

# Conjugacy Classes in Loop Groups and G-Bundles on Elliptic Curves

Vladimir Baranovsky and Victor Ginzburg

## 1 Introduction

Let  $\mathbb{C}[[z]]$  be the ring of formal power series and  $\mathbb{C}((z))$  the field of formal Laurent power series, the field of fractions of  $\mathbb{C}[[z]]$ . Given a complex algebraic group  $G$ , we will write  $G((z))$  for the group of  $\mathbb{C}((z))$ -rational points of  $G$ , thought of as a formal “loop group,” and  $a(z)$  for an element of  $G((z))$ . Let  $q$  be a fixed nonzero complex number. Define a “twisted” conjugation action of  $G((z))$  on itself by the formula

$$g(z): a(z) \mapsto {}^g a = g(q \cdot z) \cdot a(z) \cdot g(z)^{-1}. \quad (1.1)$$

We are concerned with the problem of classifying the orbits of the twisted conjugation action. If  $q = 1$ , twisted conjugation becomes the ordinary conjugation, and the problem reduces to the classification of conjugacy classes in  $G((z))$ .

In this paper we will be interested in the case  $|q| < 1$ . Let  $G[[z]] \subset G((z))$  be the subgroup of  $\mathbb{C}[[z]]$ -points of  $G$ . A twisted conjugacy class in  $G((z))$  is called *integral* if it contains an element of  $G[[z]]$ .

Introduce the elliptic curve  $\mathcal{E} = \mathbb{C}^*/q^{\mathbb{Z}}$ . Our main result is the following.

**Theorem 1.2.** Let  $G$  be a complex connected semisimple algebraic group. Then there is a natural bijection between the set of integral twisted conjugacy classes in  $G((z))$  and the set of isomorphism classes of semistable holomorphic principal  $G$ -bundles on  $\mathcal{E}$ .  $\square$

The main reason we are interested in this result is that twisted conjugacy classes in  $G((z))$  may be interpreted as ordinary conjugacy classes in a larger group. Specifically,

the group  $\mathbb{C}^*$  acts on  $\mathbb{C}((z))$  by field automorphisms rescaling the variable  $z$ , i.e.,  $t \in \mathbb{C}^*$  acts by  $a(z) \mapsto a(t \cdot z)$ . This gives a  $\mathbb{C}^*$ -action on the group  $G((z))$  called “rotation of the loop.” Write  $\mathbb{C}^* \ltimes G((z))$  for the corresponding semidirect product. Then, for  $(q, a), (1, g) \in \mathbb{C}^* \ltimes G((z))$ , we have  $(1, g) \cdot (q, a) \cdot (1, g)^{-1} = (q, {}^a a)$ . Thus, twisted conjugacy classes in  $G((z))$  are essentially the same as ordinary conjugacy classes in a Kac-Moody group (= standard central extension of  $\mathbb{C}^* \ltimes G((z))$ ).

We arrived at Theorem 1.2 while trying to find an algebraic version of the following unpublished analytic result due to Looijenga (cf. [EFK]). Let  $G$  be a connected complex Lie group, let  $G(\mathbb{C}^*)_{\text{hol}}$  be the group of all holomorphic maps  $\alpha: \mathbb{C}^* \rightarrow G$  (possibly with an essential singularity at  $z = 0$ ), and let  $q$  be a fixed nonzero complex number such that  $|q| < 1$ . Then Looijenga showed the following proposition.

**Proposition 1.3.** There is a natural bijection between the set of all twisted conjugacy classes in  $G(\mathbb{C}^*)_{\text{hol}}$  and the set of isomorphism classes of arbitrary holomorphic  $G$ -bundles on  $\mathcal{E}$ .  $\square$

*Proof.* Observe that the pull-back via the projection  $\pi: \mathbb{C}^* \rightarrow \mathbb{C}^*/q^{\mathbb{Z}} = \mathcal{E}$  establishes an equivalence between the category of  $G$ -bundles on  $\mathcal{E}$  and the category of  $q^{\mathbb{Z}}$ -equivariant holomorphic  $G$ -bundles on  $\mathbb{C}^*$ . We associate to  $\alpha \in G(\mathbb{C}^*)_{\text{hol}}$  the trivial holomorphic  $G$ -bundle  $\mathbb{C}^* \times G \rightarrow \mathbb{C}^*$  on  $\mathbb{C}^*$  with  $q^{\mathbb{Z}}$ -equivariant structure given by the action  $q: (z, g) \mapsto (q \cdot z, \alpha(z) \cdot g)$ . The corresponding  $G$ -bundle on  $\mathcal{E}$  will be referred to as the  $G$ -bundle with multiplier  $\alpha$ . It is easy to see that two  $G$ -bundles on  $\mathcal{E}$  associated to two different multipliers are isomorphic if and only if the multipliers are twisted conjugate. Conversely, it is known that any holomorphic  $G$ -bundle on  $\mathbb{C}^*$  is trivial. The action of the element  $q$  on such a trivial bundle has to be of the form  $q: (z, g) \mapsto (q \cdot z, \alpha(z) \cdot g)$ , where  $\alpha: \mathbb{C}^* \rightarrow G$  is a holomorphic map. (Changing trivialization has the effect of replacing  $\alpha$  by a twisted conjugate map.) Hence, every  $q^{\mathbb{Z}}$ -equivariant holomorphic  $G$ -bundle on  $\mathbb{C}^*$  can be obtained via the above construction.  $\blacksquare$

Although motivation for Theorem 1.2 came from loop groups, the result itself is most adequately understood in the framework of  $q$ -difference equations. To explain this, assume, for simplicity, that  $G = \text{GL}_n$ . Given  $q \in \mathbb{C}^*$  and  $a(z) \in \text{GL}_n((z))$ , we consider a difference equation

$$x(q \cdot z) = a(z) \cdot x(z), \tag{1.4}$$

where  $x(z) \in \mathbb{C}^n((z))$  is the unknown  $\mathbb{C}^n$ -valued formal power series. It is clear that if  $x(z)$  is a solution to (1.4) and  $g(z) \in \text{GL}_n((z))$ , then  $\tilde{x}(z) := g(z)x(z) \in \mathbb{C}^n((z))$  is a solution to a similar equation with  $a(z)$  being replaced by  $\tilde{a}(z) = g(qz) \cdot a(z) \cdot g(z)^{-1}$ , a twisted conjugate loop.

Therefore classification of equations (1.4) modulo transformations  $x(z) \mapsto \tilde{x}(z)$  reduces to the classification of the twisted conjugacy classes in  $GL_n((z))$ .

Equation (1.4) should be regarded as a  $q$ -analogue of the first-order differential equation

$$z \frac{dx}{dz} = a(z) \cdot x(z), \tag{1.5}$$

and twisted conjugation (1.1) should be regarded as a  $q$ -analogue of the gauge transformation:  $a(z) \mapsto g(z) \cdot a(z) \cdot g(z)^{-1} + z(dg/dz)g(z)^{-1}$ . It is well known that the classification of equivalence classes of equations like (1.5) depends in an essential way on the type of functions  $a$  and  $g$  one is considering. If one puts oneself into analytic framework, then  $a$  and  $g$  are taken to be elements of  $gl_n(\mathbb{C}^*)_{\text{hol}}$  and  $GL_n(\mathbb{C}^*)_{\text{hol}}$ , respectively. It is well known and easy to prove that in this case the differential equation is completely determined (up to equivalence) by the monodromy of its fundamental solution. Thus, there is a natural bijection between the set of equivalence classes of differential equations of type (1.5) and the set of conjugacy classes in  $G$ . This is a differential equation analogue of Proposition 1.3.

The situation changes drastically if  $gl_n(\mathbb{C}^*)_{\text{hol}}$  and  $GL_n(\mathbb{C}^*)_{\text{hol}}$  are replaced by formal loops  $gl_n((z))$  and  $GL_n((z))$ , respectively. The classical theory says that for the equation to be determined by its monodromy, it should have a *regular singularity* at  $z = 0$ . This is a differential analogue of the “integrality” condition in Theorem 1.2. Thus, the  $G$ -bundle in Theorem 1.2 should be thought of as a  $q$ -analogue of the monodromy of a differential equation.

Classification of  $q$ -difference equations (1.4) is equivalent to the classification of  $D_q$ -modules, where by a  $D_q$ -module we mean a finite-rank  $\mathbb{C}((z))$ -module  $M$  equipped with an invertible  $\mathbb{C}$ -linear operator  $\mathbf{q}: M \rightarrow M$  such that

$$\mathbf{q}(f(z) \cdot m) = f(qz) \cdot (\mathbf{q}m), \quad \forall f \in \mathbb{C}((z)), m \in M.$$

A  $D_q$ -module is a module over a smash product of the group algebra of the group  $q^{\mathbb{Z}}$  with  $\mathbb{C}((z))$ . A  $D_q$ -module  $M$  is said to be integral if there exists a free  $\mathbb{C}[[z]]$ -submodule  $L \subset M$  of maximal rank such that  $\mathbf{q}(L) \subset L$  and  $\mathbf{q}^{-1}(L) \subset L$ . Integral  $D_q$ -modules form an abelian category,  $\mathcal{M}_q$ . It is easy to verify that tensor product over  $\mathbb{C}((z))$  makes  $\mathcal{M}_q$  into a tensor category. On the other hand, it is known that degree-zero semistable holomorphic vector bundles on  $\mathcal{E}$  form an abelian category,  $\text{Vect}_{\text{ss},o}(\mathcal{E})$ , with Hom’s given by arbitrary vector bundle morphisms. Tensor product of vector bundles makes this category into a tensor category. The following result is a natural strengthening of Theorem 1.2 in the  $G = GL =$  case.

