



## Quiver Varieties and a Noncommutative $\mathbb{P}^2$

V. BARANOVSKY<sup>1</sup>, V. GINZBURG<sup>2</sup> and A. KUZNETSOV<sup>3</sup>

<sup>1</sup>Caltech, Mathematics 253-37, Caltech, Pasadena, CA 91125, U.S.A.

e-mail: baranovs@caltech.edu

<sup>2</sup>Department of Mathematics, University of Chicago, Chicago, IL 60637, U.S.A.

e-mail: ginzburg@math.uchicago.edu

<sup>3</sup>Institute for Information Transmission Problems, Russia. e-mail: sasha@kuznetsov.mccme.ru

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**Abstract.** To any finite group  $\Gamma \subset SL_2(\mathbb{C})$  and each element  $\tau$  in the center of the group algebra of  $\Gamma$ , we associate a category,  $\text{Coh}(\mathbb{P}_{\Gamma, \tau}^2, \mathbb{P}^1)$ . It is defined as a suitable quotient of the category of graded modules over (a graded version of) the *deformed preprojective algebra* introduced by Crawley-Boevey and Holland. The category  $\text{Coh}(\mathbb{P}_{\Gamma, \tau}^2, \mathbb{P}^1)$  should be thought of as the category of coherent sheaves on a ‘noncommutative projective space’,  $\mathbb{P}_{\Gamma, \tau}^2$ , equipped with a framing at  $\mathbb{P}^1$ , the line at infinity. Our first result establishes an isomorphism between the moduli space of *torsion free* objects of  $\text{Coh}(\mathbb{P}_{\Gamma, \tau}^2, \mathbb{P}^1)$  and the Nakajima quiver variety arising from  $\Gamma$  via the McKay correspondence. We apply the above isomorphism to deduce a generalization of the Crawley-Boevey and Holland conjecture, saying that the moduli space of ‘rank 1’ projective modules over the deformed preprojective algebra is isomorphic to a particular quiver variety. This reduces, for  $\Gamma = \{1\}$ , to the recently obtained parametrisation of the isomorphism classes of right ideals in the first Weyl algebra,  $A_1$ , by points of the Calogero–Moser space, due to Cannings and Holland and Berest and Wilson. Our approach is algebraic and is based on a monadic description of torsion free sheaves on  $\mathbb{P}_{\Gamma, \tau}^2$ . It is totally different from the one used by Berest and Wilson, involving  $\tau$ -functions.

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### 1. Introduction

#### 1.1. NONCOMMUTATIVE ALGEBRAIC GEOMETRY

A standard construction of algebraic geometry associates to any graded, finitely generated commutative  $\mathbb{C}$ -algebra  $A = \bigoplus_{i \geq 0} A_i$ , with  $A_0 = \mathbb{C}$ , a projective scheme,  $\text{Proj } A$ . A well-known theorem essentially due to Serre says that the Abelian category of coherent sheaves on  $\text{Proj } A$  is equivalent to  $\text{qgr}(A) := \text{gr}(A)/\text{tor}(A)$ , the quotient of the Abelian category of finitely generated graded  $A$ -modules by the Serre subcategory of finite-dimensional modules. Now let  $A$  be a *noncommutative* graded algebra, let  $\text{gr}(A)$  stand for the category of finitely generated *right*  $A$ -modules, and put  $\text{qgr}(A) := \text{gr}(A)/\text{tor}(A)$ . According to the philosophy of *noncommutative geometry*, see, e.g., [AZ], [SVB], and [Ve], one thinks of  $\text{qgr}(A)$  as the category of coherent sheaves on a ‘noncommutative scheme’  $\text{Proj } A$ .

Throughout this paper we assume that  $A = \bigoplus_{n \geq 0} A_n$  is a positively graded not necessarily commutative Noetherian  $\mathbb{C}$ -algebra such that all graded components,  $A_n$ , are finite dimensional over  $\mathbb{C}$ . A minor novelty of our approach to the ‘noncommutative algebraic geometry’ is that we do not restrict to the case:  $A_0 = \mathbb{C}$ , and allow  $A_0$  to be any finite dimensional semisimple  $\mathbb{C}$ -algebra, e.g., the group algebra of a finite group.

Write  $\text{mod-}A$  for the category of all (not necessarily graded) finitely generated right  $A$ -modules. We will use the following definition, cf. [AZ]:

**DEFINITION 1.1.1.** An algebra  $A = \bigoplus_{n \geq 0} A_n$  is called strongly regular of dimension  $d$  if  $A_0$  is a semisimple  $\mathbb{C}$ -algebra and, in addition, we have:

- (i) The algebra  $A$  has finite global dimension equal to  $d$ , that is  $d$  is the minimal integer, such that
 
$$\text{Ext}_{\text{mod-}A}^{>d}(M, N) = 0, \quad \text{for all } M, N \in \text{mod-}A.$$
- (ii) The algebra  $A$  is Noetherian of polynomial growth, i.e., there exist integers:  $m, n > 0$  such that:  $\dim_{\mathbb{C}} A_i \leq m \cdot (i + 1)^n$ , for all  $i \geq 0$ ;
- (iii)  $\text{Ext}_{\text{mod-}A}^i(A_0, A) = \begin{cases} A_0(l), & \text{if } i = d \\ 0, & \text{else} \end{cases}$ , i.e., the algebra  $A$  is Gorenstein with parameters  $(d, l)$ .

*Remark.* In [AZ], the authors only consider the algebras  $A$  such that  $A_0 = \mathbb{C}$ , in which case they used the notion of a *regular algebra* similar (but not identical) to Definition 1.1.1. If  $A_0 = \mathbb{C}$ , it has been effectively shown in [AZ] that any strongly regular algebra in the sense of Definition 1.1.1 satisfies the following property.

**$\chi$ -condition:**  $\dim_{\mathbb{C}} \text{Ext}_{\text{mod-}A}^i(A_0, M) < \infty$ , for all  $i \geq 0$  and  $M \in \text{mod-}A$ .

Here is a sketch of proof of this property in the general case of an arbitrary finite-dimensional semisimple algebra  $A_0$ . Observe that any finitely generated right  $A$ -module  $M$  can be included in a short exact sequence  $0 \rightarrow M' \rightarrow P \rightarrow M \rightarrow 0$ , where  $P$  is a free  $A$ -module of finite rank. Here  $M'$  is automatically finitely generated due to the Noetherian property. Thus, the  $\chi$ -condition can be proved by considering the long exact sequence of Ext-groups and using the descending induction starting at  $i = \text{gl.dim}(A)$ , once we know the Gorenstein property. □

Given a strongly regular algebra  $A$ , we let  $\pi: \text{gr}(A) \rightarrow \text{qgr}(A)$  be the projection functor. The objects of  $\text{qgr}(A)$  will be referred to as ‘sheaves on  $\text{Proj } A$ ’, and we will often write  $\text{coh}(\text{Proj } A)$  instead of  $\text{qgr}(A)$ .

