The Cohomology Ring of the Moduli Space of Stable Vector Bundles with Odd Determinant *

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Abstract

The multiplicative structure of the cohomology ring of the moduli space of stable rank 2 bundles on a smooth projective curve is computed.

INTRODUCTION

Let C be a complete smooth curve of genus $g \geq 2$ over the field of complex numbers $\mathbb{C}$. Consider the moduli space N of stable rank 2 vector bundles on C with fixed determinant L of odd degree $2k-1$. Then N is a smooth projective variety of dimension $3g-3$. On the direct product $N \times C$ one has a universal bundle V with Chern classes (cf. [1])

$$c_1(V) = pr_N^*(A_1) + (2k-1)pr_C^*(\omega),$$

$$c_2(V) = pr_N^*(A_2) + \sum_{i=1}^{2g} pr_N^*(D_i) \cup pr_C^*(\beta_i) + k \cdot pr_N^*(A_1) \cup pr_C^*(\omega),$$

where $A_j \in H^{2j}(N, \mathbb{C}), D_i \in H^3(N, \mathbb{C}), \omega \in H^2(C, \mathbb{C})$, and (β_i) is a symplectic basis of $H^1(C, \mathbb{C})$ (i.e., $\beta_i \cup \beta_{g+s} = \delta_{is}\omega$ and $\beta_i \cup \beta_s = 0$ for $i, s = 1, \ldots, s$). One can show that A_i and D_j generate $H^*(N, \mathbb{C})$ as a ring (cf. [1]). Hence to describe the cohomology ring it suffices to compute the intersection numbers $\int_N A_1^n A_2^m \prod_{i=1}^{2g} D_i^{r_i}$ where $r_i \in \{0, 1\}$ and all the degrees add up to $6g-6$ (the real dimension of N), and to determine relations between A_1, A_2 and D_i.

Let $W = H^3(N, \mathbb{C})$, and let $\Lambda^*(W)$ be the exterior algebra over W. Note that the symplectic group $Sp(2g, \mathbb{C})$ acts on $\Lambda^*(W)$ (a symplectic transformation of $H^1(C, \mathbb{C})$ induces a transformation of W). The intersection form on N is Sp-invariant and, if we denote $D = \sum_{i=1}^{g} D_i \cup D_{g+i}$, the cohomology ring $H^*(N, \mathbb{C})$ as a module over $\mathbb{C}[A_1, A_2, D]$ splits into direct sum of submodules generated by primitive components of the Lefschetz decomposition for $\Lambda^*(W)$. We will show that each of these submodules can be recovered from the cohomology of the moduli spaces corresponding to smaller genera.

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Let $\widetilde{\pi}_1(C)$ be the universal central extension of the fundamental group $\pi_1(C)$. By Narasimhan-Seshadri Theorem, as a topological space, N parametrizes (up to congugacy) group homomorphisms $\widetilde{\pi}_1(C) \to SU(2)$ mapping the central element to $-1 \in SU(2)$. In particular, the structure of the cohomology ring of N does not depend on the complex structure of C.

Using the explicit realization of N for hyperelliptic curves given in [2], we succeed in carrying out the computations within the framework of the classical Schubert calculus (in particular, we do not use the Verlinde formula). A similar method was used by Laszlo [4] to compute the dimension of the space of conformal blocks. We refer to [6] for computation of the intersection form using the Verlinde formula.

In §1 we compute the intersection form on the cohomology of N. In §2 we construct a resolution of $H^*(N, \mathbb{C})$ as a module over the polynomial ring $C[A_1, A_2, D]$ (this involves determining relations between A_1, A_2, D_i).

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§1. THE INTERSECTION FORM

1.1. The moduli space and the Grassmann variety. We recall some results on the moduli space of bundles on a hyperelliptic curve [2].

Consider a general pencil of quadrics in $\mathbb{P}^{2g+1}$. A general element of this pencil is a smooth quadric containing two families of g-dimensional linear subspaces. The degenerate elements of this pencil (there are $2g+2$ of them) have one singular point and one family of g-dimensional linear subspaces. Hence the data

$$\{\text{quadric in the pencil} + \text{distinguished family of linear subspaces}\}$$

defines a double covering of $\mathbb{P}^1$ branched at $2g+2$ points, that is a hyperelliptic curve of genus g. Conversely, to each hyperelliptic curve C of genus g there corresponds a pencil of quadrics $\{\alpha_0 Q_0 + \alpha_1 Q_1\}, \ (\alpha_0 : \alpha_1) \in \mathbb{P}^1$.