**Theorem 1.6.** The tensor category  $\mathcal{M}_q$  is equivalent to  $\text{Vect}_{\text{ss},o}(\mathcal{E})$ . □

## 2 From loop groups to $G$ -bundles on $\mathcal{E}$

The ring homomorphism  $\mathbb{C}[[z]] \rightarrow \mathbb{C}$ ,  $f \mapsto f(0)$  induces, for any algebraic group  $H$ , a natural group homomorphism  $H[[z]] \rightarrow H$ . Let  $H_1[[z]]$  denote the kernel of this homomorphism, a “congruence subgroup.” We use the notation  $H[z]$  and  $H[z, z^{-1}]$  for the groups of  $\mathbb{C}[z]$ - and  $\mathbb{C}[z, z^{-1}]$ -points of  $H$ , respectively. Thus,  $H[z] \subset H[[z]]$  and  $H[z, z^{-1}] \subset H((z))$ . Elements of  $H[z, z^{-1}]$  will be referred to as *polynomial loops*.

From now on we fix a complex connected semisimple algebraic group  $G$  with Lie algebra  $\mathfrak{g}$ , and  $q \in \mathbb{C}^*$  such that  $|q| < 1$ .

Our proof of Theorem 1.2 consists of several steps. We first assign to an integral element  $a \in G((z))$  a  $G$ -bundle on  $\mathcal{E}$ . The naive idea of using  $a$  as a multiplier (cf. proof of Proposition 1.3) cannot be applied here directly, for  $a$  is only a *formal* loop, and hence, does not give a holomorphic map, in general. To overcome this difficulty, we prove the following result.

**Proposition 2.1.** For any  $a \in G[[z]]$ , there exists a Borel subgroup  $B \subset G$  with unipotent radical  $U$ , such that  $a$  is twisted conjugate to a polynomial loop of the form  $a_0 \cdot a_1(z)$  where  $a_0 \in B$  and  $a_1 \in U[z]$ .  $\square$

To prove the proposition we need some preparations. Recall that for a semisimple element  $s \in G$ , the adjoint action of  $s$  on  $\mathfrak{g}$  has a weight space decomposition  $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}$  where  $\mathfrak{g}_{\lambda}$  is the eigenspace corresponding to an eigenvalue  $\lambda \in \mathbb{C}^*$ .

Let  $a(z) = a_0 \cdot a_1(z) \in G[[z]]$ , where  $a_0 \in G$  is a constant loop and  $a_1(z) \in G_1[[z]]$ . Write  $a_0^{ss} \in G$  for the semisimple part in the Jordan decomposition of  $a_0$ , and let  $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}$  be the weight space decomposition with respect to the adjoint action of  $a_0^{ss}$ .

**Definition.** The element  $a(z) = a_0 \cdot a_1(z)$  is called *aligned* if it can be written as a product  $a_0 \exp(x_1 z) \exp(x_2 z^2) \cdot \dots$ , where  $x_i \in \mathfrak{g}_{q^i}$ .

Note that the product above is finite and gives an element of  $G[z]$ . Hence any aligned element is a polynomial loop.

**Lemma 2.2.** For any  $a \in G[[z]]$ , one can find  $g \in G_1[[z]]$  such that  ${}^q a$  is aligned.  $\square$

**Proof.** Following [BV, pp. 31, 68], we will construct a sequence of elements  $x_i \in \mathfrak{g}$  and  $y_i \in \mathfrak{g}_{q^i}$  as follows. Note that the exponential map gives a bijection  $z \cdot \mathfrak{g}[[z]] \xrightarrow{\sim} G_1[[z]]$ . Therefore we can write  $a$  in the form  $a = a_0 \exp(a_1 z) \exp(a' z^2)$ , where  $a_1 \in \mathfrak{g}$  and  $a' \in \mathfrak{g}[[z]]$ .

Since the operator  $(q \cdot \text{Ad}_{a_0^{-1}} - \text{Id})$  is invertible on  $\bigoplus_{\lambda \neq q} \mathfrak{g}_{\lambda}$ , there are uniquely determined elements  $x_1 \in \bigoplus_{\lambda \neq q} \mathfrak{g}_{\lambda}$  and  $y_1 \in \mathfrak{g}_q$  such that

$$(q \cdot \text{Ad}_{a_0^{-1}} - \text{Id})(x_1) + a_1 = y_1.$$

We next find  $y_2$ . To that end, set  $g_1 = \exp(x_1 z)$ . Then the above equation implies  ${}^q a = a_0 \exp(y_1 z) \exp(a_2 z^2) \exp(a' z^3)$ , where  $a_2 \in \mathfrak{g}$  and  $a' \in \mathfrak{g}[[z]]$ . Hence there exist uniquely determined elements  $x_2 \in \bigoplus_{\lambda \neq q^2} \mathfrak{g}_\lambda$  and  $y_2 \in \mathfrak{g}_{q^2}$  such that

$$(q^2 \cdot \text{Ad}_{a_0^{-1}} - \text{Id})(x_2) + a_2 = y_2.$$

Set  $g_2 = \exp(x_2 z^2) \exp(x_1 z)$ . Then the above equation ensures that

$${}^q a = a_0 \exp(y_1 z) \exp(y_2 z^2) \exp(a_3 z^3) \exp(a'' z^4),$$

where  $a_3 \in \mathfrak{g}$  and  $a'' \in \mathfrak{g}[[z]]$ . Iterating this process, we construct a sequence  $\{x_i \in \mathfrak{g}, i = 1, 2, \dots\}$ , such that, setting  $g_k := \exp(x_k z^k) \exp(x_{k-1} z^{k-1}) \cdots \exp(x_1 z)$ , we get

$${}^q a = a_0 \cdot \exp(y_1 z) \exp(y_2 z^2) \cdots \exp(y_k z^k) \exp(y z^{k+1}), \tag{2.3}$$

where  $y_i \in \mathfrak{g}_{q^i}$  and  $y \in \mathfrak{g}[[z]]$ . Then the product  $g := \lim g_k = \cdots \exp(x_k z^k) \cdot \exp(x_{k-1} z^{k-1}) \cdots \exp(x_1 z)$  stabilizes since  $\mathfrak{g}_{q^k} = 0$  for all  $k \gg 0$ . Equation (2.3) shows that  ${}^q a$  is aligned. ■

**Proof of Proposition 2.1.** Choose a maximal torus  $T \subset G$  containing  $a_0^{ss}$ . Let  $R \subset \text{Hom}(T, \mathbb{C}^*)$  be the set of roots of  $(G, T)$ . The subset consisting of the roots  $\gamma \in R$  such that  $|\gamma(a_0^{ss})| \leq 1$  defines a parabolic  $P \subset G$ . Then, for any  $i > 0$ , the subspace  $\mathfrak{g}_{q^i}$  is contained in the nilradical of  $\text{Lie} P$ , for  $|q| < 1$ .

Further, we may choose a Borel subgroup  $B \subset P$  that contains the unipotent part of  $a_0$ . Let  $U$  denote the unipotent radical of  $B$ . Then the element  $\exp(y_1 z) \exp(y_2 z^2) \cdots \exp(y_k z^k)$  constructed in the proof of Lemma 2.2 belongs to  $U[[z]]$ , and the proposition follows. ■

**Lemma 2.5.** Let  $B$  and  $\tilde{B}$  be two Borel subgroups with unipotent radicals  $U, \tilde{U}$ . Let  $a = a_0 \cdot a_1, (a_0 \in B, a_1 \in U_1[[z]])$ , and  $\tilde{a} = \tilde{a}_0 \cdot \tilde{a}_1, (\tilde{a}_0 \in B, \tilde{a}_1 \in \tilde{U}_1[[z]])$ . Then any element  $g \in G((z))$  such that  ${}^q a = \tilde{a}$  is a polynomial loop, i.e.,  $g \in G[z, z^{-1}]$ . □

**Proof.** Multiplying  $g$  by an element of  $G$ , we may assume that  $B = \tilde{B}$ . Further, we find a faithful rational representation  $\rho: G \rightarrow \text{SL}_n(\mathbb{C})$  such that  $B$  is the inverse image of the subgroup of upper triangular matrices in  $\text{SL}_n(\mathbb{C})$ . Thus, applying  $\rho$ , we are reduced to proving the lemma in the case  $G = \text{SL}_n(\mathbb{C})$  and  $B =$  upper triangular matrices. Thus, from now on,  $a$  and  $\tilde{a}$  are assumed to be upper triangular polynomial matrices. Set  $M = \max(\deg a, \deg \tilde{a})$ , the maximum of the degrees of the corresponding matrix-valued polynomials. Note that, by assumption, the diagonal entries  $a_{ii}$  and  $\tilde{a}_{ii}$  of the matrices  $a$  and  $\tilde{a}$  are independent of  $z$ .