For a graded  $A$ -module  $M$  and  $k \in \mathbb{Z}$ , write  $M(k)$  for the same module with the grading being shifted by  $k$ . For each  $k \in \mathbb{Z}$ , we put  $\mathcal{O}(k) := \pi(A(k))$ , a sheaf on  $\text{Proj } A$ . Similarly, for any sheaf  $E = \pi(M)$  we write  $E(k)$  for the sheaf  $\pi(M(k))$ . Further, for any sheaves  $E, F \in \text{qgr}(A)$ , let  $\text{Ext}^p(E, F)$  be the  $p$ th derived functor of the Hom-functor:  $\text{Hom}(E, F) = \text{Hom}_{\text{qgr}(A)}(E, F)$ .

The crucial properties of the category  $\text{coh}(\text{Proj } A) = \text{qgr}(A)$  that follow from the strong regularity of  $A$  are:

*Ampleness*, see [AZ]: The sequence  $\{\mathcal{O}(i)\}_{i \in \mathbb{Z}}$  is ample, that is, for any  $E \in \text{coh}(\text{Proj } A)$ , there exists an epimorphism:  $\mathcal{O}(-n)^{\oplus m} \twoheadrightarrow E$ , and for any epimorphism:  $E \twoheadrightarrow F$ , the morphism:  $\text{Hom}(\mathcal{O}(-n), E) \rightarrow \text{Hom}(\mathcal{O}(-n), F)$  is surjective for  $n \gg 0$ .<sup>\*</sup>

*Serre duality*, see [YZ]: There are integers  $d \geq 0$  (dimension) and  $l$  (index of the canonical class) such that one has functorial isomorphisms

$$\text{Ext}^i(E, F) \cong \text{Ext}^{d-i}(F, E(-l))^\vee, \quad \forall E, F \in \text{coh}(\text{Proj } A) \tag{1.1.2}$$

where  $(-)^\vee$  stands for the dual in the category of  $\mathbb{C}$ -vector spaces.

Write  $\text{gr}_{\text{left}}(A)$  for the Abelian category of finitely generated graded *left*  $A$ -modules, and  $\pi_{\text{left}}: \text{gr}_{\text{left}}(A) \rightarrow \text{qgr}_{\text{left}}(A) := \text{gr}_{\text{left}}(A)/\text{tor}_{\text{left}}(A)$ , for the projection to the quotient by the Serre subcategory of finite dimensional modules. Observe that the left action of the algebra  $A$  on itself by multiplication induces, for each  $i$ , natural morphisms:  $A_k \rightarrow \text{Hom}_{\text{qgr}(A)}(\mathcal{O}(i), \mathcal{O}(i+k))$ . This gives, for any  $E \in \text{qgr}(A)$ , a graded *left*  $A$ -module structure on the graded space  $\bigoplus_{k \geq 0} \text{Hom}(E, \mathcal{O}(k))$ . Thus,  $\underline{\mathcal{H}om}(E, \mathcal{O}) := \pi_{\text{left}}(\bigoplus_{k \geq 0} \text{Hom}(E, \mathcal{O}(k)))$  is a well defined object of  $\text{qgr}_{\text{left}}(A)$ . This way we have defined an *internal Hom-functor*  $\underline{\mathcal{H}om}(-, \mathcal{O}): \text{qgr}(A) \rightarrow \text{qgr}_{\text{left}}(A)$ . Note that it takes right modules to left modules, and the other way around. The functor  $\underline{\mathcal{H}om}(-, \mathcal{O})$  is left exact, and we write  $\underline{\mathcal{E}xt}^p(-, \mathcal{O}): \text{qgr}(A) \rightarrow \text{qgr}_{\text{left}}(A)$  for the corresponding derived functors. One can check that

$$\underline{\mathcal{E}xt}^p(E, \mathcal{O}) = \pi \left( \bigoplus_{k \geq 0} \text{Ext}^p(E, \mathcal{O}(k)) \right), \quad \forall p \geq 0. \tag{1.1.3}$$

For a sheaf  $E \in \text{coh}(\text{Proj } A)$ , we define  $H^p(\text{Proj } A, E) := \text{Ext}^p(\mathcal{O}, E)$ . One has a functorial isomorphism:  $H^0(\text{Proj } A, \underline{\mathcal{H}om}(E, \mathcal{O})) \cong \text{Hom}(E, \mathcal{O})$ , and also a functorial spectral sequence:

$$E_2^{p,q} = H^p(\text{Proj } A, \underline{\mathcal{E}xt}^q(E, \mathcal{O})) \implies \text{Ext}^\bullet(E, \mathcal{O}).$$

DEFINITION 1.1.4. (i) A sheaf  $E \in \text{qgr}(A)$  is called *locally free* if  $\underline{\mathcal{E}xt}^p(E, \mathcal{O}) = 0$ ,  $\forall p > 0$ .

(ii) A sheaf  $E \in \text{qgr}(A)$  is called *torsion free* if it admits an embedding into a locally free sheaf.

The sheaves  $\mathcal{O}(i)$  are locally free. Moreover,  $\underline{\mathcal{H}om}(\mathcal{O}(i), \mathcal{O}(j)) = \mathcal{O}(j-i)$ .

*Remark.* It is easy to see that in the case of a *smooth* commutative projective variety  $X$  the definitions given above are equivalent to the standard ones.  $\square$

<sup>\*</sup>For a graded algebra  $A$  generated by its degree one component, this is equivalent to condition (4.2.1) in [AZ].

1.2. NONCOMMUTATIVE  $\mathbb{P}_\Gamma^2$

In Section 3 we will study torsion-free sheaves on a particular two-dimensional noncommutative scheme analogous to  $\mathbb{P}^2$ . Specifically, let  $(L, \omega)$  be a two-dimensional symplectic vector space, and  $\Gamma \subset \text{Sp}(L)$  a finite subgroup. We form the graded algebra:  $TL[z] = TL \otimes_{\mathbb{C}} \mathbb{C}[z]$  denote the polynomial algebra in a dummy variable  $z$  (placed in degree 1) with coefficients in  $TL$ , the tensor algebra of the vector space  $L$ . Let  $(TL[z])\#\Gamma$  be the smash product of  $TL[z]$  with  $\mathbb{C}\Gamma$ , the group algebra of  $\Gamma$ , acting naturally on  $TL$  and trivially on  $z$  (thus, as a vector space, one has:  $(TL[z])\#\Gamma \simeq (TL[z]) \otimes_{\mathbb{C}} \mathbb{C}\Gamma$ ).

To any element  $\tau$  in  $Z(\mathbb{C}\Gamma)$ , the center of  $\mathbb{C}\Gamma$ , we associate following Crawley-Boevey and Holland [CBH], a graded algebra  $A^\tau$  as follows.

DEFINITION 1.2.5.  $A^\tau = ((TL[z])\#\Gamma)/\langle\langle u \otimes v - v \otimes u = \omega(u, v) \cdot \tau \cdot z^2 \rangle\rangle_{u, v \in L}$ , where  $\langle\langle \dots \rangle\rangle$  stands for the two-sided ideal generated by the indicated relation.