Theorem 1. *Let N be the moduli space of rank two bundles with a fixed odd determinant on a hyperelliptic curve C, and let $\alpha_0 Q_0 + \alpha_1 Q_1$ be the corresponding pencil of quadrics in $\mathbb{P}^{2g+1}$. Then*

$$N \simeq \{\textit{variety of (g-2)-dimensional linear subspaces on } Q_0 \cap Q_1\}$$

The *proof* is given in [2]. □

Thus, N is naturally embedded in the Grassmann variety $Gr(g-1, 2g+2)$. In what follows this Grassmann variety will be denoted simply by Gr_g. Let S and Q denote, respectively, the universal sub- and quotient bundle on Gr_g.

Let $x_1, \ldots, x_{g+3}$ be formal variables. We recall the definitions of some symmetric functions in x_i: the Newton's symmetric function p_n is equal to $\sum x_i^n$, while the elementary symmetric functions e_n and the full symmetric functions h_n are best defined via generating functions:

$$E(t) = \sum_{i=1}^{g+3} e_n t^n, \qquad H(t) = \sum_{i=1}^{g+3} h_n t^n.$$

By definition

$$E(t) = \prod_{i=1}^{g+3} (1 + x_i t), \qquad H(t) = \prod_{i=1}^{g+3} (1 - x_i t)^{-1}.$$

Recall that the ring of symmetric functions in $x_1, \ldots, x_{g+3}$ can be mapped surjectively onto $H^*(Gr_g, \mathbb{C})$. This map sends e_n to $c_n(Q)$, h_n to $c_n(S^*)$, and p_n maps to the n-th component of the Chern character $ch_n(Q)$. Recall further that any partition $\lambda = (\lambda_i)_{i=1,\ldots,g-1}$ gives rise to the Schur symmetric function $det(e_{\lambda_i+j-i})$, which is mapped to the Schubert class $\{\lambda_1, \ldots, \lambda_{g-1}\} = det(c_{\lambda_i+j-i}(Q))_{1\le i,j\le g-1}$. In what follows we denote by the same letters p_i and h_j the symmetric functions, their images in $H^*(Gr_g, \mathbb{C})$, as well as restrictions to $H^*(N, \mathbb{C})$ (this should not lead to confusion).

The quadrics Q_0 and Q_1 of the pencil in $\mathbb{P}^{2g+1}$ define two sections of the bundle $Sym^2(S^*)$ on Gr_g , and N is the common zero set of these two sections. From ([3], Example 14.7.15) we conclude that the class in $H^*(Gr_g, \mathbb{C})$ Poincaré dual to N is $2^{2g-2}\{g-1, g-2, \ldots, 1\}^2$.

Now we want to express the cohomology classes A_1, A_2, D_i on N, via the classes p_i coming from Gr_g. By the adjunction formula, we have the following equality on N:

$$ch(T_{Gr_g}|_N) = ch(T_N) + 2ch(Sym^2(S^*|_N)).$$

A direct computation yields:

$$ch(T_N) = 2\Big((g-1) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1} p_i}{i!}\Big)\Big(2 + \sum_{i=1}^{\infty} \frac{p_{2i}}{(2i)!}\Big) - (g-1) + \sum_{i=1}^{\infty} \frac{(-2)^i p_i}{i!}.$$

On the other hand, let $\pi : N \times C \to C$ be a projection. Then $T_N = -\pi_!(End_0 V)$ (cf. [1]). By Grothendieck-Riemann-Roch one has

$$ch(\pi_!(End_0 V)) = \pi_*(ch(End_V)(1 - (g-1)\omega)).$$

Let $\Delta = 4A_2 - A_1^2$. Then for $k = 1, 2 \ldots$ one has:

$$ch_{2k}(T_N) = \frac{2(-1)^k}{(2k)!}(g-1)\Delta^k,$$

$$ch_{2k-1}(T_N) = \frac{2(-1)^k}{(2k-1)!}(8(k-1)D\Delta^{k-2} - A_1\Delta^{k-1}).$$