Let  ${}^g a = \tilde{a}$ . We can write  $g(z) = \sum_{k \geq k_0} g(k)z^k$ , where  $g(k)$  are complex  $(n \times n)$ -matrices. Computing the bottom-left corner matrix entry of each side of the equation  $g(qz)a(z) = \tilde{a}(z)g(z)$  yields

$$g(k)_{n,1} \cdot (q^k a_{1,1} - \tilde{a}_{n,n}) = 0.$$

It follows, since the diagonal entries of  $a, \tilde{a}$  are nonzero, that there exists at most one  $k$ , say  $k = K_1$ , such that  $g(k)_{n,1} \neq 0$ . Using this, we now compute the two matrix entries standing on  $(n-1) \times 1$  and  $n \times 2$  places of each side of the equation  $g(qz)a(z) = \tilde{a}(z)g(z)$ . We find that, for any  $k > K_1 + M$ ,

$$g(k)_{n-1,1} \cdot (q^k a_{1,1} - \tilde{a}_{n-1,n-1}) = 0, \quad g(k)_{n,2} \cdot (q^k a_{2,2} - \tilde{a}_{n,n}) = 0.$$

We deduce, as before, that there exists  $K_2 \gg 0$  such that for all  $k \geq K_2$ , we have  $g(k)_{n-1,1} = g(k)_{n,2} = 0$ .

Continuing the process of computing the entries of each side of the equation  $g(qz)a(z) = \tilde{a}(z)g(z)$  along the diagonals (moving from bottom-left corner to top-right corner) we prove by descending induction on  $(i-j)$  that  $g(k)_{i,j} = 0$ , for all  $k \gg 0$ .  $\blacksquare$

We define a map from integral twisted conjugacy classes in  $G((z))$  to  $G$ -bundles on the elliptic curve  $\mathcal{E} = \mathbb{C}^*/q^{\mathbb{Z}}$  as follows. Given an element  $a \in G((z))$  in an integral twisted conjugacy class, we find (Proposition 2.1) an aligned element  $f \in G((z))$  which is twisted conjugate to  $a$ . The loop  $f$  being polynomial, it gives a well-defined holomorphic map  $f: \mathbb{C}^* \rightarrow G$ . Hence, we can associate to  $f$  the holomorphic  $G$ -bundle on  $\mathcal{E}$  with multiplier  $f$ ; see the proof of Proposition 1.3. If  $f'$  is another aligned element which is twisted conjugate to  $a$ , then by Lemma 2.5,  $f$  and  $f'$  are twisted conjugate to each other via a Laurent polynomial, hence a holomorphic loop. It follows that the  $G$ -bundles with multipliers  $f$  and  $f'$  are isomorphic. Thus, we have associated to  $a$  a well-defined isomorphism class of  $G$ -bundles on  $\mathcal{E}$ .

Remark 2.6. Note that if  $a$  is a polynomial loop, the above map does *not* necessarily take the twisted conjugacy class of  $a$  (in  $G((z))$ ) to the holomorphic  $G$ -bundle on  $\mathcal{E}$  with multiplier  $a$ , even though the latter is defined. We do not claim that, if  $a$  and  $a'$  are two polynomial loops that are twisted conjugate in  $G((z))$ , then the  $G$ -bundles on  $\mathcal{E}$  with multipliers  $a$  and  $a'$  are isomorphic. Moreover, the polynomial loops

$$a = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad a' = \begin{pmatrix} z & z^{-1} \\ 0 & z^{-1} \end{pmatrix}$$

are twisted conjugate in  $SL_2((z))$  by a *divergent* element, while the holomorphic  $SL_2$ -bundles on  $\mathcal{E}$  with multipliers  $a$  and  $a'$  are not isomorphic. This is an obstacle for trying to extend the correspondence of Theorem 1.2 beyond the set of *integral* twisted conjugacy classes on the one hand, and *semistable*  $G$ -bundles on the other.

### 3 Going to a finite covering

Recall that for any positive integer  $m$ , the field imbedding  $\mathbb{C}((z)) \hookrightarrow \mathbb{C}((w))$ ,  $z \mapsto w^m$ , makes  $\mathbb{C}((w))$  a Galois extension of  $\mathbb{C}((z))$  with the Galois group  $\mathbb{Z}/m\mathbb{Z}$ . From now on we will write  $z^{1/m}$  instead of  $w$ , so that  $(z^{1/m})^m = z$ , and the generator of the Galois group acts as  $\omega: z^{1/m} \mapsto e^{2\pi i/m} z^{1/m}$ . Let  $G((z^{1/m}))$  denote the group of  $\mathbb{C}((z^{1/m}))$ -rational points of  $G$ . We view  $G((z))$  as the subgroup of  $\omega$ -fixed points in  $G((z^{1/m}))$ . We will sometimes write  $a = a(z^{1/m})$  for an element of  $G((z^{1/m}))$ .

Further, we fix  $\tau$  in the upper half-plane,  $\text{Im}\tau > 0$ , such that  $q = e^{2\pi i\tau}$ . The automorphism  $f(z) \mapsto f(q \cdot z)$  of the field  $\mathbb{C}((z))$  can be extended to an automorphism of  $\mathbb{C}((z^{1/m}))$  via the assignment  $z^{1/m} \mapsto e^{2\pi i\tau/m} z^{1/m}$ . This gives rise to a twisted conjugation action  $g: a \mapsto {}^q a$  on  $G((z^{1/m}))$  that extends the one on  $G((z))$ .

**Definition.** An element  $s \in G$  is said to be *reduced* if, for any finite-dimensional rational representation  $\rho: G \rightarrow \text{GL}(V)$ , and any eigenvalue  $\lambda$  of the operator  $\rho(s)$ , the equation  $\lambda^k = q^l$ , (for some  $k, l \in \mathbb{Z} \setminus \{0\}$ ) implies  $\lambda = 1$ , i.e., if there are no eigenvalues of the form  $\lambda = e^{2\pi i\tau \cdot r}$ ,  $r \in \mathbb{Q}$ , except  $\lambda = 1$ .

View  $G$  as the subgroup of “constant loops” in  $G((z^{1/m}))$ .

**Theorem 3.1.** Let  $a(z) \in G[[z]]$  be an aligned element. Then one can find a positive integer  $m$  and  $g \in G((z^{1/m}))$  such that  ${}^g a$  is a constant loop and, moreover, the element  ${}^g a \in G$  is reduced. □

To prove the theorem, we fix a maximal torus  $T \subset G$ , and let  $X^*(T) = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$  and  $X_*(T) = \text{Hom}_{\text{alg}}(\mathbb{C}^*, T)$  denote the weight and coweight lattices, respectively. There is a canonical perfect pairing  $\langle \cdot, \cdot \rangle: X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$ . We first prove the following lemma.

**Lemma 3.2.** For any  $s \in T$ , there exists a  $\phi \in X_*(T)$  and an integer  $m \neq 0$  such that the following holds:

- (i)  $s = \phi(e^{2\pi i\tau/m}) \cdot s_{\text{red}}$  where  $s_{\text{red}}$  is reduced.
- (ii) Let  $\alpha \in X^*(T)$ . If  $\alpha(s) = q^l$  for some  $l \in \mathbb{Z}$ , then  $\langle \alpha, \phi \rangle/m = l$  and  $\alpha(s_{\text{red}}) = 1$ . □

**Proof of Lemma 3.2.** Fix  $s \in T$ . In  $\mathbb{C}^*$  consider the subgroup

$$\Gamma = \{z \in \mathbb{C}^* \mid \exists k, l \in \mathbb{Z} \text{ such that } z^k = q^l\}.$$

Let  $L$  be the subgroup of the weights  $\alpha \in X^*(T)$  such that  $\alpha(s) \in \Gamma$ . Clearly, if  $\alpha \in X^*(T)$  and  $m \cdot \alpha \in L$  for some integer  $m \neq 0$ , then  $\alpha \in L$ . Hence, by the well-known structure theorem about subgroups in  $\mathbb{Z}^n$ , we deduce that  $L$  splits off as a direct summand in  $X^*(T)$ . Therefore, there is another lattice  $L_{\text{red}} \subset X^*(T)$  such that  $X^*(T) = L \oplus L_{\text{red}}$ . This direct-sum

decomposition of lattices must be induced by a direct product decomposition  $\Gamma = \Gamma_1 \times \Gamma_{\text{red}}$ , where  $\Gamma_1$  and  $\Gamma_{\text{red}}$  are subtori in  $\Gamma$  such that  $X^*(\Gamma_1) = L$  and  $X^*(\Gamma_{\text{red}}) = L_{\text{red}}$ . Thus, we have  $s = s_1 \cdot s'_{\text{red}}$ , where  $s_1 \in \Gamma_1$  and  $s'_{\text{red}} \in \Gamma_{\text{red}}$ .

For any  $\alpha \in X^*(\Gamma)$ , we have by construction  $\alpha(s_1) \in \Gamma$ . Furthermore,  $\alpha(s) \in \Gamma$  implies  $\alpha \in L$ , and hence  $\alpha(\Gamma_{\text{red}}) = 1$ . Therefore, for  $\alpha \in X^*(\Gamma)$  such that  $\alpha(s'_{\text{red}}) \in \Gamma$ , we have  $\alpha(s_1 \cdot s'_{\text{red}}) \in \Gamma$ , and hence  $\alpha \in L$ , and hence  $\alpha(s'_{\text{red}}) = 1$ . Thus,  $s'_{\text{red}}$  is reduced.