It is convenient to choose and fix a symplectic basis  $\{x, y\}$  in  $L$ , such that  $\omega(x, y) = 1$ , and to identify  $L$  with  $\mathbb{C}^2$  and  $\text{Sp}(L)$  with  $SL_2(\mathbb{C})$ . Then  $TL$  gets identified with a free associative algebra on two generators. Writing  $\mathbb{C}\langle x, y, z \rangle$  for the free associative algebra generated by  $x, y$  and  $z$ , we have

$$A^\tau = (\mathbb{C}\langle x, y, z \rangle\#\Gamma)/\langle\langle [x, z] = [y, z] = 0, [y, x] = \tau z^2 \rangle\rangle$$

The algebra  $A^\tau$  has a natural grading defined by  $\deg x = \deg y = \deg z = 1$  and  $\deg \gamma = 0$ , for any  $\gamma \in \Gamma$ . Thus  $A^\tau = \bigoplus_{n \geq 0} A_n^\tau$  is a positively graded algebra, such that  $A_0^\tau = \mathbb{C}\Gamma$ . We will see in Appendix B that the algebra  $A^\tau$  is strongly regular in the sense of Definition 1.1.1.

EXAMPLE 1.2.6. (i) If the group  $\Gamma = \{1\}$  is trivial then  $\tau$  reduces to a complex number. The corresponding algebra,  $A^\tau$ , is a noncommutative deformation of the polynomial algebra  $\mathbb{C}[x, y, z]$ .

(ii) If the group  $\Gamma$  is arbitrary and  $\tau = 0$  then the algebra  $A^{\tau=0}$  is  $\mathbb{C}[x, y, z]\#\Gamma$ , the smash product of the polynomial algebra  $\mathbb{C}[x, y, z]$  with the group  $\Gamma$ .

We\* set  $\mathbb{P}_\Gamma^2 = \text{Proj}(A^\tau)$ , and write  $\text{coh}(\mathbb{P}_\Gamma^2) = \text{qgr}(A^\tau)$ , for the corresponding category of ‘coherent sheaves’ on the noncommutative scheme  $\mathbb{P}_\Gamma^2$ . Further, let  $\mathbb{P}_\Gamma^1 = \text{Proj}(\mathbb{C}[x, y]\#\Gamma)$  be a noncommutative variety corresponding to the graded algebra  $\mathbb{C}[x, y]\#\Gamma$ . The projection  $A^\tau \twoheadrightarrow A^\tau/A^\tau z \cong \mathbb{C}[x, y]\#\Gamma$  can be considered as a closed embedding  $i: \mathbb{P}_\Gamma^1 \hookrightarrow \mathbb{P}_\Gamma^2$ . Let  $i^*: \text{coh}(\mathbb{P}_\Gamma^2) \rightarrow \text{coh}(\mathbb{P}_\Gamma^1)$  and  $i_*: \text{coh}(\mathbb{P}_\Gamma^1) \rightarrow \text{coh}(\mathbb{P}_\Gamma^2)$  be

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\*Noncommutative projective planes have been considered earlier by various authors, see, e.g., [SVB]; the only new feature of our present version is that the degree zero component of the algebra  $A^\tau$  is the group algebra of  $\Gamma$  rather than the field  $\mathbb{C}$ .

the corresponding pull-back and push-forward functors given in terms of modules by the formulas

$$i^*(\pi(M)) = \pi(M/Mz) \quad \text{resp.}, \quad i_*(\pi(N)) = \pi(N), \quad \text{with } z \text{ acting by zero.}$$

In Section 4, we will give a description of torsion free sheaves on  $\mathbb{P}_\Gamma^2$  in terms of monads, similar to the well-known description of vector bundles on the commutative  $\mathbb{P}^2$ , cf. [OSS]. We will see that the linear algebra data given by a monad associated with a sheaf on  $\mathbb{P}_\Gamma^2$  is nothing but a Nakajima quiver. This way one obtains a bijection between the moduli space of torsion free sheaves on  $\mathbb{P}_\Gamma^2$  and the Nakajima quiver variety, see Theorem 1.3.10 below. Specifically, we introduce

**DEFINITION 1.2.7.** Let  $\mathcal{M}_\Gamma^\tau(V, W)$  be the set of isomorphism classes of coherent torsion free sheaves  $E$  on  $\mathbb{P}_\Gamma^2$  equipped with an isomorphism  $i^*E \cong W \otimes_{\mathbb{C}\Gamma} \mathcal{O}$  (framing at infinity) and such that  $H^1(\mathbb{P}_\Gamma^2, E(-1)) \cong V$ , as  $\Gamma$ -modules.

### 1.3. QUIVER VARIETIES

We keep the above setup, in particular,  $\Gamma$  is a finite subgroup of  $SL_2(\mathbb{C})$  and  $L$  is the tautological 2-dimensional representation of  $\Gamma$ . Given a pair  $(V, W)$  of finite dimensional  $\Gamma$ -modules, and an element  $\tau \in Z(\mathbb{C}\Gamma)$ , consider the set of *quiver data*

$$\mathbb{M}_\Gamma^\tau(V, W) \subset \text{Hom}_\Gamma\left(V, V \otimes_{\mathbb{C}} L\right) \oplus \text{Hom}_\Gamma(W, V) \oplus \text{Hom}_\Gamma(V, W), \quad (1.3.8)$$

a locally closed subvariety formed by all triples  $(B, I, J)$  satisfying the following two conditions:

*Moment Map Equation:*  $[B, B] + IJ = \tau|_V$  (here  $[B, B] \in \text{End}_\Gamma V \otimes \wedge^2 L \simeq \text{End}_\Gamma V$ )

*Stability Condition:* if  $V' \subset V$  is a  $\Gamma$ -submodule such that  $B(V') \subset V' \otimes L$  and  $I(W) \subset V'$  then  $V' = V$ .

The group  $G_\Gamma(V)$ , the centralizer of  $\Gamma$  in  $GL(V)$ , acts on  $\mathbb{M}_\Gamma^\tau(V, W)$  by the formula  $g: (B, I, J) \mapsto (gBg^{-1}, gI, Jg^{-1})$ . This  $G_\Gamma(V)$ -action is free, due to the stability condition.

**DEFINITION 1.3.9.** The quiver variety is defined as the geometric invariant theory quotient:  $\mathfrak{M}_\Gamma^\tau(V, W) = \mathbb{M}_\Gamma^\tau(V, W)/G_\Gamma(V)$ .

Thus, the quiver variety above depends only on  $[V]$ , the isomorphism class of the  $\Gamma$ -module  $V$ , but abusing the notation we will write  $\mathfrak{M}_\Gamma^\tau(V, W)$  rather than  $\mathfrak{M}_\Gamma^\tau([V], W)$ .