Comparing the components of the two formulas for $ch(T_N)$ we get:

$$A_1 = p_1, \qquad A_2 = \frac{1}{4}(p_1^2 - p_2), \qquad D = \frac{1}{4}(p_1 p_2 - p_3). \tag{1}$$

Moreover, we obtain the following relations between restrictions of cohomology classes to N:

$$p_{2k} = p_2^k, \qquad p_{2k-1} = p_3 p_2^{k-2}, \qquad k = 2, 3, \ldots \tag{2}$$

Denote by $\langle \ldots \rangle_g$, the intersection form on the moduli space N corresponding to genus a g curve (for our computations we also need the intersection forms for smaller genera), and by $\langle \ldots \rangle_{Gr_g}$ the intersection form on the Grassman variety Gr_g.

1.2. Computation of $\langle \mathbf{A^{3g-3}} \rangle_{\mathbf{g}}$. The Schubert class dual to $\{g-1, g-2, \ldots, 1\}$ is $\{g+2, g+1, \ldots, 4\}$. Hence to compute

$$\langle A^{3g-3} \rangle_g = \langle 2^{2g-2} \{g-1, g-2, \ldots, 1\}^2 p_1^{3g-3} \rangle_{Gr_g}$$

it suffices to find the coefficient of $\{g+2, g+1, \ldots, 4\}$ in the decomposition of $\{g-1, g-2, \ldots, 1\} p_1^{3g-3}$ into Schubert classes. A coefficient of this kind can be found using the Schubert determinant formula (cf. [31, Example 14.7.11, formula (iv)). In our case it is equal to

$$(3g-3)! \begin{vmatrix} \frac{1}{3!} & \frac{1}{1!} & 0 & \ldots & 0 \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(2g-3)!} & \frac{1}{(2g-5)!} & \frac{1}{(2g-7)!} & \ldots & \frac{1}{1!} \\ \frac{1}{(2g-1)!} & \frac{1}{(2g-3)!} & \frac{1}{(2g-5)!} & \ldots & \frac{1}{3!} \end{vmatrix}$$

Denote this determinant by a_{g-1} . Decompose it with respect to the last column. The second summand in this decomposition is equal to $\frac{1}{3!} a_{g-2}$ while the first can be decomposed again with respect to the last column, and so on. This procedure leads to a relation

$$0 = -a_{g-1} + \frac{1}{3!} a_{g-2} - \frac{1}{5!} a_{g-3} + \ldots + (-1)^g \frac{1}{(2g-1)!}.$$

Put $f(t) = \sum_{g=1}^{\infty} a_{g-1} t^{2g-2}$, where it is assumed that $a_0 = 1$. The above relation means that $f(t)\sin t = 1$, i.e.,

$$f(t) = \sum_{k=0}^{\infty} (-1)^{k+1}(2^{2k} - 2)B_{2k}\frac{t^{2k}}{(2k)!},$$

where B_{2k} is the Bernoulli number. Finally we get:

$$\langle A_1^{3g-3}\rangle_g = \frac{(3g-3)!}{(2g-2)!}(-1)^g 2^{2g-2}(2^{2g-2} - 2)B_{2g-2}.$$

1.3. Computation of $\langle \mathbf{A_1^{3r}(A_1A_2)^q}\rangle_{\mathbf{r+q+1}}$. By virtue of (1) it suffices to compute $\langle p_1^{3r}(p_1p_2)^q\rangle_{r+q+1}$. To this end we need a multiplication rule for Schur and Newton functions. For each partition $\rho = (\rho_i)_{i=1,\dots,m}$ we put $p_\rho = p_{\rho_1}p_{\rho_2}\dots p_{\rho_m}$. Recall that any partition λ can be represented by a Young diagram which has λ_i squares in the i-th row. Then, according to [5], we have

$$\{\lambda_1, \dots, \lambda_{g-1}\}p_\rho = \sum_{\lambda'} \alpha_{\lambda'}\{\lambda'_1, \dots, \lambda'_{g-1}\},$$