View the groups  $X^*(\Gamma_1)$  and  $X_*(\Gamma_1)$  as lattices in  $(\text{Lie}\Gamma_1)^*$  and  $\text{Lie}\Gamma_1$ , respectively, so that  $X_*(\Gamma_1)$  is the kernel of the exponential map. Write  $s_1 = \exp(h)$ , where  $h \in \text{Lie}\Gamma_1$ . Since  $\alpha(s_1) \in \Gamma$  for any  $\alpha \in X^*(\Gamma)$ , and elements of  $\Gamma$  have the form  $z = e^{2\pi i(\tau \cdot r + r')}$ ,  $r, r' \in \mathbb{Q}$ , it follows that  $\alpha(h) \in \mathbb{Q} \cdot \tau + \mathbb{Q}$ , for any  $\alpha \in X^*(\Gamma)$ . Hence,  $h \in \tau \cdot \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\Gamma_1) + \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\Gamma_1)$ . Therefore, there exist  $\phi, \psi \in X_*(\Gamma_1)$  and an integer  $m$  such that  $h = (\tau/m)\phi + (1/m)\psi$ . Thus,  $s_1 = \exp(h) = \epsilon \cdot \phi(e^{2\pi i \cdot \tau/m})$ , where  $\epsilon = \psi(e^{2\pi i/m})$  is an element of order  $m$ . We put  $s_{\text{red}} = \epsilon \cdot s'_{\text{red}}$ . Clearly,  $s_{\text{red}}$  is reduced, and  $s = s_1 \cdot s'_{\text{red}} = \phi(e^{2\pi i \cdot \tau/m}) \cdot s_{\text{red}}$ .

To prove part (ii), let  $\alpha \in X^*(\Gamma)$  be such that  $\alpha(s) = q^l$  for some  $l \in \mathbb{Z}$ . Then  $\alpha \in L$ , and hence  $\alpha(s'_{\text{red}}) = 1$ . Furthermore, the equation

$$e^{2\pi i \cdot \tau \cdot l} = q^l = \alpha(s) = e^{2\pi i \cdot \tau \cdot \langle \alpha, \phi \rangle / m + 2\pi i \cdot \langle \alpha, \psi \rangle / m}$$

yields  $\tau \cdot (l - \langle \alpha, \phi \rangle / m) + \langle \alpha, \psi \rangle / m \in \mathbb{Z}$ . It follows, since  $\langle \alpha, \phi \rangle$  and  $\langle \alpha, \psi \rangle$  are integers, that  $l = \langle \alpha, \phi \rangle / m$  and that  $\langle \alpha, \psi \rangle / m \in \mathbb{Z}$ . Hence  $\alpha(\epsilon) = \alpha(\psi(e^{2\pi i/m})) = 1$ . Thus,  $\alpha(s_{\text{red}}) = \alpha(\epsilon) \cdot \alpha(s'_{\text{red}}) = 1$ , and (ii) follows. ■

**Proof of Theorem 3.1.** We choose the Borel subgroup  $B = T \cdot U$  as constructed in the proof of Proposition 2.1. Thus we have  $a(z) = a_0 \exp(x_1 z) \exp(x_2 z^2) \cdot \dots \cdot \exp(x_k z^k)$ , where  $a_0^{ss} \in T$  and  $x_i \in \mathfrak{g}_{q^i} \subset \text{Lie}B$ , where  $\mathfrak{g}_{q^i}$  stands for the  $q^i$ -eigenspace of  $\text{Ad } a_0^{ss}$ .

Applying Lemma 3.2 to  $s = a_0^{ss}$ , we find an integer  $m$  and an algebraic group homomorphism  $\phi: \mathbb{C}^* \rightarrow T$  such that  $a_0^{ss} = \phi(e^{2\pi i \cdot \tau/m}) \cdot s_{\text{red}}$ .

For any integer  $i \geq 1$  we can write  $x_i = \sum_{\alpha} x_{\alpha,i}$ , where  $\alpha$  is a positive root of  $(G, T)$  such that  $\alpha(a_0^{ss}) = q^i$  and  $x_{\alpha,i}$  is a nonzero root vector corresponding to  $\alpha$ . For such an  $\alpha$ , part (ii) of Lemma 3.2 yields  $\alpha(\phi(z^{1/m})) = z^{\langle \alpha, \phi \rangle / m} = z^i$ . We set  $g = \phi(z^{1/m})$ , a well-defined element of the group  $G((z^{1/m}))$ . Then we obtain

$$(\text{Ad } g)(x_{\alpha,i}) = \alpha(\phi(z^{1/m})) \cdot x_{\alpha,i} = z^i \cdot x_{\alpha,i}.$$

It follows that a similar equation holds for  $x_i$  instead of  $x_{\alpha,i}$ . From this we deduce

$$g^{-1} \cdot \exp(x_i z^i) \cdot g = \exp(x_i). \tag{3.2.1}$$

Further, let  $u$  be the unipotent part of the Jordan decomposition of  $a_0$ . Write  $u = \exp(y)$ . We have  $y = \sum_{\alpha} y_{\alpha}$ , where  $y_{\alpha} \in \text{Lie } U$  are root vectors. Since  $a_0^{ss}$  commutes with  $y$ , we



deduce similarly, using Lemma 3.2 (ii), that  $\alpha(\phi) = 0$  for any root  $\alpha$  such that  $y_\alpha \neq 0$ . It follows that  $g^{-1} \cdot u \cdot g = u$ . From this and (3.2.1) we obtain

$$\begin{aligned} {}^g a &= \phi(e^{2\pi i \cdot \tau/m} z^{1/m})^{-1} \cdot a_0^{ss} \cdot u \cdot \exp(x_1 z) \exp(x_2 z^2) \cdot \dots \cdot \exp(x_k z^k) \cdot \phi(z^{1/m}) \\ &= \phi(e^{2\pi i \cdot \tau/m})^{-1} \cdot a_0^{ss} \cdot u \cdot \exp(x_1) \cdot \exp(x_2) \cdot \dots \cdot \exp(x_k). \end{aligned}$$

Using Lemma 3.2 (i), we see that  $\phi(e^{2\pi i \cdot \tau/m})^{-1} \cdot a_0^{ss} \cdot u = s_{\text{red}} \cdot u$ . This element is reduced, and the theorem follows. ■

**Lemma 3.3.** Let  $s \in G$  be reduced. Then any element  $g \in G((z))$  such that  ${}^g s = s$  is a constant loop. □

*Proof.* Consider the adjoint representation  $\rho: G \rightarrow GL(\mathfrak{g})$ . We choose a basis in  $\mathfrak{g}$  such that  $\rho(s)$  is an upper-triangular matrix. Given  $g$  such that  ${}^g s = s$ , we write

$$\rho(g) = \sum_{k \geq k_0} g(k) z^k,$$

where  $g(k)$  are complex  $(n \times n)$ -matrices. The same process as in the proof of Lemma 2.5 gives equations of the type

$$g(k)_{m,n} \cdot (q^k s_{n,n} - s_{m,m}) = 0, \quad k \in \mathbb{Z}.$$

Since  $s$  is reduced, this implies  $g(k)_{m,n} = 0$  for all  $k \neq 0$ . Hence the image of  $g$  in  $GL(\mathfrak{g})((z))$  is constant. It follows that  $g$  is itself constant, for the kernel of the adjoint representation  $G \rightarrow GL(\mathfrak{g})$  is finite. ■

**Corollary 3.4.** Let  $a \in G[[z]]$  be aligned and  $s \in G$  be reduced. Assume  $g \in G((z^{1/m}))$  is such that  ${}^g a = s$ . Then  $g \in G[z^{1/m}, z^{-1/m}]$  is a Laurent polynomial loop in  $z^{1/m}$ . Furthermore,  $\theta = g(e^{2\pi i \cdot \tau/m} z^{1/m}) g(z^{1/m})^{-1}$  is a constant loop, and  $\theta^m = 1$ . □

*Proof.* The first claim follows from Lemma 2.5. To prove the second claim, recall the Galois automorphism  $\omega: f(z^{1/m}) \mapsto f(e^{2\pi i \cdot \tau/m} z^{1/m})$  on  $\mathbb{C}((z^{1/m}))$ . We apply the induced automorphism of  $G((z^{1/m}))$  to the equation  ${}^g a = s$ . The right-hand side being independent of  $z$ , and  $a$  being fixed by  $\omega$ , we get  ${}^{\omega g} a = s$ . This equation together with the original one,  ${}^g a = s$ , yields  ${}^\theta s = s$ , where  $\theta = g(e^{2\pi i \cdot \tau/m} z^{1/m}) g(z^{1/m})^{-1}$ . Hence,  $\theta$  is a constant loop, by Lemma 3.3. Further, applying the automorphism  $\omega$  several times to the first equation below we get a sequence of equations

$$\theta = (\omega g) \cdot g^{-1}, \quad \theta = (\omega^2 g) \cdot (\omega g)^{-1}, \dots, \quad \theta = (\omega^m g) \cdot (\omega^{m-1} g)^{-1}.$$

Since  $\omega^m = \text{Id}$ , taking the product of all these equations yields  $\theta^m = 1$ . ■

We fix two generators, an “a-cycle” and “b-cycle,” of the fundamental group  $\pi_1(\mathcal{E})$  as follows. The a-cycle is defined to be the image of a generator of  $\pi_1(\mathbb{C}^*) = \mathbb{Z}$  under the imbedding  $\pi_1(\mathbb{C}^*) \hookrightarrow \pi_1(\mathcal{E})$  induced by the projection  $\mathbb{C}^* \rightarrow \mathbb{C}^*/q^{\mathbb{Z}} = \mathcal{E}$ . The b-cycle is the image of the segment  $[1, q] \subset \mathbb{C}^*$  under the projection.