The relation of the above definition of quiver variety with the original definition of Nakajima is provided by McKay correspondence. Recall that McKay correspondence, cf. [Re], establishes a bijection between affine Dynkin graphs of *ADE*-type, and finite subgroups  $\Gamma \subset SL_2(\mathbb{C})$  (up to conjugacy). Given such a subgroup  $\Gamma$ , the corresponding affine Dynkin graph can be recovered as follows.

Let  $R_i, i = 0, \dots, n$ , be the complete set of the isomorphism classes of complex irreducible representations of  $\Gamma$ . For any  $i, j \in \{0, \dots, n\}$ , let  $a_{ij}$  be the multiplicity of  $R_j$  in the  $\Gamma$ -module  $R_i \otimes_{\mathbb{C}} L$ , where  $L$  is the tautological two-dimensional representation. Since  $L$  is self-dual, we have  $a_{ij} = a_{ji}$ . Following McKay, we attach to  $\Gamma$  a graph  $Q = Q(\Gamma)$  by taking  $\{0, \dots, n\}$  as the set of vertices of  $Q$ , and connecting  $i$  and  $j$  by  $a_{ij}$  edges. The graph  $Q(\Gamma)$  turns out to be an affine Dynkin graph of ADE-type, with Cartan matrix  $C(Q) = 2 \cdot \mathbb{I} - \|a_{ij}\|$ . The Nakajima's quiver variety corresponding to  $Q(\Gamma)$  coincides with the variety  $\mathfrak{M}_{\Gamma}^{\tau}(V, W)$  defined above.

*Remark.* The case:  $\Gamma = \{1\}$  is somewhat degenerate. In this case  $Q(\Gamma)$  should stand for the quiver with one vertex, and *one* (rather than two) loop.

Our first important result relates torsion free sheaves on  $\mathbb{P}_{\Gamma}^2$  with the Nakajima quiver variety:

**THEOREM 1.3.10.** *There exists a natural bijection  $\mathfrak{M}_{\Gamma}^{\tau}(V, W) \xrightarrow{\sim} \mathcal{M}_{\Gamma}^{\tau}(V, W)$ .*

*Remark.* It is natural to expect that the bijection of the theorem establishes, in effect, an isomorphism of schemes. We are unaware, however, of any construction of a scheme (or stack) structure on the set  $\mathcal{M}_{\Gamma}^{\tau}(V, W)$ , since the formalism of moduli spaces of sheaves on noncommutative varieties is not yet developed.

The above theorem can be viewed in the following context. Let  $\mathfrak{h} := \mathbb{Z}^0(\mathbb{C}\Gamma) \subset \mathbb{Z}(\mathbb{C}\Gamma)$  be the codimension one hyperplane in  $\mathbb{Z}(\mathbb{C}\Gamma)$  formed by all central elements which have trace zero in the regular representation  $\mathbb{C}\Gamma$ . According to McKay correspondence,  $\mathfrak{h}^*$ , the dual space, carries a root system associated to the Dynkin graph  $Q(\Gamma)$ . Write  $\mathbb{W} \subset \text{GL}(\mathfrak{h})$  for the Weyl group of that root system. The Kleinian singularity,  $\mathbb{C}^2/\Gamma$ , has a semiuniversal deformation, a family  $\{\mathcal{X}_{\bar{\tau}}\}$  of surfaces parametrized by the points  $\bar{\tau} \in \mathfrak{h}/\mathbb{W}$ , such that  $\mathcal{X}_{\bar{\tau}=0} = \mathbb{C}^2/\Gamma$ . Making a finite base change  $\mathfrak{h} \rightarrow \mathfrak{h}/\mathbb{W}$ ,  $\tau \mapsto \bar{\tau}$ , one gets a family  $\{\mathcal{X}_{\tau}\}_{\tau \in \mathfrak{h}}$ . Now, let  $W = \text{triv}$  be the trivial  $\Gamma$ -module, and  $V = \mathbb{C}\Gamma$ , the regular representation. For each  $\tau \in \mathfrak{h}$  put  $\tilde{\mathcal{X}}_{\tau} = \mathfrak{M}_{\Gamma}^{\tau}(\mathbb{C}\Gamma, \text{triv})$ . According to Kronheimer [Kr], the family  $\{\tilde{\mathcal{X}}_{\tau}\}_{\tau \in \mathfrak{h}}$  gives a simultaneous resolution:  $\tilde{\mathcal{X}}_{\tau} \rightarrow \mathcal{X}_{\tau}$  of the family  $\{\mathcal{X}_{\tau}\}_{\tau \in \mathfrak{h}}$ . In particular,  $\tilde{\mathcal{X}}_0 \rightarrow \mathbb{C}^2/\Gamma$  is the minimal resolution.

Kronheimer and Nakajima [KN] showed further that, for arbitrary  $V$  and  $W$ , and  $\tau \in \mathfrak{h} = \mathbb{Z}^0(\mathbb{C}\Gamma)$ , the quiver variety  $\mathfrak{M}_{\Gamma}^{\tau}(V, W)$  is isomorphic to the moduli space of anti-self-dual connections on  $\tilde{\mathcal{X}}_{-\tau}$  with a suitable decay condition at infinity. If  $\tau = 0$ , then there is an alternative, purely algebraic, interpretation of  $\mathfrak{M}_{\Gamma}^{\tau}(V, W)$  as the moduli space of framed torsion-free sheaves (satisfying an appropriate stability condition) on  $\hat{\mathcal{X}}_0$ , a natural projective completion of  $\mathcal{X}_0$ . Such an interpretation of  $\mathfrak{M}_{\Gamma}^{\tau}(V, W)$  in terms of moduli spaces of framed torsion-free sheaves on  $\hat{\mathcal{X}}_{\tau}$  doesn't seem to be possible, however, for other  $\tau \in \mathfrak{h}/\{0\}$ .

A way to circumvent this difficulty has been suggested by Kapranov and Vasserot, who proved in [KV] that considering framed torsion-free sheaves on  $\hat{\mathcal{X}}_0$  is equivalent

to considering  $\Gamma$ -equivariant framed sheaves on  $\mathbb{P}^2$ . Furthermore, Varagnolo and Vasserot proved in [VV] that  $\Gamma$ -equivariant framed torsion-free sheaves on  $\mathbb{P}^2$  (equivalently, framed torsion-free sheaves on the noncommutative Proj-scheme corresponding to the graded algebra  $A^0 = \mathbb{C}[x, y, z]\#\Gamma$ ) are parametrized by the points of  $\mathfrak{M}_\tau^0(V, W)$ . Thus, our theorem above is a very natural extension of [VV] to the case of an arbitrary  $\tau$ .