and $\alpha_{\lambda'} = \sum_{s\in S}(-1)^{l(s)}$. Here s runs through the set S formed by sequences of partitions $(\lambda = \lambda^0, \lambda^1, \dots, \lambda^{m-1}, \lambda^m = \lambda')$ such that the Young digram of each λ^{j-1} is contained in the diagram of λ^j and its complement is a skew ρ_j-hook, i.e. a connected set of ρ_j squares which does not contain any 2×2 squares. The length l of a hook is defined as the number of its columns minus one and $l(s)$ in the previous formula denotes $\sum_j l(\lambda^j - \lambda^{j-1})$. Since we want to determine the coefficient of $\{g+2, g+1, \dots, 4\}$ in the decomposition of $\{g-1, g-2, \dots, 1\}p_1^{3r}(p_1p_2)^q$ into a sum of Schubert classes, in our case:

$$\lambda = (g-1, \dots, 1), \qquad p = (\underbrace{2, \dots, 2}_{q}, \underbrace{1, \dots, 1}_{q+3r}), \qquad \lambda' = (g+2, g+1, \dots, 4).$$

Hence all $l(s)$ are equal to q and the set S represents the number of ways to fill the space between two "staircases" in Diagram 1, by skew hooks made of 2 or 1 squares:

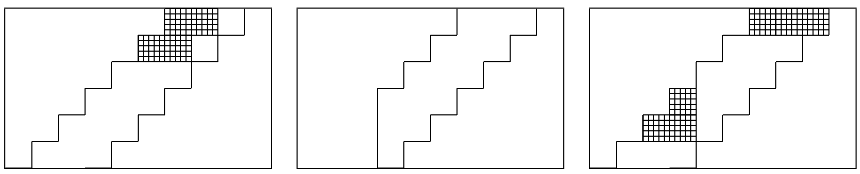

Diagram 1 Diagram 2 Diagram 3

It is clear from the picture that the hooks with two squares can be added in a unique way (each of them goes horizontally to the top possible position). The

1×1 squares to the right of them can be added at any point of the sequence $\{\lambda^0, \dots, \lambda^m\} \in S$. Since this involves choosing q ordered squares out of $(3r+q)$, the cardinality of S is equal to $\frac{(3r+q)!}{(3r)!} \cdot N_r$, where N_r is the number of ways to fill the remaining space between the Young diagrams with 1×1-squares (obtaining a Young diagram at each step). It follows from the above formula for symmetric functions that N_r, up to a power of 2, coincides with $\langle p_1^{3r} \rangle_{r+1}$ computed in the previous section. Hence

$$\langle p_1^{3r}(p_1p_2)^q \rangle_{r+q+1} = \frac{(3r+q)!}{(3r)!}(-4)^q \langle p_1^{3r} \rangle_{r+1} =$$

$$= \frac{(3r+q)!}{(2r)!}(-1)^{r+q+1}2^{2(r+q)}(2^{2r}-2)B_{2r}.$$

Similar considerations show that $\langle p_1^m p_2^n \rangle_{1+(m+2n)/3}$ vanishes if $m < n$ (there will be too many skew 2-hooks).

1.4. The remaining intersection numbers. Computation of the intersection numbers involving D_i is slightly more complicated. Recall that the intersection form is Sp-invariant; therefore it vanishes on the non-invariant part of $\Lambda^*(W)$. Since the subspace of invariants is generated by the powers of $D = \sum_{i=1}^{g} D_i \cup D_{g+i}$, it suffices to compute the intersection numbers $\langle p_1^{3r}(p_1p_2)^q D^s \rangle_{s+r+q+1}$. Due to (1) it is enough to evaluate the intersection numbers involving p_1, p_2 and p_3.

