following polynomial has all real, negative roots:

$$f_{t,k+2}(x) = \cdots + (1+o(1))x^k + \frac{2(\beta_k + o(1))}{k+2} t x^{k+1} + \frac{\beta_k \beta_{k+1} + o(1)}{(k+2)^2} t^2 x^{k+2}. \quad (4.3.21)$$

As the hidden coefficients are $o(1)$, Rouché's Theorem implies that k roots of $f_{t,k+2}$ are also $o(1)$. Thus the remaining two roots a, b satisfy

$$\begin{aligned} a + b &= \frac{2(k+2) + o(1)}{\beta_{k+1} t} \\ ab &= \frac{(k+2)^2 + o(1)}{\beta_k \beta_{k+1} t^2}. \end{aligned}$$

Solving for real a, b implies that the discriminant

$$4t^{-2}(k+2)^2[\beta_{k+1}^{-2} - (\beta_k \beta_{k+1})^{-1} + o(1)] \geq 0, \quad \text{for small } t > 0. \quad (4.3.22)$$

Taking $t \to 0$, we conclude that $\beta_k \geq \beta_{k+1}$. □

We now concentrate on the "if" part of Theorem 4.1.1. Here is a useful fact about quadratic death rates:

Proposition 4.3.4. *Suppose μ_t is the evolving distribution of a birth-death chain with rates $\beta_k = 0$, $\delta_k = k(k-1)$, and with finite initial distribution μ. Then the generating function*

$$\phi(t, z) = \sum_{k \geq 0} \mu_t(k) z^k \quad (4.3.23)$$

satisfies the Wright-Fisher partial differential equation:

$$\frac{\partial}{\partial t}\phi(t, z) = z(1-z)\frac{\partial^2}{\partial z^2}\phi(t, z).$$

Proof. Let $p_t(l, k) = P(X_t = k | X_0 = l)$ be the transition probabilities. Since the process stays bounded we can use the forward equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(l,k) = (k+1)kp_t(l,k+1) - k(k-1)p_t(l,k) \tag{4.3.24}$$

Then, since $\mu_t(k) = \sum_l \mu(l)p_t(l,k)$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\phi(t,z) =& \sum_{k,l}\mu(l)\frac{\mathrm{d}}{\mathrm{d}t}p_t(l,k)z^k \\ =& \sum_{k,l}\mu(l)[(k+1)kp_t(l,k+1) - k(k-1)p_t(l,k)]z^k \\ =& z(1-z)\sum_{k,l}\mu(l)k(k-1)p_t(l,k)z^{k-2} = z(1-z)\frac{\partial^2}{\partial z^2}\phi(t,z).\end{aligned}$$

□

Proposition 4.3.5. *The birth-death chain with rates* $\beta_k = 0$, $\delta_k = k(k-1)$ *preserves stability.*

Proof. As above, let $\phi(t,z)$ be the generating function of the chain at time t, and now assume that $\phi(0,z)$ has only real roots. Setting

$$\tau = \inf\{t \geq 0;\ \phi(t,z) \text{ is not stable}\}, \tag{4.3.25}$$

by Hurwitz's Theorem (Theorem 2.0.2) $\phi(\tau,z)$ is stable. Hence it suffices to prove that for any stable initial distribution there exists an $\epsilon > 0$ such that $\phi(t,z)$ is stable for all $0 < t < \epsilon$.

For any root $w < 0$ of $\phi(0,t)$, we can write

$$\phi(0,z) = (z-w)^n q(z), \tag{4.3.26}$$

where n is the multiplicity and $q(w) \neq 0$. We will show that for all small enough t the generating function $\phi(t,z)$ has n distinct real roots of distance approximately $\sqrt{t}$ from w. This follows directly from showing that

$$\phi(t, w + \alpha t^{1/2}) = t^{n/2}p(\alpha) + o(t^{n/2}),\ \text{as } t \downarrow 0, \tag{4.3.27}$$

where $p(\alpha)$ has only simple real roots. Indeed, (4.3.27) immediately implies that for all small enough t, $\phi(t,z)$ changes sign n times near w. Since we can do this for each w, this shows that $\phi(t,z)$ has only real roots.

Set $\kappa = \lfloor n/2 \rfloor$. To show (4.3.27), we Taylor expand $\phi(t,z)$ in the first variable, i.e., for small $t > 0$

$$\phi(t,z) = \sum_{k=0}^{\kappa} \frac{\partial^k \phi}{\partial t^k}(0,z)\frac{t^k}{k!} + o(t^{n/2}). \tag{4.3.28}$$