Notice further, that while the spaces  $\tilde{\mathcal{X}}_\tau$  are only defined for  $\tau \in \mathfrak{h} = \mathbb{Z}^0(\mathbb{C}\Gamma)$ , the quiver varieties  $\mathfrak{M}_\tau^c(V, W)$  exist for all  $\tau \in \mathbb{Z}(\mathbb{C}\Gamma)$ . This discrepancy has been resolved by Crawley-Boevey and Holland, who constructed in [CBH] a family of associative algebras  $\mathcal{A}_\tau$  parametrized by all  $\tau \in \mathbb{Z}(\mathbb{C}\Gamma)$ , and such that the algebra  $\mathcal{A}_\tau$  is isomorphic to the coordinate ring of the variety  $\mathcal{X}_\tau$ , if  $\tau \in \mathfrak{h}$ , and is noncommutative otherwise. From this perspective, our theorem can also be viewed as describing framed torsion-free sheaves on a projective completion of  $\text{Spec } \mathcal{A}_\tau$ , the noncommutative affine scheme corresponding to the algebra  $\mathcal{A}_\tau$ .

Next we introduce the algebra

$$\mathcal{B}^\tau = A^\tau / (z - 1) \cdot A^\tau \simeq (\mathbb{C}\langle x, y \rangle \#\Gamma) / \langle\langle [y, x] = \tau \rangle\rangle,$$

which can be thought of as the ‘coordinate ring’ of  $\mathbb{P}_\tau^2 / \mathbb{P}_\tau^1$ , a noncommutative affine plane.

The algebra  $\mathcal{B}^\tau$  comes equipped with a natural increasing filtration,  $F_\bullet \mathcal{B}^\tau$ , such that  $F_0 \mathcal{B}^\tau = \mathbb{C}\Gamma$ . It follows from [Q] that the assignment  $R \mapsto R \otimes_{\mathbb{C}\Gamma} \mathcal{B}^\tau$  gives a natural isomorphism:  $K(\Gamma) \xrightarrow{\sim} K(\mathcal{B}^\tau)$ , between the Grothendieck group of the category of finite-dimensional  $\Gamma$ -modules and that of finitely generated projective  $\mathcal{B}^\tau$ -modules. Further, the map assigning the integer  $\dim_{\mathbb{C}} R$  to a finite-dimensional  $\Gamma$ -module  $R$  extends to a canonical group homomorphism  $\dim: K(\Gamma) \rightarrow \mathbb{Z}$ . Composing this with the isomorphism  $K(\mathcal{B}^\tau) \simeq K(\Gamma)$ , one gets a function  $\dim_{\mathcal{B}}: K(\mathcal{B}^\tau) \rightarrow \mathbb{Z}$ .

The second major result of this paper is a classification of isomorphism classes of finitely generated projective  $\mathcal{B}^\tau$ -modules  $N$  such that  $\dim_{\mathcal{B}} N = 1$ . In order to formulate it we need the following technical result whose proof is given in Section 8.

**PROPOSITION 1.3.11.** *Let  $R \in K(\Gamma)$  be such that  $\dim R = 1$ . Then there exist uniquely determined (up to isomorphism)  $\Gamma$ -modules  $W$  and  $V$ , such that in  $K(\Gamma)$  we have  $R = [W] + [V] \cdot ([L] - 2[\text{triv}])$  and, moreover,  $\dim_{\mathbb{C}} W = 1$  and  $V$  does not contain the regular representation,  $\mathbb{C}\Gamma$ , as a submodule.*

The Theorem below is a generalization of a conjecture of Crawley-Boevey and Holland, see [BLB], Example 5.7. The conjecture corresponds, when reformulated in terms of deformed preprojective algebras, see Section 6, to the special case of the theorem  $R = [W]$ ,  $V = 0$ .

**THEOREM 1.3.12.** *Assume that  $\tau \in \mathbb{Z}(\mathbb{C}\Gamma)$  is generic in the sense of [CBH], i.e., it does not belong to root hyperplanes. Let  $R \in K(\Gamma)$  be such that  $\dim R = 1$ , and  $V, W$  the  $\Gamma$ -modules attached to this class by Proposition 1.3.11. Then there exists a natural bijection:*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of finitely generated projective} \\ \mathcal{B}^\tau\text{-modules } N \text{ such that } [N] = R \text{ in } K(\mathcal{B}^\tau) = K(\Gamma) \end{array} \right\} \simeq \bigsqcup_{k=0}^{\infty} \mathfrak{M}_\Gamma^\tau(V \oplus \mathbb{C}\Gamma^{\oplus k}, W).$$

Recall that to define ‘root hyperplanes’ in  $Z(\mathbb{C}\Gamma)$ , one has to identify the  $\mathbb{C}$ -dual of  $Z(\mathbb{C}\Gamma)$  with the underlying vector space of the affine root system,  $\Delta_{\text{aff}} \subset Z(\mathbb{C}\Gamma)^*$ , associated to the Dynkin graph  $Q(\Gamma)$ , see also sections 5 and 6 of [CBH]. The vertices of  $Q$  may be identified with the base of the space  $Z(\mathbb{C}\Gamma)^*$  formed by the irreducible characters of  $\Gamma$ , and the root system  $\Delta_{\text{aff}}$  is given in this basis by the incidence matrix  $\|a_{ij}\|$  of the graph  $Q$ .

To prove Theorem 1.3.12 we observe first that, for generic  $\tau \in Z(\mathbb{C}\Gamma)$ , any torsion-free sheaf on  $\mathbb{P}_\Gamma^2$  is in effect locally free. We show further that ‘restriction’ from  $\mathbb{P}_\Gamma^2$  to  $\mathbb{P}_\Gamma^2 \setminus \mathbb{P}_\Gamma^1$ , gives, for generic  $\tau \in Z(\mathbb{C}\Gamma)$ , a bijection between locally free sheaves on  $\mathbb{P}_\Gamma^2$  and  $\mathbb{P}_\Gamma^2 \setminus \mathbb{P}_\Gamma^1$ , respectively. The latter objects being nothing but finitely generated projective  $\mathcal{B}^\tau$ -modules, the result follows from Theorem 1.3.10.

Let  $\Gamma = \{1\}$  so that:  $R = \text{triv}$  is the trivial one-dimensional class, and  $Z(\mathbb{C}\Gamma) = \mathbb{C}$ . The parameter  $\tau \in \mathbb{C}$  is generic in the above sense if and only if  $\tau \neq 0$ . The algebra  $\mathcal{B}^\tau$  becomes, for  $\tau \neq 0$ , the first Weyl algebra with two generators,  $x, y$ , subject to the relation:  $[y, x] = \tau \cdot 1$ . Further, the variety  $\mathfrak{M}_\Gamma^\tau(\mathbb{C}\Gamma^{\oplus k}, \text{triv})$  becomes in this case the Calogero-Moser variety,  $\mathfrak{M}^\tau(\mathbb{C}^k) = \mathfrak{M}_{\Gamma=\{1\}}^\tau(\mathbb{C}^{\oplus k}, \mathbb{C})$ , introduced in [KKS] and studied further in [W]. Explicitly, we have

$$\mathfrak{M}^\tau(\mathbb{C}^k) = \{(B_1, B_2) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \mid [B_1, B_2] - \tau \cdot \text{Id}_{\mathbb{C}^n} = \text{rank 1 matrix}\} / \text{Ad } GL_n.$$

Since any rank 1 projective module over the Weyl algebra  $\mathcal{B}^\tau$  is isomorphic to a right ideal in  $\mathcal{B}^\tau$ , Theorem 1.3.12 reduces, in the special case of the trivial group  $\Gamma$ , to the following result proved recently by Berest and Wilson [BW].