The computation of $\langle p_1^{3r} p_3^s \rangle_{s+r+1}$ can be carried out using the skew hooks formula of 1.3. We will also need another auxiliary intersection number

$$[p_1^{3r}p_3^s] := \langle p_1^{3r}p_3^s\{g-1, \dots, 4, 3, 3, 3\}\{g-1, \dots, 4, 3, 2, 1\} \rangle_{Gr_{s+r-1}}$$

This number can be evaluated as a coefficient of $\{g+2, \dots, 5, 4\}$ in the decomposition of $p_1^{3r}p_3^s\{g-1, \dots, 4, 3, 3, 3\}$ into the sum of Schubert classes. As before, we need to count the number of ways to fill the "snake" shaped figure in Diagram 2 by skew hooks with 3 or 1 squares.

Introduce the following generating functions

$$F(t_1, t_2) = \sum_{s,r \geq 0} \langle p_1^{3r} p_3^s \rangle \frac{t_1^{2r} t_2^s}{2^{2(s+r)}(3r)!s!},$$

$$G(t_1, t_2) = 1 - \sum_{\substack{s,r \geq 0 \\ s+r \neq 0}} [p_1^{3r} p_3^s] \frac{t_1^{2r} t_2^s}{(3r)!s!}$$

(the numerical coefficients come from the coefficients in the formulas of 1.2).

We have shown in 1.2 that $F(t, 0) = f(t) = t/\sin t$. In a similar way one can verify that $G(t, 0) = t \cdot \cos t/\sin t$. Now we compute $\langle p_3^s p_1^{3r} \rangle_{s+r+1}$ by the skew hooks rule. The first skew 3-hook can be of two types (cf. Diagram 3): either a "rectangle" (in this case the number or ways to fill the rest with skew hooks is $\frac{1}{2^{2(s+r-1)}}\langle p_3^{s-1}p_1^{3r} \rangle_{s+r}$); or a "corner" (in this case the shape is split into two pieces as in Diagrams 1 and 2, and the number of ways to fill the rest with hooks is equal to $\sum_{s_1,r_1} \binom{s-1}{s_1}\binom{3r}{3r_1}\frac{1}{2^{2(s_1+r_1)}}\langle p_3^{s_1}p_1^{3r_1}\rangle_{s_1+r_1+1}[p_3^{s-s_1-1}p_1^{3r-3r_1}])$. A similar argument applies to $[p_3^s p_1^{3r}]$. It is convenient to describe this inductive step in terms of generating functions:

$$\begin{cases} \frac{\partial}{\partial t_2}F = FG, \\ \frac{\partial}{\partial t_2}G = t_1^2 + G^2. \end{cases}$$

(The first equation explains why we had to consider $[p_3^s p_1^{3r}]$.) We conclude that

$$F(t_1,t_2) = \frac{t_1}{\sin(t_1 - t_1t_2)}, \qquad G(t_1,t_2) = \frac{t_1\cos(t_1 - t_1t_2)}{\sin(t_1 - t_1t_2)}.$$

Expanding the expression for F we get

$$\langle p_3^s p_1^{3r}\rangle_{s+r+1} = (-4)^s(2r-1)\dots(2r-s)\langle p_1^{3r}\rangle_{r+1}.$$

To compute $\langle p_3^s(p_1p_2)^q p_1^{3r}\rangle_{s+r+q+1}$ we multiply $\{g-1,\dots,1\}$ first by p_2^q and then by $p_3^s p_1^{3r}$. Since we are interested in the coefficient of $\{g+2,\dots,5,4\}$ in the resulting product, it follows from the pictures that all the 2-hooks have to be added in a unique way (only one per each row). Thus we get

$$\langle p_3^s(p_1p_2)^q p_1^{3r}\rangle_{s+q+r+1} = (-4)^s\frac{(3r+q)!}{(3r)!}\langle p_3^s p_1^{3r}\rangle_{s+r+1}.$$

Again we see that $\langle\, p_3^s p_1^m p_2^n\rangle_{1+s+(m+2n)/3}$ vanishes if $m < n$ (we need a skew 1-hook to put to the immediate right of each skew 2-hook). Finally, a direct computation using (1) shows that

$$\langle D^s(p_1p_2)^q p_1^{3r}\rangle_{s+r+q+1} = (-1)^s\frac{(s+q+r+1)!}{(r+q+1)!}\langle p_1^{3r}(p_1p_2)^q\rangle_{r+q+1}.$$

Put $s=1$ and note that the number of summands in the definition of D is equal to $g = q+r+2$. By Sp-invariance of the intersection form we deduce that:

If $R(A_i,D_j)$ is an expression not involving D_g and D_{2g} then

$$\langle D_gD_{2g}R(A_i,D_j)\rangle_g = -\langle R(A_i,D_j)\rangle_{g-1}$$

$$\langle D_gR(A_i,D_j)\rangle_g = \langle D_{2g}R(A_i,D_j)\rangle_{g-1} = 0.$$

§2. RELATIONS IN THE COHOMOLOGY RING

Consider $H^*(N,\mathbb{C})$ as a module over the ring of weighted polynomials As we have already observed, this module splits into a direct sum of submodules generated by primitive components of the Lefschetz decomposition (this follows from the fact that the action of the group $Sp(2g,\mathbb{C})$ on them is irreducible). Put

$$P^i = Ker\Big((\cdot\wedge D^{i+1}) : \Lambda^{g-i}(W) \to \Lambda^{g+i+2}\Big),$$

$$\Psi_i = D_g \wedge \ldots \wedge D_{1+i} - D_{2g} \wedge \ldots \wedge D_{g+1+i}, \qquad \Psi_i \in P^i, \qquad i = 0, \ldots, g.$$

Let h_i be the cohomology class on N corresponding to the symmetric function h_i. Note that the relations (2) allow us to express h_i as a polynomial in p_1, p_2, p_3. Recall h_i are nothing but restrictions of the Chern classes of the universal subbundle S on the Grassmann variety. Since $rk(S) = g - 1$, in $H^*(N, \mathbb{C})$ we have $h_i = 0$ for $i \geq g$. In general, let H_i^* be the $\mathbb{C}[A_1, A_2, D]$-submodule generated by P^i. Using the computations in the end of §1, we see that $\Psi_i h_j = 0$ for $j \geq i$ and, in view of Sp-invariance $P^i h_j = 0$ for $j \geq i$. We claim that all the relations in H_i^* follow from these relations (and in fact it is enough to consider only $j = i, i+1, i+2$).

To prove this we use the reccurence relation $nh_n = \sum_{r=1}^n p_r h_{n-r}$ in the ring of symmetric functions (cf. [5]). In view of (1) and (2) this can be rewritten as follows:

$$\begin{aligned}(i+3)h_{i+3} = p_1 h_{i+2} + p_2 h_{i+1} + p_3 h_i + p_2((i+1)h_{i+1} - p_1 h_i) = \\ = p_1 h_{i+2} + (i+2) p_2 h_{i+1} - 4D h_i \qquad (3)\end{aligned}$$

Thus, we aleady know the relations $D^{i+1} = h_i = h_{i+1} = h_{i+2} = 0$ in H_i^* (those for $h_{\geq i+3}$ follow from them).

Lemma 1. *The relations $h_i = h_{i+1} = h_{i+2} = 0$ imply that $D^i = 0$.*

Proof. We have $h_1 = p_1$, $h_2 = \frac{1}{2}(p_1^2 + p_2)$, $h_3 = \frac{1}{6}(p_1^3 + 5p_1p_2 - 8D)$. Therefore, our assertion holds for $i = 1$. Suppose for $i = k - 1$ we have

$$D^{k-1} = R_{k-1} h_{k-1} + R_k h_k + R_{k+1} h_{k+1},$$

where R_i are polynomials in p_1, p_2, D. Then

$$D^k = \frac{1}{4} R_{k-1}(p_1 h_{k+1} + (k+1) p_2 h_k - (k+2) h_{k+2}) + D R_k h_k + D R_{k+1} h_{k+1}.$$

□

Let $\widetilde{h}_i$ be a polynomial in p_1, p_2 obtained from h_i by substituting $D = 0$ (or, equivalently, $p_3 = p_1 p_2$).