Given an integer  $m \neq 0$ , set  ${}^m\mathcal{E} = \mathbb{C}^*/q^{\mathbb{Z}/m}$ , and let  ${}^m\pi: {}^m\mathcal{E} \rightarrow \mathcal{E}$ ,  $z \mapsto z^m$  be the natural projection. Thus,  ${}^m\mathcal{E}$  is an elliptic curve and the map  ${}^m\pi$  is an  $m$ -sheeted Galois covering with the Galois group  $\mathbb{Z}/m\mathbb{Z}$  acting as monodromy around the a-cycle.

**Proposition 3.5.** Let  $\alpha \in G[z]$  be an aligned element and  $P$  the principal  $G$ -bundle on  $\mathcal{E}$  with multiplier  $\alpha$ . Then there exists a positive integer  $m$  such that

(i) The bundle  ${}^m\pi^*P$  is isomorphic to the holomorphic  $G$ -bundle on  ${}^m\mathcal{E}$  with a reduced constant multiplier  $s \in G$ .

(ii) Let  $\tilde{\nabla}$  be the holomorphic connection on  ${}^m\pi^*P$  transported via the isomorphism (i) from the trivial connection  $d$  on the  $G$ -bundle with multiplier  $s$ . Then  $\tilde{\nabla}$  descends to a well-defined holomorphic connection  $\nabla$  on  $P$ . The latter has finite monodromy around the a-cycle and a reduced monodromy around the b-cycle.  $\square$

*Proof.* By Theorem 3.1 there exists an element  $g \in G(z^{1/m})$  such that  ${}^0\alpha = s$  is a constant loop, and  $s \in G$  is reduced. By Corollary 3.4,  $g = g(z^{1/m})$  is a Laurent polynomial in  $z^{1/m}$ . Hence,  $g$  may be viewed as a well-defined  $G$ -valued regular function on the  $m$ -fold covering of  $\mathbb{C}^*$ . Let  $P$  be the  $G$ -bundle on  $\mathcal{E}$  with multiplier  $\alpha$ . It follows that the pull-back,  ${}^m\pi^*P$ , has a multiplier which is a twisted conjugate of  $s$ . This proves part (i).

To prove (ii), recall that any  $G$ -bundle with a constant multiplier  $s$  has a natural flat holomorphic connection which is given (in the trivialization on  $\mathbb{C}^*$  corresponding to  $s$ ) by the de Rham differential  $d$ . We transport this connection to  ${}^m\pi^*P$  via the isomorphism given by the loop  $g$ . The connection  $\tilde{\nabla} = g^{-1} \circ d \circ g$  thus obtained descends to a connection on  $P$  if and only if it is invariant under the Galois action of  $\mathbb{Z}/m\mathbb{Z}$ . But by Corollary 3.4 we have  $\omega g = \theta \cdot g$ , and hence we get

$$\omega(\tilde{\nabla}) = (\theta \cdot g)^{-1} \circ d \circ (\theta \cdot g) = g^{-1} \cdot (\theta^{-1} \circ d \circ \theta) \cdot g = g^{-1} \circ d \circ g = \tilde{\nabla},$$

since  $\theta$  commutes with  $d$ . Thus,  $\tilde{\nabla}$  is fixed by the Galois action.

To compute the monodromy, note that  $g^{-1}$  is a flat section of the connection  $\tilde{\nabla}$ . Hence the monodromy of  $\tilde{\nabla}$  around the b-cycle equals  $g(e^{2\pi i \cdot \tau/m} z^{1/m}) \cdot g(z^{1/m})^{-1} = \theta$ . Since the covering  ${}^m\pi: {}^m\mathcal{E} \rightarrow \mathcal{E}$  has no monodromy around the b-cycle and has finite monodromy around the a-cycle, it follows that  $\nabla$  also has monodromy  $\theta$  around the b-cycle and has finite monodromy around the a-cycle.  $\blacksquare$

Given a finite-dimensional rational  $G$ -module  $V$ , write  $V_p$  for the associated vector bundle on  $\mathcal{E}$  corresponding to a principal  $G$ -bundle  $P$ .

**Lemma 3.6.** Let  $P$  be the  $G$ -bundle with an aligned multiplier, and  $\nabla$  the connection on  $P$  constructed in Proposition 3.5. Then, for any rational representation  $\phi: G \rightarrow GL(V)$ , every holomorphic section of the associated vector bundle  $V_p$  is flat with respect to the induced connection on  $V_p$ .  $\square$

*Proof.* Since the connection  $\nabla$  constructed in Proposition 3.5 was obtained from a connection  $\tilde{\nabla}$  on  ${}^m\pi^*P$ , the claim for  $V_p$  is equivalent to a similar claim for the vector bundle  ${}^m\pi^*V_p$ . This vector bundle is isomorphic to the vector bundle  $\mathcal{V}$  on  ${}^m\mathcal{E}$  with multiplier  $\phi(s)$ , so that the connection  $\tilde{\nabla}$  is isomorphic to the trivial connection  $d$ . Thus, proving the claim amounts to showing that any holomorphic section of the vector bundle  $\mathcal{V}$  with multiplier  $\phi(s)$  is constant.

To that end, write the matrix  $\phi(s)$  in Jordan form  $\phi(s) = \bigoplus_i J(\lambda_i, n_i)$ , where  $J(\lambda_i, n_i)$  is the  $(n_i \times n_i)$  Jordan block with eigenvalue  $\lambda_i$ . This gives the corresponding vector bundle decomposition  $\mathcal{V} = \bigoplus_i \mathcal{V}_i$  where  $\mathcal{V}_i$  is the vector bundle with multiplier  $J(\lambda_i, n_i)$ . If  $L_i$  denotes the line bundle with multiplier  $\lambda_i$ , then there is a canonical vector bundle imbedding  $L_i \hookrightarrow \mathcal{V}_i$ . Furthermore, one can prove (using, e.g., the Fourier-Mukai transform) that the imbedding induces an isomorphism  $\Gamma({}^m\mathcal{E}, L_i) \xrightarrow{\sim} \Gamma({}^m\mathcal{E}, \mathcal{V}_i)$  of the spaces of global sections. Hence, any holomorphic section of  $\mathcal{V}_i$  comes from a holomorphic section of  $L_i$ . But  $L_i$  is a degree-zero line bundle, and hence has a nonzero section only if it is a trivial bundle, i.e., if  $\lambda_i = q^m$ . Observe now that  $\lambda_i$  is an eigenvalue of the matrix  $\phi(s)$ . Since  $s \in G$  is reduced, equation  $\lambda_i = q^m$  implies  $\lambda_i = 1$ . But then the only holomorphic section of  $L_i$  is a constant section. The latter is annihilated by the de Rham differential  $d$ , and the lemma is proved.  $\blacksquare$

**Proposition 3.7.** Let  $a, a_1 \in G((z))$  be two aligned elements. If the  $G$ -bundle on  $\mathcal{E}$  with multiplier  $a$  is isomorphic to the  $G$ -bundle on  $\mathcal{E}$  with multiplier  $a_1$ , then  $a$  is twisted conjugate to  $a_1$  via a polynomial loop.  $\square$

*Proof.* By Theorem 3.1, there exist an integer  $m \geq 1$  and elements  $g, g_1 \in G((z^{1/m}))$  such that

$${}^g a = s, \quad {}^{g_1} a_1 = s_1, \quad \text{where } s, s_1 \in G \text{ are reduced.} \tag{3.7.1}$$

Let  $\nabla, \nabla_1$  be the holomorphic connections on the  $G$ -bundles on  $\mathcal{E}$  with multipliers  $a$  and  $a_1$ , respectively, constructed in Proposition 3.5. The monodromies of the connections around the  $a$ -cycle are equal to  $s$  and  $s_1$ , respectively, and the monodromies around the  $b$ -cycle are equal to  $\theta$  and  $\theta_1$ , respectively. By Proposition 3.5 we have  $\theta^m = \theta_1^m = 1$ . If the  $G$ -bundles with multipliers  $a$  and  $a_1$  are isomorphic, then we may view  $\nabla_1$  as another holomorphic connection on the  $G$ -bundle  $P$  with multiplier  $a$ .