**COROLLARY 1.3.13.** *The set of isomorphism classes of right ideals (viewed as  $\mathcal{B}^\tau$ -modules) in the Weyl algebra  $\mathcal{B}^\tau$  is in a natural bijection with the set  $\bigsqcup_{k=0}^{\infty} \mathfrak{M}^\tau(\mathbb{C}^k)$ .*

The proof of this result given in [BW] is totally different from ours, and relies heavily on the earlier results of [W] and [CaH].

### 2. Sheaves on a Noncommutative Proj-Scheme

In this section we fix a strongly regular graded  $\mathbb{C}$ -algebra  $A = \bigoplus_{k \geq 0} A_k$ , set  $X = \text{Proj } A$ , and write  $\text{coh}(X) = \text{qgr}(A)$ . Recall that the quotient category  $\text{qgr}(A)$  is defined as follows. The objects of  $\text{qgr}(A)$  are the same as those of  $\text{gr}(A)$ , while

$$\text{Hom}_{\text{qgr}(A)}(M, N) = \varinjlim \text{Hom}_{\text{gr}(A)}(M', N),$$

where the limit is taken over all submodules  $M' \subset M$ , with  $M/M' \in \text{tor}(A)$ .

Recall the integer  $d$  entering Definition 1.1.1 of a strongly regular algebra and the Serre duality property (1.1.2.).



PROPOSITION 2.0.1. *For any coherent sheaves  $E$  and  $F$  on  $X$  we have*

$$\text{Ext}^{>d}(E, F) = 0, \quad \underline{\mathcal{E}xt}^{>d}(E, \mathcal{O}) = 0.$$

*Proof.* The first part follows from Serre duality and the second follows from the first and from (1.1.3).  $\square$

PROPOSITION 2.0.2. *Any coherent sheaf  $E$  on  $X$  admits a resolution of the form*

$$\dots \rightarrow \mathcal{O}(-n_k)^{\oplus m_k} \xrightarrow{\phi_k} \dots \xrightarrow{\phi_1} \mathcal{O}(-n_0)^{\oplus m_0} \rightarrow E \rightarrow 0. \quad (2.0.3)$$

*Proof.* Follows immediately from the ampleness property.  $\square$

Recall the notions of locally free and torsion free sheaves on  $X$ , see Definition 1.1.4. We write  $E^*$  instead of  $\underline{\mathcal{H}om}(E, \mathcal{O})$ , for brevity.

PROPOSITION 2.0.4. *Let  $\mathcal{E}, \mathcal{E}_1, \dots, \mathcal{E}_d$  be locally free sheaves on  $X$ .*

- (1) *For any complex:  $\mathcal{E}_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{d-1}} \mathcal{E}_d$ , which is exact at the terms:  $\mathcal{E}_2, \dots, \mathcal{E}_{d-1}$ , the sheaf  $\mathbf{Ker} \phi_1$  is locally free.*
- (2) *The sheaf  $\mathcal{E}^*$  is locally free.*
- (3) *The canonical morphism  $\mathcal{E} \rightarrow \mathcal{E}^{**}$  is an isomorphism.*
- (4) *For any  $k$ , the sheaf  $\mathcal{E}$  has a resolution of the form*

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}(n_0)^{\oplus m_0} \rightarrow \dots \rightarrow \mathcal{O}(n_k)^{\oplus m_k} \rightarrow \dots \quad (2.0.5)$$

- (5) *If  $d \leq 2$ , then for any sheaf  $E$ , the sheaf  $E^*$  is locally free.\**

*Proof.* (1) Let  $K = \mathbf{Ker} \phi_1$  and  $C = \mathbf{Coker} \phi_{d-1}$ . Applying the functor  $\underline{\mathcal{H}om}(-, \mathcal{O})$  to the sequence  $0 \rightarrow K \rightarrow \mathcal{E}_1 \rightarrow \dots \rightarrow \mathcal{E}_d \rightarrow C \rightarrow 0$  we get a spectral sequence converging to zero. The first term of the sequence looks as

$$\begin{array}{ccccccc} \underline{\mathcal{E}xt}^d(K, \mathcal{O}) & 0 & \dots & 0 & \dots & \underline{\mathcal{E}xt}^d(C, \mathcal{O}) & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ K^* & \leftarrow \mathcal{E}_1^* & \leftarrow \dots & \leftarrow \mathcal{E}_d^* & \leftarrow C^* & & \end{array}$$

Hence  $\underline{\mathcal{E}xt}^{>0}(K, \mathcal{O}) = 0$ .

(2) Choose a resolution (2.0.3) for  $\mathcal{E}$  with  $k = d$  and let  $\mathcal{F} = \mathbf{Ker} \phi_d$ . Then by (1) the sheaf  $\mathcal{F}$  is locally free. Applying the functor  $\underline{\mathcal{H}om}(-, \mathcal{O})$  to this resolution we get a resolution

$$0 \rightarrow \mathcal{E}^* \rightarrow \mathcal{O}(n_0)^{\oplus m_0} \rightarrow \dots \rightarrow \mathcal{O}(n_d)^{\oplus m_d} \rightarrow \mathcal{F}^* \rightarrow 0$$

Hence  $\mathcal{E}^*$  is locally free by (1).

\*We will usually denote arbitrary sheaves by roman letters, and locally free sheaves by script letters.

(3) For  $\mathcal{E}$ , choose a resolution of type (2.0.3) with  $k = d$ , and apply to it the functor  $\underline{\mathcal{H}om}(\underline{\mathcal{H}om}(-, \mathcal{O}), \mathcal{O})$ . Since the canonical morphism:  $\mathcal{O}(n) \rightarrow \mathcal{O}(n)^{**}$  is an isomorphism it follows that  $\mathcal{E} \xrightarrow{\sim} \mathcal{E}^{**}$  is also an isomorphism.

(4) For  $\mathcal{E}^*$ , choose a resolution of type (2.0.3), and apply  $\underline{\mathcal{H}om}(-, \mathcal{O})$ .