Now k iterations of Proposition 4.3.4 shows that

$$\frac{\partial^k \phi}{\partial t^k}(0,z) = [z(1-z)]^k \frac{\partial^{2k}\phi}{\partial z^{2k}}(0,z) + \sum_{j=1}^{2k} f_{j,k}(z)\frac{\partial^{2k-j}\phi}{\partial z^{2k-j}}(0,z), \tag{4.3.29}$$

where the $f_{j,k}(z)$ are polynomials in z. With $z = w + \alpha t^{1/2}$ we have, by the product rule,

$$\frac{\partial^{2k-j}\phi}{\partial z^{2k-j}}(0, w+\alpha t^{1/2}) = \frac{n!}{(n-2k+j)!}(\alpha t^{1/2})^{n-2k+j} q(w) + o(t^{\frac{n-2k+j}{2}}). \tag{4.3.30}$$

We've also used the fact that $q(w+\alpha t^{1/2}) = q(w) + o(1)$, because q has no zeros at w. From (4.3.30) we see that

$$f_{j,k}(w+\alpha t^{1/2})\frac{\partial^{2k-j}\phi}{\partial z^{2k-j}}(0, w+\alpha t^{1/2}) = o(t^{\frac{n-2k}{2}}), \tag{4.3.31}$$

and can be ignored for $j > 0$. Putting this all together, we have

$$\sum_{k=0}^{\kappa} \frac{t^k}{k!}\frac{\partial^k \phi}{\partial t^k}(0, w+\alpha t^{1/2}) = t^{n/2}\sum_{k=0}^{\kappa} \frac{n!}{k!(n-2k)!}(-1)^k [w(w-1)]^k \alpha^{n-2k} q(w) + o(t^{n/2}). \tag{4.3.32}$$

Recalling the (physics) Hermite polynomial of order n,

$$H_n(\alpha) = \sum_{k=0}^{\kappa} (-1)^k \frac{\alpha^{n-2k} n!}{k!(n-2k)!}, \tag{4.3.33}$$

we obtain

$$\phi(t, w+\alpha t^{1/2}) = t^{n/2}[w(w-1)]^{n/2} H_n(\alpha/2\sqrt{w(w-1)})q(w) + o(t^{n/2}). \tag{4.3.34}$$

H_n is well known (e.g. [50, §3.3]) to have n distinct real roots, which, in the case that $w < 0$ shows (4.3.27).

The case $w = 0$ requires an additional argument. It will turn out, assuming $\phi(0, z) = z^n q(z)$, that $\phi(t, z)$ has $n - 1$ distinct negative zeros with distance of order t from zero, not $t^{1/2}$.

To show this we again Taylor expand in t. By Proposition 4.3.4 we see that

$$\frac{\partial^k \phi}{\partial t^k}(0, z) = z(1 - z)\frac{\partial^2}{\partial z^2}\left[\cdots z(1 - z)\frac{\partial^2}{\partial z^2} z^n q(z)\right],$$

where the operator $z(1 - z)\partial_z^2$ is applied k times to $z^n q(z)$. If we evaluate this at $z = \alpha t$, we obtain

$$\frac{\partial^k \phi}{\partial t^k}(0, \alpha t) = [n(n - 1)] \cdots [(n - k + 1)(n - k)](\alpha t)^{n-k}\phi(0) + o(t^{n-k}). \quad (4.3.35)$$

Thus the Taylor expansion gives

$$\phi(t, \alpha t) = t^n \sum_{k=0}^{n-1} \frac{[n]_k [n - 1]_k}{k!} \alpha^{n-k} + o(t^n), \quad (4.3.36)$$

where

$$[n]_k = n(n - 1) \cdots (n - k + 1) \quad (4.3.37)$$

is the falling factorial. The sum can be rewritten as

$$\alpha^n n! {}_1F_1[1 - n, 1, -1/\alpha], \quad (4.3.38)$$

where ${}_1F_1[a, b, x]$ is Kummer's hypergeometric function of the first kind. Now ${}_1F_1[1 - n, 1, -1/x]$ has $n - 1$ distinct negative zeros $\{z_1, \ldots z_{n-1}\}$ [47, pp. 103]. Hence for t small enough we can interlace $n - 1$ zeros of $\phi(t, z)$ between the points $\{0, z_1, \ldots, z_{n-1}\}$, and so the single remaining zero of $\phi(t, z)$ near 0 must also be real.

□

Proof of Theorem 4.1.2. As in the proof of Proposition 4.2.13, the pure death chain with rates $\beta_k = b_1 k$ maps the class of stable polynomials of degree N onto itself; similarly, we just showed the same for the pure quadratic death chain. On our finite state space we can apply Trotter's product formula to combine the two processes, hence the pure death chain with rates $\beta_k = b_1 k + b_2 k^2$ preserves stable measures on $\{0, 1, \ldots, N\}$.