Since the cotangent bundle on  $\mathcal{E}$  is trivial, the difference  $X = \nabla_1 - \nabla$  may be viewed as a holomorphic section,  $X$ , of the adjoint bundle  $\mathfrak{g}_p$ . Since  $s$  is reduced the section  $X$  is flat with respect to  $\nabla$ , by Lemma 3.6. Let  $p: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$  be a universal cover of  $\mathcal{E}$ . The bundle  $p^*P$  on  $\tilde{\mathcal{E}}$  has a horizontal holomorphic section. This section gives a trivialization of  $p^*P$  such that, in the induced trivialization of  $p^*\mathfrak{g}_p$ , the pull-back  $p^*X$  is a constant element  $x \in \mathfrak{g}$ . Observe that, in general, any element  $y \in \mathfrak{g}$  gives rise in this way to a flat multi-valued section of  $\mathfrak{g}_p$ , and the monodromy of this section around  $a$ - and  $b$ -cycles is equal to  $\text{Ad}\theta(y)$  and  $\text{Ad}s(y)$ , respectively. It follows, since  $X$  is a single-valued flat section of  $\mathfrak{g}_p$  without monodromy, that  $x$  commutes with both  $\theta$  and  $s$ . Hence, the equation  $\nabla_1 = \nabla + X$  shows that the monodromy of the connection  $\nabla_1$  is given by the formulas

$$\theta_1 = \exp(x) \cdot \theta, \quad s_1 = \exp(\tau x) \cdot s. \quad (3.7.2)$$

From these formulas and the equations  $\theta_1^m = \theta^m = 1$ , we deduce  $\exp(m \cdot x) = 1$ . Thus, we may find a maximal torus  $T$  containing  $\theta, \theta_1$ , and  $\phi \in X_*(T)$ , such that  $x = \phi/m$  (cf. proof of Lemma 3.2).

Clearly,  $\phi(z^{1/m})$  is a well-defined element of  $G((z^{1/m}))$ , and from formulas (3.7.2) we deduce  $\phi(e^{2\pi i \cdot \tau/m} z^{1/m}) \cdot s \cdot \phi(z^{-1/m}) = s_1$ . Recall the notation of (3.7.1), and put  $f(z^{1/m}) = g_1(z^{1/m})^{-1} \cdot \phi(z^{1/m}) \cdot g(z^{1/m}) \in G((z^{1/m}))$ . We claim that  $f \in G((z))$ . To prove this, it suffices to show that  $f(e^{2\pi i \cdot \tau/m} z^{1/m}) = f(z^{1/m})$ . The latter follows from the chain of equalities:

$$\begin{aligned} f(e^{2\pi i \cdot \tau/m} z^{1/m}) &= g_1^{-1}(e^{2\pi i \cdot \tau/m} z^{1/m}) \cdot \phi(e^{2\pi i \cdot \tau/m} z^{1/m}) \cdot g(e^{2\pi i \cdot \tau/m} z^{1/m}) \\ &= g_1^{-1}(z^{1/m}) \cdot \theta_1^{-1} \cdot \exp(x) \cdot \phi(z^{1/m}) \cdot \theta \cdot g(z^{1/m}) \\ &= g_1^{-1}(z^{1/m}) \cdot \theta_1^{-1} \cdot \exp(x) \cdot \theta \cdot \phi(z^{1/m}) \cdot g(z^{1/m}) \\ &= g_1^{-1}(z^{1/m}) \cdot \phi(z^{1/m}) \cdot g(z^{1/m}) = f(z^{1/m}). \end{aligned}$$

Finally, using (3.7.1) we calculate

$$f_a = g_1^{-1} \cdot \phi \cdot g_a = g_1^{-1} \cdot \phi_s = g_1^{-1} s_1 = a_1.$$

Thus,  $a$  and  $a_1$  are twisted conjugate by an element of  $G((z))$ . Lemma 2.5 completes the proof.  $\blacksquare$

#### 4 Semistable $G$ -bundles and holomorphic connections

Recall that  $G$  is a complex connected semisimple group. For the definition and properties of semistable holomorphic  $G$ -bundles on an elliptic curve, we refer to [R] and [RR].

**Proposition 4.1.** A holomorphic principal  $G$ -bundle over an elliptic curve is semistable if and only if it has a holomorphic connection (necessarily flat).  $\square$

Proof. The “if” part is a corollary of the main result of [B]. The “only if” part follows from Theorem 4.2 below.  $\blacksquare$

**Theorem 4.2.** For any semistable  $G$ -bundle  $P$  on  $\mathcal{E}$ , there exists a holomorphic connection on  $P$  with finite-order monodromy around the  $\alpha$ -cycle and such that, for any rational  $G$ -module  $V$ , every holomorphic section of the associated vector bundle  $V_p$  is flat with respect to the induced connection on  $V_p$ .  $\square$

Proof. We choose and fix a faithful rational representation  $G \rightarrow GL(V)$ . By a theorem of Ramanan and Ramanathan [RR], semistability of  $P$  implies semistability of  $V_p$ . By the classification of semistable vector bundles on  $\mathcal{E}$ , due to Atiyah [A], any semistable vector bundle is isomorphic to the vector bundle with a constant multiplier. Hence, the bundle  $V_p$  has constant multiplier  $\alpha \in GL(V)$ . View  $\alpha$  as an element of the semisimple group  $PGL = PGL(V)$ , and let  $P_\alpha$  be the principal  $PGL$ -bundle with (constant) multiplier  $\alpha$ . By construction, the  $PGL(V)$ -bundle  $P_\alpha$  is induced from the  $G$ -bundle  $P$  via the composition  $G \rightarrow GL(V) \rightarrow PGL(V)$ .

We may regard the element  $\alpha \in PGL$  as a constant aligned loop in  $PGL((z))$ . Applying Proposition 3.5, we see that there is an integer  $m \geq 1$  and a reduced element  $s \in PGL$  such that the bundle  ${}^m\tau^*P$  on  ${}^m\mathcal{E}$  is isomorphic to the  $PGL$ -bundle on  ${}^m\mathcal{E}$  with multiplier  $s \in PGL$ . Let  $\nabla$  be the connection on  $P_\alpha$  constructed in Proposition 3.5.

We claim that the connection  $\nabla$  on  $P_\alpha$  arises from a holomorphic connection on the  $G$ -bundle  $P$  via the composite homomorphism  $\rho: G \rightarrow GL(V) \rightarrow PGL(V)$ . Note that this composition has finite kernel, so that the induced canonical map  $i: P \rightarrow P_\alpha$  is an immersion. Let  $TP_\alpha$  be the tangent bundle on  $P_\alpha$ . Our claim is equivalent to saying that the distribution in  $TP_\alpha$  formed by the horizontal subspaces of the connection  $\nabla$  is tangent to the immersed submanifold  $i(P) \subset P_\alpha$ . Observe that the canonical map  $i: P \rightarrow P_\alpha$  gives rise to a holomorphic section  $\nu: \mathcal{E} = P/G \rightarrow P_\alpha/\rho(G)$ . The horizontal distribution is tangent to  $i(P)$  if and only if  $\nu$  is a horizontal section.

To show the latter, we apply Chevalley’s theorem [S, Theorem 5.1.3] to the algebraic subgroup  $\rho(G) \subset PGL$ . The theorem says that we can find a rational representation  $\phi: PGL \rightarrow GL(E)$  and a one-dimensional subspace  $\mathbf{l} \subset E$  such that  $\rho(G) = \{g \in PGL \mid \phi(g)(\mathbf{l}) = \mathbf{l}\}$ . Notice that since  $G$  is semisimple, it stabilises a vector  $l \in \mathbf{l}$ . Hence, the assignment  $g \mapsto g(l)$  gives rise to an imbedding  $PGL/\rho(G) \hookrightarrow E$ . Now let  $E_{P_\alpha}$  be the associated vector bundle corresponding to  $E$ , equipped with the connection induced by  $\nabla$ . The imbedding  $PGL/\rho(G) \hookrightarrow E$  gives rise to an imbedding  $P_\alpha/\rho(G) \hookrightarrow E_{P_\alpha}$  compatible with the connections. To show that  $\nu$  is horizontal, it suffices to show that its image under

the above imbedding is a flat section. But this image is a holomorphic section of  $E_{p_\alpha}$ . By Lemma 3.6, any holomorphic section of the vector bundle  $E_{p_\alpha}$  is flat with respect to the connection on  $E_{p_\alpha}$  induced by  $\nabla$ . This proves that  $v$  is horizontal, so that the horizontal distribution on  $\mathbb{TP}_\alpha$  is tangent to  $i(P)$  and the connection  $\nabla$  comes from a holomorphic  $G$ -connection on  $P$ .

Observe further that the connection  $\nabla$  on  $P_\alpha$  has finite monodromy around the  $\alpha$ -cycle. The map  $i: P \rightarrow P_\alpha$  being an immersion with finite fibers, it follows that the  $G$ -connection on  $P$  also has finite monodromy around the  $\alpha$ -cycle.

Finally, it remains to show that there exists a holomorphic connection on  $P$  with finite-order monodromy around the  $\alpha$ -cycle such that, for any rational  $G$ -module  $V$ , every holomorphic section of the associated vector bundle  $V_p$  is flat with respect to the induced connection on  $V_p$ . We do not claim that the connection we have constructed has this property. Instead, we proceed as follows. We first use the connection that we have constructed above to prove that any semistable  $G$ -bundle on  $\mathcal{E}$  is isomorphic to a  $G$ -bundle with an aligned multiplier. This will be done in the proof of Theorem 4.3 below. We can then apply Proposition 3.5 (ii) and Lemma 3.6 to get a connection on  $P$  with all the required properties. ■

**Theorem 4.3.** A  $G$ -bundle on  $\mathcal{E}$  is semistable if and only if it is isomorphic to the  $G$ -bundle with an aligned multiplier  $a \in G[z]$ . □

*Proof.* By Proposition 3.5 (ii), any  $G$ -bundle  $P$  with an aligned multiplier has a holomorphic connection. Then the “if” part of Proposition 4.1 (due to Biswas) implies that  $P$  is semistable.