(5) For  $\mathcal{E}$ , choose a resolution of type (2.0.3) with  $k = 1$ . Apply to it the functor  $\underline{\mathcal{H}om}(-, \mathcal{O})$ , and use (1). □

**PROPOSITION 2.0.6.** *If  $E$  is a torsion free sheaf then  $\underline{\mathcal{E}xt}^d(E, \mathcal{O}) = 0$ .*

*Proof.* Given  $E$ , choose an embedding  $E \hookrightarrow \mathcal{E}$  into a locally free sheaf. Applying the functor  $\underline{\mathcal{H}om}(-, \mathcal{O})$  to the exact sequence  $0 \rightarrow E \rightarrow \mathcal{E} \rightarrow \mathcal{E}/E \rightarrow 0$  we get an epimorphism  $\underline{\mathcal{E}xt}^d(\mathcal{E}, \mathcal{O}) \twoheadrightarrow \underline{\mathcal{E}xt}^d(E, \mathcal{O}) \rightarrow 0$ . □

**LEMMA 2.0.7.** *For  $n \gg 0$ , there is a canonical isomorphism*

$$H^0(X, (\underline{\mathcal{E}xt}^k(E, \mathcal{O}))(n-l)) \cong H^{d-k}(X, E(-n))^\vee, \quad \forall k \geq 0.$$

*Proof.* It is clear that

$$H^p(X, (\underline{\mathcal{E}xt}^k(E, \mathcal{O}))(n-l)) = H^p(X, \underline{\mathcal{E}xt}^q(E(-n), \mathcal{O}(-l))).$$

Further, there is a standard spectral sequence for the derived functor of composition

$$E_2^{p,q} = H^p(X, \underline{\mathcal{E}xt}^q(E(-n), \mathcal{O}(-l))) \implies \text{Ext}^{p+q}(E(-n), \mathcal{O}(-l)).$$

On the other hand, for  $n \gg 0$  we have

$$H^{>0}(X, (\underline{\mathcal{E}xt}^k(E, \mathcal{O}))(n-l)) = 0,$$

hence the spectral sequence degenerates and

$$H^0(X, (\underline{\mathcal{E}xt}^k(E, \mathcal{O}))(n-l)) = \text{Ext}^k(E(-n), \mathcal{O}(-l)).$$

Finally, Serre duality gives

$$\text{Ext}^k(E(-n), \mathcal{O}(-l)) \cong H^{d-k}(X, E(-n))^\vee. \quad \square$$

**DEFINITION 2.0.8.** A sheaf  $F$  on  $X$  is called an *Artin sheaf* if  $H^{>0}(X, F(k)) = 0$ ,  $\forall k \in \mathbb{Z}$ .

*Remark.* In the situations studied in this paper, any Artin sheaf has finite length. We don't know if this is true in general, as well as whether a subsheaf of an Artin sheaf is Artin.

**PROPOSITION 2.0.9.** *Assume that  $d = \dim X \leq 2$ .*

- (1) *If  $F$  is an Artin sheaf and  $\mathcal{E}$  is locally free then  $\text{Ext}^{>0}(\mathcal{E}, F) = 0$ .*
- (2) *For any sheaf  $E$  the sheaf  $\underline{\mathcal{E}xt}^d(E, \mathcal{O})$  is Artin.*
- (3)  *$F$  is an Artin sheaf if and only if  $\underline{\mathcal{E}xt}^{<d}(F, \mathcal{O}) = 0$ .*
- (4) *If  $F$  is an Artin sheaf then  $\underline{\mathcal{E}xt}^d(F, \mathcal{O}) = 0$  implies  $F = 0$ .*
- (5) *Any extension of Artin sheaves is Artin.*

*Proof.* (1) First note that  $\text{Ext}^p(\mathcal{O}(k), F) = H^p(X, F(-k)) = 0$ , for  $p > 0$ . Now, assume  $\mathcal{E}$  is locally free and take a resolution of  $\mathcal{E}$  of type (2.0.5) with  $k = d$ . Applying the functor  $\text{Hom}(-, F)$  to this resolution we deduce that  $\text{Ext}^{>0}(\mathcal{E}, F) = 0$ .

(2) We have a spectral sequence

$$E_2^{p,q} = H^p(X, \underline{\mathcal{E}xt}^q(E, \mathcal{O}(k))) \implies \text{Ext}^{p+q}(E, \mathcal{O}(k)).$$

It is easy to see that any nonzero element in  $H^p((X, \underline{\mathcal{E}xt}^d(E, \mathcal{O}(k))))$  with  $p > 0$  gives a nonzero contribution in  $\text{Ext}^{d+p}(E, \mathcal{O}(k))$ . But the latter space is zero for  $p > 0$ . Hence  $H^p((X, \underline{\mathcal{E}xt}^d(E, \mathcal{O}(k)))) = H^p((X, \underline{\mathcal{E}xt}^d(E, \mathcal{O}(k)))) = 0$ , that is  $\underline{\mathcal{E}xt}^d(E, \mathcal{O})$  is an Artin sheaf.

(3) If  $F$  is an Artin sheaf then it follows from Lemma 2.0.7. and ampleness that  $\underline{\mathcal{E}xt}^{<d}(F, \mathcal{O}) = 0$ . Conversely, assume that  $\underline{\mathcal{E}xt}^{<d}(F, \mathcal{O}) = 0$ . We have a spectral sequence

$$E_2^{p,q} = \underline{\mathcal{E}xt}^p(\underline{\mathcal{E}xt}^{-q}(F, \mathcal{O}), \mathcal{O}) \implies E_\infty^i = \begin{cases} F, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\underline{\mathcal{E}xt}^{<d}(F, \mathcal{O}) = 0$ , the spectral sequence degenerates at the second term and yields an isomorphism:  $F \cong \underline{\mathcal{E}xt}^d(\underline{\mathcal{E}xt}^d(F, \mathcal{O}), \mathcal{O})$ . Hence,  $F$  is an Artin sheaf by (2).

(4) It follows from the proof of (3) that for any Artin sheaf  $F$  we have a canonical isomorphism:

$$F \cong \underline{\mathcal{E}xt}^d(\underline{\mathcal{E}xt}^d(F, \mathcal{O}), \mathcal{O}).$$

Hence,  $\underline{\mathcal{E}xt}^d(F, \mathcal{O}) = 0$  implies  $F = 0$ . (5) is clear. □

### 3. Torsion-Free Sheaves on $\mathbb{P}_\Gamma^2$

In this section we set

$$A^\tau = (C\langle x, y, z \rangle \# \Gamma) / \langle [x, z] = [y, z] = 0, [y, x] = \tau z^2 \rangle,$$

see Definition 1.2.5. We write  $A^\tau = \bigoplus_{i \geq 0} A_i$ , and put  $\mathbb{P}_\Gamma^2 = \text{Proj}(A^\tau)$ .

#### 3.1. BEILINSON SPECTRAL SEQUENCE

The noncommutative projective plane  $\mathbb{P}_\Gamma^2$  shares a lot of properties with the commutative projective plane. In particular, we have, see Appendix B for the definition of  ${}^*A_j$ :

$$H^p(\mathbb{P}_\Gamma^2, \mathcal{O}(i)) = \begin{cases} A_i, & \text{if } p = 0 \text{ and } i \geq 0, \\ {}^*A_{-i-3}, & \text{if } p = 2 \text{ and } i \leq -3, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.1}$$

Our approach to the classification of torsion free sheaves on  $\mathbb{P}_\Gamma^2$  mimics the standard approach, see [OSS], [KKO], to the study of coherent sheaves on  $\mathbb{P}^2$  by means of Beilinson’s spectral sequence, see Appendix B. The latter allows one to describe coherent sheaves in terms of linear algebra data, sometimes called the ‘ADHM-equations’.