Now, a pure birth chain on $\{0, 1, \ldots, N\}$ with the rates $\beta_k = b_1(N - k) + b_2(N - k)^2$ is the same as the above pure death chain, after we relabel the sites such that $k \mapsto N - k$. Furthermore, the transformation $z \mapsto 1/z$ shows that

$$\sum_{k=0}^{N} a_k z^k \text{ is stable if and only if } \sum_{k=0}^{N} a_k z^{N-k} \text{ is stable,} \tag{4.3.39}$$

so we can conclude that this pure birth chain also preserves stable measures on $\{0, 1, \ldots, N\}$. We again apply Trotter's product formula to conclude Theorem 4.1.2. □

Proof of "if" direction in Theorem 4.1.1. By Proposition 4.2.13, the birth-death chain with constant birth and linear death rates preserves t-stability, and we just showed that the pure quadratic death chain preserves stability (and hence t-stability). However, the latter chain is no longer a Feller process, so we cannot immediately apply Trotter's product formula as we did in the proof of Theorem 4.1.2. Indeed, it is well known that a pure quadratic death chain comes down from infinity in finite time, in the sense that $\liminf_{k\to\infty} p_t(k, 1) > 0$ for each $t > 0$ [23].

We rectify this situation by considering the Banach space $l^1(\mathbb{N})$ of absolutely summable sequences. Let $X_t^{(1)}$, $X_t^{(2)}$, and $X_t^{(3)}$ be the birth-death chains with respective rates $\{\beta_k^{(1)} = b_0,\ \delta_k^{(1)} = d_1 k\}$, $\{\beta_k^{(2)} = 0,\ \delta_k^{(2)} = d_2 k(k-1)\}$, and

$\{\beta_k^{(3)} = b_0,\ \delta_k^{(3)} = d_1 k + d_2 k(k-1)\}$. With

$$P^{(i)}(t)f(x) = \sum_y f(y)P(X_t^{(i)} = x | X_0^{(i)} = y) \tag{4.3.40}$$

as the (adjoint) strongly continuous contraction semigroups on $l^1(\mathbb{N})$, we consider the infinitesimal generators as the l^1 limit

$$\Omega^{(i)}f = \lim_{t \downarrow 0} \frac{P^{(i)}(t)f - f}{t}. \tag{4.3.41}$$

See [43] for the theory of adjoint semigroups of Markov chains.

Let

$$\begin{aligned} D_0 &= \{f \in l^1(\mathbb{N});\ f(x) = 0 \text{ for all but finitely many } x\}, \\ D_e &= \{f \in l^1(\mathbb{N});\ |f(x)| \le Ce^{-x}\},\ C \text{ depending only on } f, \text{ and} \\ \mathcal{D}(\Omega^{(i)}) &= \{f \in l^1(\mathbb{N});\ \lim_{t \downarrow 0} t^{-1}(P^{(i)}(t)f - f) \text{ exists as an } l^1 \text{ limit.}\} \end{aligned}$$

By explicit calculation, it can be seen that $D_0 \subset D_e \subset \mathcal{D}(\Omega^{(i)})$ for each i, $P^{(i)}(t) : D_0 \to D_e$, and for $f \in D_e$,

$$\Omega^{(i)}f(x) = \delta_{x+1}^{(i)}f(x+1) + \beta_{x-1}^{(i)}f(x-1) - [\beta_x^{(i)} + \delta_x^{(i)}]f(x). \tag{4.3.42}$$

By [17, Prop. 3.3 of Ch. 1], D_e is a core for all three generators. Also,

$$\Omega^{(1)} + \Omega^{(2)} = \Omega^{(3)} \quad \text{on } D_e, \tag{4.3.43}$$

so we can apply Trotter's product formula to conclude preservation of t-stability for $X_t^{(3)}$. □

CHAPTER 5

Further directions

1. To derive the distributional limit of the current in the symmetric exclusion process, it was enough to show that the limiting variance is infinite. It is also natural to ask what the asymptotics of the current variance is. Assuming step-initial distribution, translation invariance and finite second moment of the jump rates, Liggett [34] showed existence of a $\delta > 0$ such that

$$\delta \leq t^{-1/2} \operatorname{Var} W_t \leq 1/\delta \text{ for all } t. \tag{5.0.1}$$

In the specific case of only nearest-neighbor jumps to either side at rate 1, Derrida and Gerschenfeld [14] showed that

$$\operatorname{Var} W_t \sim \frac{1}{\sqrt{\pi}} \left(1 - \frac{1}{\sqrt{2}}\right) t^{1/2} \text{ as } t \to \infty. \tag{5.0.2}$$

In general, it seems likely that $\operatorname{Var} W_t$ is always of the same order as $\mathbb{E}W_t$, the latter being easy to compute using duality. For example, in the case that the one-particle chain X_t is in the domain of attraction of a stable law of index $\alpha > 1$, we expect $\operatorname{Var} W_t$ to be of order $t^{1/\alpha}$.

2. In section 4.3 we fully classified the birth-death chains on the non-negative integers that preserve stability. Do any other Markov processes on the integers preserve stability? A tentative guess would be no. Indeed, suppose one allows more than two particles to die at a time in a death process. Doing a similar calculation to the one in Example 4.2.14 shows that starting from exactly two particles yields a generating function at small later times with negative discriminant, hence real zeros are not preserved.

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