Lemma 2. *$\widetilde{h}_i$ and $\widetilde{h}_{i+1}$ are relatively prime for every i.*

Proof. Induction on i. For $i = 1$ the assertion if true. If it holds for $i = k$ but $\widetilde{h}_{k+1}$ and $\widetilde{h}_{k+2}$ have a common factor h' then (3) shows that h' is also a divisor of $p_2 \widetilde{h}_k$. It is easy to check that none of $\widetilde{h}_i$ is divisible by p_2 (in fact, the coefficient of p_1^i in $\widetilde{h}_i$ is $\frac{1}{i!}$). Hence p_2 does not divide h' therefore h' divides $\widetilde{h}_k$ contrary to the induction hypothesis. □

Lemma 3. *h_i and h_{i+1} are relatively prime for every i.*

Proof. Suppose h'' is a common factor of h_i and h_{i+1}. Then h'' is homogeneous in the grading given by $\deg p_1 = 2$, $\deg p_2 = 4$, $\deg D = 6$. Therefore its reduction modulo D cannot be a non-zero constant. It cannot be zero either, since none of the h_i is divisible by D. Hence the reduction of h'' modulo D is a non-trivial common factor of $\widetilde{h}_i$ and $\widetilde{h}_{i+1}$ which contradicts Lemma 2. $\square$

Lemma 4. *The polynomials h_i, h_{i+1} and h_{i+2} form a regular sequence for every $i \geq 1$.*

Proof. Induction on i. For $i = 1$ the statement is clearly true. Suppose it is true for $i = k - 1$. In view of Lemma 3, to show the regularity for $i = k$ it suffices to prove that if $Ah_{k+2} = Bh_k + Ch_{k+1}$ then A belongs to the ideal in $\mathbb{C}[p_1, p_2, D]$ generated by h_k and h_{k+1}. In fact, (3) shows that $ADh_{k-1} = B'h_k + C'h_{k+1}$ for some polynomials B' and C'. By the inductive assumption this implies that

$$AD = Fh_k + Gh_{k+1} \tag{4}$$

for some polynomials F, G, and we want to show that a similar decomposition holds for A itself. To that end, reduce (4) modulo D. By Lemma 2 we have: $\widetilde{F} = H\widetilde{h}_{k+1}$ and $\widetilde{G} = -H\widetilde{h}_k$ for some polynomial H in p_1 and p_2. Then $F = Hh_{k+1} + DF'$, $G = -Hh_k + DG'$, for some polynomials F' and G' in p_1, p_2 and D. From (4) we conclude that $A = F'h_k + G'h_{k+1}$. $\square$.

Theorem 2. *Let M_i be a graded module over the ring $\mathbb{C}[A_1, A_2, D]$ with generators $P^i(\Lambda^*(W))$ and relations $h_i = h_{i+1} = h_{i+2} = 0$. Then the Poincaré series (in fact, a polynomial) of M_i is equal to*

$$Q_i(t) = \left[\binom{2g}{i} - \binom{2g}{i-2}\right] t^{3g-3i} \frac{(1 - t^{2i})(1 - t^{2i+2})(1 - t^{2i+4})}{(1 - t^2)(1 - t^4)(1 - t^6)}.$$

Proof. It follows from Lemmas 1 - 4 that M_i has a graded free resolution

$$0 \to P^i \otimes_{\mathbb{C}} \left(\Lambda^3 U \to \Lambda^2 U \to U \to \mathbb{C}[A_1, A_2, D]\right) \to M_i \to 0$$

where U is the free rank 3 graded $\mathbb{C}[A_1, A_2, D]$-module with generators corresponding to h_i, h_{i+1}, h_{i+2}. Since the dimension of P^i is equal to the difference of the binomial coefficients in the formula, the theorem follows from the above resolution. $\square$.

Theorem 3. *As a module over $\mathbb{C}[A_1, A_2, D]$, the cohomology ring $H^*(N, \mathbb{C})$ is isomorphic to the direct some $\bigoplus_{i=0}^{g} M_i$ of the submodules defined in Theorem 2.*

Proof. Since A_1, A_2 and D_i generate $H^*(N, \mathbb{C})$ multiplicatively, $\bigoplus_{i=0}^{g} M_i$ maps surjectively onto $H^*(N, \mathbb{C})$. Since both spaces have Poincaré polynomial

$$\frac{(1 + t^3)^{2g} - t^{2g}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)}$$

(due to the formula in [1] and Theorem 2), this map is an isomorphism. In particular, $H_i^* \simeq M_i$. $\square$

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