Conversely, let  $P$  be a semistable  $G$ -bundle. By Theorem 4.2, we can equip  $P$  with a holomorphic connection that has monodromies  $\theta, b \in G$  around the  $\alpha$ - and  $b$ -cycles, respectively, such that  $\theta^m = 1$  for some integer  $m \geq 1$ . Observe that the elements  $\theta$  and  $b$  commute, for  $\pi_1(\mathcal{E})$  is an abelian group. Hence, there is a maximal torus  $T \subset G$  such that  $\theta, b^{ss} \in T$ . As in the proof of Proposition 2.1, we choose a Borel subgroup  $B \supset T$  such that  $b \in B$  and  $|\alpha(b^{ss})| \leq 1$ , for any positive (with respect to  $B$ ) root  $\alpha$ .

Further, since  $\theta^m = 1$ , there exists a  $\phi \in X_*(T)$  such that  $\theta = \phi(e^{2\pi i/m})$ . Set  $g = \phi(z^{1/m})^{-1}$ , a well-defined polynomial loop in  $G((z^{1/m}))$ . We put  $a = {}^g b \in G((z^{1/m}))$ . We have  $g(e^{2\pi i/m} z^{1/m}) = \theta^{-1} g(z^{1/m})$ . Since  $\theta$  commutes with  $b$ , we deduce that  $a(e^{2\pi i/m} z^{1/m}) = a(z^{1/m})$ . It follows that  $a$  is fixed by the Galois group, and hence,  $a \in G((z))$ .

Let  $U$  be the unipotent radical of  $B$ . We have  $b = b^{ss} \cdot u$  where  $u \in U$ . Hence, the condition  $|\alpha(b^{ss})| \leq 1$ , for any positive root  $\alpha$ , ensures that  $a = {}^g b = b^{ss} \cdot a_1$  where  $a_1 \in U_1[[z]]$ . Moreover, since  $g$  is a polynomial loop, we have  $a_1 \in U_1[z]$ . By Proposition 1.3, the element  $a$  is twisted conjugate in  $G((z))$  to an aligned element  $a'$ . Using Lemma 2.5

and the fact that  $a \in B \cdot U[z]$ , we see that  $a$  is twisted conjugate to  $a'$  via a polynomial loop. Thus, there is an element  $f \in G[z^{1/m}, z^{-1/m}]$  such that

$${}^f a' = b, \quad f(e^{2\pi i/m} z^{1/m}) f(z^{1/m})^{-1} = \theta.$$

These equations show (see the proof of Proposition 3.5) that the  $G$ -bundle  $P'$  with multiplier  $a'$  has a holomorphic connection with the monodromies  $\theta$  and  $b \in G$  around the  $a$ - and  $b$ -cycle, respectively. Thus,  $P$  and  $P'$  are two  $G$ -bundles with connections that have the same monodromy. Since a holomorphic  $G$ -bundle with connection is determined, up to isomorphism, by the monodromy representation, we deduce that  $P \simeq P'$ , and the theorem follows. ■

Proof of Theorem 1.2. Proposition 3.5 shows that the  $G$ -bundle associated to any integral twisted conjugacy class in  $G((z))$ , via the procedure described at the end of Section 2, has a holomorphic connection and hence is semistable, due to Proposition 4.1. Theorem 4.3 ensures that the map  $\{\text{integral twisted conjugacy classes}\} \rightarrow \{\text{isomorphism classes of semistable } G\text{-bundles}\}$  is surjective. Injectivity of the map follows from Proposition 3.7. ■

Sketch of proof of Theorem 1.6. We note first that although  $GL_n$  is not a semisimple group, Lemma 3.2 and all other results of Sections 3–4 hold for  $G = GL_n$ . In particular, any holomorphic degree-zero semistable vector bundle on  $\mathcal{E}$  has a holomorphic connection with finite-order monodromy around the  $a$ -cycle and a reduced monodromy around the  $b$ -cycle. (Since any semistable vector bundle is isomorphic to the one with a constant multiplier [A], it has a holomorphic connection without monodromy around the  $a$ -cycle at all. Moreover, it is easy to achieve that the monodromy matrix around the  $b$ -cycle has no eigenvalues of the form  $q^k$ ,  $k \in \mathbb{Z} \setminus \{0\}$ . But in order to ensure that the monodromy around the  $b$ -cycle is reduced, one still has to allow a finite monodromy around the  $a$ -cycle.) In this way, one shows that an analogue of Theorem 1.2 holds for  $G = GL_n$ .

Now let  $V$  be a rank- $n$  and degree-zero semistable vector bundle on  $\mathcal{E}$ , and  $\nabla$  a holomorphic connection on  $V$  with order- $m$  monodromy around the  $a$ -cycle. Let  $\pi: \mathbb{C}^* \rightarrow \mathcal{E}$  and  ${}^m p: \mathbb{C}^* \rightarrow \mathbb{C}^*$ ,  $z \mapsto z^m$  denote the projections. We know that the pull-back  ${}^m p^*(\pi^*V)$  can be trivialized by means of a flat (with respect to the pull-back connection) frame. If  $\nabla'$  is another holomorphic connection on  $V$  with finite-order monodromy around the  $a$ -cycle, then we can choose a large enough integer  $m$  and two trivializations of  ${}^m p^*(\pi^*V)$  that are flat with respect to  ${}^m p^*\nabla$  and  ${}^m p^*\nabla'$ , respectively. We claim that the transition matrix between any two such trivializations is given by a holomorphic map  $\alpha: \mathbb{C}^* \rightarrow GL_n$  with a pole at  $z = 0$ . To prove the claim, we may choose and fix one holomorphic connection  $\nabla^\circ$  on  $V$  that has finite-order monodromy around the  $a$ -cycle and, moreover, a reduced

monodromy around the  $b$ -cycle. Using an argument similar to that in the proof of Proposition 3.7, one shows that the transition matrix between the trivialization corresponding to a flat frame with respect to  $\nabla^\circ$  and the trivialization corresponding to a flat frame with respect to any other connection with finite monodromy around the  $a$ -cycle is given by a holomorphic loop with a pole at  $z = 0$ . This proves the claim.

Next, we define a class of “moderate” holomorphic sections of  $\pi^*V$  as follows. Write  $\mathcal{R}$  for the ring of holomorphic functions on  $\mathbb{C}^*$  with a pole at  $z = 0$ . Let  $s$  be a holomorphic section of  $\pi^*V$ , let  $\nabla$  be a holomorphic connection on  $V$  with finite-order monodromy around the  $a$ -cycle, and let  $s_1, \dots, s_n$  be a frame of flat sections of  ${}^m p^*(\pi^*V)$ , for some  $m$ . We say that  $s$  is moderate with respect to  $\nabla$  if  ${}^m p^*s$  is expressed as a linear combination of the sections  $s_1, \dots, s_n$  with coefficients in  $\mathcal{R}$ . It is clear that this definition does not depend on the choice of a flat frame and the choice of finite covering  ${}^m p: \mathbb{C}^* \rightarrow \mathbb{C}^*$ . Moreover, the claim proved in the previous paragraph ensures that  $s$  is moderate with respect to  $\nabla$  if and only if it is moderate with respect to  $\nabla'$ , provided both connections have finite-order monodromy around the  $a$ -cycle. Thus, there is a well-defined notion of a *moderate* holomorphic section of  $\pi^*V$ . Clearly, moderate sections form an  $\mathcal{R}$ -module, to be denoted  $V(\mathcal{R})$ .

In the previous setting, assume that  $V$  is the vector bundle with a constant multiplier  $a \in GL_n(\mathbb{C})$ , and  $\nabla = d$  is the trivial connection on  $V$ . Then the tautological trivialization of  $\pi^*V$  is flat. Furthermore, it is clear that a section of  $\pi^*V$  is moderate if and only if all its coordinates (relative to the trivialization) belong to  $\mathcal{R}$ . It follows that  $V(\mathcal{R}) \simeq \mathcal{R}^n$  is a free  $\mathcal{R}$ -module. The  $q^{\mathbb{Z}}$ -equivariant structure on  $\pi^*V$  (see the proof of Proposition 1.3) provides an operator  $\mathbf{q}: V(\mathcal{R}) \rightarrow V(\mathcal{R})$ . Viewed as an operator  $\mathcal{R}^n \rightarrow \mathcal{R}^n$  via the isomorphism  $V(\mathcal{R}) \simeq \mathcal{R}^n$ , this operator has the form

$$\mathbf{q}: f(z) \mapsto a^{-1} \cdot f(q \cdot z), \quad \text{where } a = \text{multiplier.} \tag{4.4}$$

Let  $\text{Vect}_{ss,o}(\mathcal{E})$  be the tensor category of degree-zero holomorphic semistable vector bundles on  $\mathcal{E}$ . We define a functor  $F: \text{Vect}_{ss,o}(\mathcal{E}) \rightarrow \mathcal{M}_q$  by the assignment

$$V \mapsto F(V) = \mathbb{C}((z)) \otimes_{\mathcal{R}} V(\mathcal{R}),$$

where the operator  $\mathbf{q}: F(V) \rightarrow F(V)$  extends the one introduced above. Isomorphism  $V(\mathcal{R}) \simeq \mathcal{R}^n$  and formula (4.4) show that  $\mathbf{q}$  and  $\mathbf{q}^{-1}$  preserve the standard lattice  $L = \mathbb{C}[[z]]^n \subset \mathbb{C}((z))^n$ . Thus,  $F(V) \in \mathcal{M}_q$ .

Let  $V, V' \in \text{Vect}_{ss,o}(\mathcal{E})$ . It is clear from construction that there is a natural map  $V(\mathcal{R}) \otimes_{\mathcal{R}} V'(\mathcal{R}) \rightarrow (V \otimes V')(\mathcal{R})$ . This map is, in effect, an isomorphism. To see this, choose trivializations of  $\pi^*V$  and  $\pi^*V'$  corresponding to constant multipliers. Then the tensor product trivialization on  $\pi^*(V \otimes V')$  corresponds to the tensor product of the multipliers. We have seen that in these trivializations one has  $V(\mathcal{R}) \simeq \mathcal{R}^{\text{rk}V}$ ,  $V'(\mathcal{R}) \simeq \mathcal{R}^{\text{rk}V'}$ , and



$(V \otimes V')(\mathcal{R}) \simeq \mathcal{R}^{\text{rk}V \cdot \text{rk}V'}$ . It follows that the natural map above becomes the standard isomorphism  $\mathcal{R}^{\text{rk}V} \otimes_{\mathcal{R}} \mathcal{R}^{\text{rk}V'} \xrightarrow{\sim} \mathcal{R}^{\text{rk}V \cdot \text{rk}V'}$ . Applying  $\mathbb{C}((z)) \otimes_{\mathcal{R}} (\bullet)$ , we deduce that  $F$  is a tensor functor.

Recall now that both  $\mathcal{M}_q$  and  $\text{Vect}_{\text{ss},o}(\mathcal{E})$  are rigid tensor categories (cf. [DM]); in particular,  $\text{Vect}_{\text{ss},o}(\mathcal{E})$  is an abelian  $\mathbb{C}$ -category with finite-dimensional Hom's. To prove that a tensor functor between two rigid tensor categories is an equivalence, it suffices to show it is fully faithful. The analogue of Theorem 1.2 for  $G = \text{GL}_n$  ensures that the functor  $F$  is full. It remains to show that, for any  $V', V \in \text{Vect}_{\text{ss},o}(\mathcal{E})$ , the functor  $F$  induces an isomorphism  $\text{Hom}_{\text{Vect}_{\text{ss},o}(\mathcal{E})}(V', V) = \text{Hom}_{\mathcal{M}_q}(F(V'), F(V))$ . Using the duality functor on rigid tensor categories, one reduces to the case  $V' = \mathbf{1}_{\mathcal{E}}$ , the unit of the tensor category  $\text{Vect}_{\text{ss},o}(\mathcal{E})$ , i.e., the trivial one-dimensional vector bundle. In that case, we have that

$$\text{Hom}_{\text{Vect}_{\text{ss},o}(\mathcal{E})}(\mathbf{1}_{\mathcal{E}}, V) = \Gamma(\mathcal{E}, V) = \Gamma_{\text{hol}}(\mathbb{C}^*, \pi^*V)^{\mathfrak{q}}$$

is the fixed-point subspace of the  $q^{\mathbb{Z}}$ -action on the space of all holomorphic sections of  $\pi^*V$ . On the other hand, we have that  $\text{Hom}(F(\mathbf{1}_{\mathcal{E}}), F(V)) = F(V)^{\mathfrak{q}}$  is the  $\mathfrak{q}$ -fixed-point space of the operator  $\mathfrak{q}$ . In the trivialization corresponding to a constant multiplier  $\alpha$ , the operator  $\mathfrak{q}$  is given by formula (4.4). Therefore, we find

$$F(V)^{\mathfrak{q}} = \{f(z) \in \mathbb{C}((z))^n \mid f(q \cdot z) = \alpha f(z)\}.$$

Expanding  $f$  as a formal Laurent series  $f(z) = \sum_{k \geq k_0} f_k z^k$  shows that  $f \in F(V)^{\mathfrak{q}}$  if and only if, for any  $k$ , the coefficient  $f_k \in \mathbb{C}^n$  is an eigenvalue of the operator  $\alpha$  with eigenvalue  $q^k$ . It follows that  $f_k$  must vanish for all  $k \gg 0$ . Hence  $f$  is a Laurent polynomial section,  $f \in V(\mathcal{R})^{\mathfrak{q}}$ . Similarly, any element of  $\Gamma_{\text{hol}}(\mathbb{C}^*, \pi^*V)^{\mathfrak{q}}$  is a Laurent polynomial. We see that the natural imbeddings

$$F(V)^{\mathfrak{q}} \hookrightarrow V(\mathcal{R})^{\mathfrak{q}} \hookrightarrow \Gamma_{\text{hol}}(\mathbb{C}^*, \pi^*V)^{\mathfrak{q}} = \Gamma(\mathcal{E}, V)$$

are both bijections. The theorem follows. ■

Added in proof.

Following a suggestion of M. Kontsevich we briefly outline an alternative approach to Theorem 1.6. Given a finitely generated abelian group  $A$ , form the algebraic group  $\text{Hom}(A, \mathbb{C}^*)$ , which is a direct product of a torus and a finite abelian group scheme. Consider the projective system of algebraic groups  $\text{Hom}(A, \mathbb{C}^*)$  induced by the direct system of all finitely generated subgroups  $A \subset \mathbb{C}^*/q^{\mathbb{Z}}$  partially ordered by inclusion. This way we get pro-algebraic groups

$$\Gamma_o = \varprojlim_{A \subset \mathbb{C}^*/q^{\mathbb{Z}}} \text{Hom}(A, \mathbb{C}^*) \quad \text{and} \quad \Gamma = \Gamma_o \times \mathbb{C}.$$

Let  $\text{Rep}(\Gamma)$  be the category of finite-dimensional representations of the group  $\Gamma$  that factor through a rational representation of some algebraic quotient. Note that simple objects of the category  $\text{Rep}(\Gamma)$  are parametrised by the points of the set  $\varinjlim A = \mathbb{C}^*/q^{\mathbb{Z}}$ .

**Theorem 1.6'.** The tensor categories  $\text{Vect}_{\text{ss},o}(\mathcal{E})$  and  $\mathcal{M}$  are equivalent to  $\text{Rep}(\Gamma)$  each.  $\square$

*Sketch of proof.* Note that the Fourier-Mukai transform gives an equivalence of the tensor category  $\text{Vect}_{\text{ss},o}(\mathcal{E})$  with the tensor category (under convolution) of coherent sheaves on  $\mathbb{C}^*/q^{\mathbb{Z}}$  with finite support. The latter is easily seen to be equivalent to  $\text{Rep}(\Gamma)$  (by the Jordan form theorem applied locally at each point of the support).

To prove the theorem for  $\mathcal{M}_q$ , we argue as follows. For any  $M \in \mathcal{M}_q$ , there is a  $\mathfrak{q}$ -stable maximal rank  $\mathbb{C}[[z]]$ -submodule  $L \subset M$ . The action of  $\mathfrak{q}$  on  $L$  gives rise to a  $\mathbb{C}$ -linear operator, still denoted by  $\mathfrak{q}$ , on the finite-dimensional  $\mathbb{C}$ -vector space  $V = L/zL$ . One first finds the lattice  $L$  in such a way that the ratio of any two distinct eigenvalues of this operator is not an integral power of  $q$ . Such an  $L$  being found, one shows using the completeness of  $L$  in the  $z$ -adic topology that the natural projection  $L \rightarrow L/zL = V$  admits a  $\mathfrak{q}$ -equivariant splitting. Thus,  $V$  may be viewed as a subspace in  $L$  so that we get a  $\mathfrak{q}$ -equivariant imbedding  $\cdots \oplus z^{-1}V \oplus V \oplus zV \oplus z^2V \oplus \cdots \hookrightarrow M$  with dense image. It follows easily that the  $\mathbb{C}$ -vector space  $M$  is  $\mathfrak{q}$ -equivariantly isomorphic to the completion of the infinite direct sum on the left. We see that giving an object of  $\mathcal{M}_q$  amounts to giving a finite-dimensional vector space  $V$  and an invertible operator  $\mathfrak{q} : V \rightarrow V$  whose eigenvalues satisfy the above-mentioned condition. Writing the Jordan decomposition of this operator, one deduces that the category of  $D_q$ -modules of the form  $\cdots \oplus z^{-i}V \oplus V \oplus zV \oplus z^2V \oplus \cdots$  is equivalent to  $\text{Rep}(\Gamma)$ .

## Acknowledgments

We are grateful to R. Bezrukavnikov and M. Kontsevich for helpful discussions.

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Baranovsky: University of Chicago, Department of Mathematics, Chicago, Illinois 60637, USA;  
barashek@math.uchicago.edu

Ginzburg: University of Chicago, Department of Mathematics, Chicago, Illinois 60637, USA;  
ginzburg@math.uchicago.edu