Beilinson’s spectral sequence for a general Koszul algebra, given in Appendix B, simplifies considerably in the ‘two-dimensional’ case of  $\mathbb{P}_\Gamma^2$ . Specifically, let  $T$  denote the coherent sheaf on  $\mathbb{P}_\Gamma^2$  defined by either of the following two exact sequences:

$${}^!A_0 \otimes_{\mathbb{C}\Gamma} \mathcal{O} \hookrightarrow {}^!A_1 \otimes_{\mathbb{C}\Gamma} \mathcal{O}(1) \twoheadrightarrow T, \quad T \hookrightarrow {}^!A_2 \otimes_{\mathbb{C}\Gamma} \mathcal{O}(2) \twoheadrightarrow {}^!A_3 \otimes_{\mathbb{C}\Gamma} \mathcal{O}(3), \tag{3.1.2}$$

where  ${}^!A_k$  stand for the graded components of the Koszul dual algebra, see Appendix B. Then, for any sheaf  $E$ , there is a spectral sequence with the  $E_1^{p,q}$ -term looking like a 3-term complex:

$$E_1^{p,q} = \{ \text{Ext}^q(\mathcal{O}(1), E) \otimes_{\mathbb{C}\Gamma} \mathcal{O}(-2) \rightarrow \text{Ext}^q(T(-1), E) \otimes_{\mathbb{C}\Gamma} \mathcal{O}(-1) \rightarrow \text{Ext}^q(\mathcal{O}, E) \otimes_{\mathbb{C}\Gamma} \mathcal{O} \}$$

The above complex corresponds to the groups  $E_1^{p,q}$  with  $p = -2, -1, 0$ , all other groups being zero. This spectral sequence converges to (see, e.g., [OSS] for details in the commutative case):

$$E_1^{p,q} \implies E_\infty^{p,q} = \begin{cases} E, & \text{for } p+q=0, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that  $K(-)$  stands for the Grothendieck group of an Abelian category.

**COROLLARY 3.1.3.** *We have  $K(\text{coh}(\mathbb{P}_\Gamma^2)) \cong K(\Gamma)^{\oplus 3}$ . In particular,  $K(\text{coh}(\mathbb{P}_\Gamma^2))$  is a free  $\mathbb{Z}$ -module with a basis given by classes of sheaves  $R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(k)$ , where  $R$  runs through the set of isomorphism classes of irreducible  $\Gamma$ -modules and  $k \in \{-2, -1, 0\}$ .*

For any  $E \in \text{coh}(\mathbb{P}_\Gamma^2)$  we define the *Hilbert function* of  $E$  as

$$h_E(t) = \sum_{p=0}^2 (-1)^p \dim_{\mathbb{C}} H^p(\mathbb{P}_\Gamma^2, E(t)).$$

**LEMMA 3.1.4.** *For any  $E \in \text{coh}(\mathbb{P}_\Gamma^2)$ , the function  $h_E(t)$  is a polynomial in  $t$  (of degree  $\leq 2$ ) of the form*

$$h_E(t) = r \frac{t^2}{2} + \dots, \quad r = 0, 1, 2, \dots$$

*Proof.* It is clear that  $h_E(t)$  depends only on the class of  $E$  in  $K(\text{coh}(\mathbb{P}_\Gamma^2))$ . Hence, by Corollary 3.1.3 it suffices to compute the Hilbert function only for the sheaves of the form  $R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(p)$ ,  $p \in \mathbb{Z}$ , where  $R$  is a  $\Gamma$ -module. Further, from (3.1.1) one deduces that for  $t+k \geq 0$  we have:  $H^{>0}(\mathbb{P}_\Gamma^2, R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(t+k)) = 0$ . Therefore, we find

$$\begin{aligned} h_{R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(k)}(t) &= \dim_{\mathbb{C}} H^0(\mathbb{P}_\Gamma^2, R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(t+k)) \\ &= \dim_{\mathbb{C}}(R \otimes_{\mathbb{C}\Gamma} A_{t+k}^\tau) = \dim_{\mathbb{C}}(R \otimes_{\mathbb{C}\Gamma} (\mathbb{C}\Gamma \otimes_{\mathbb{C}} \text{Sym}^{t+k}(x, y, z))) \\ &= \dim_{\mathbb{C}}(R \otimes_{\mathbb{C}} \text{Sym}^{t+k}(x, y, z)) = \frac{(t+k+1)(t+k+2)}{2} \cdot \dim_{\mathbb{C}} R \end{aligned}$$

*A similar computation shows that this formula also holds for  $t+k < 0$ .*

Thus we see that, for any  $E$ , the Hilbert function  $h_E(t)$  is a quadratic polynomial of the form  $h_E(t) = r(t^2/2) + \dots$ , with some  $r \in \mathbb{Z}$ . Now if  $r$  were negative then  $h_E(t)$  would have been negative for  $t \gg 0$  which is nonsense, because by the ampleness we have  $h_E(t) = \dim_{\mathbb{C}} H^0(\mathbb{P}_{\Gamma}^2, E(t)) \geq 0$ , for  $t \gg 0$ . Thus  $r$  is nonnegative and we are done.  $\square$

**DEFINITION 3.1.5.** Define the rank  $r(E)$  of  $E \in \text{coh}(\mathbb{P}_{\Gamma}^2)$  to be the leading coefficient  $r$  of the Hilbert polynomial  $h_E(t)$ , see Lemma 3.1.4.

**COROLLARY 3.1.6.** *The rank,  $r(-)$ , is a well-defined linear function on  $K(\text{coh}(\mathbb{P}_{\Gamma}^2))$ . Moreover,  $r(R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(i)) = \dim_{\mathbb{C}} R$ , for any  $\Gamma$ -module  $R$  and any  $i \in \mathbb{Z}$ .*

### 3.2. SHEAVES ON $\mathbb{P}_{\Gamma}^1$

Recall that in Section 1.2 we have defined the noncommutative projective line as the Proj-scheme corresponding to the algebra  $\mathbb{C}[x, y] \# \Gamma$ . The following result is clear:

**PROPOSITION 3.2.7.** *The embedding of graded algebras  $\mathbb{C}[x, y] \subset \mathbb{C}[x, y] \# \Gamma$  induces an equivalence of categories  $\text{coh}(\mathbb{P}_{\Gamma}^1) \rightarrow \text{coh}_{\Gamma}(\mathbb{P}^1)$ , where  $\text{coh}_{\Gamma}(\mathbb{P}^1)$  is the category of  $\Gamma$ -equivariant coherent sheaves on the commutative  $\mathbb{P}^1$ .*

Note that the equivalence of Proposition 3.2.7 commutes with the functors  $\text{Ext}^p$ ,  $\mathcal{E}xt^p$ , and  $H^p$ . It follows that a sheaf  $E$  on  $\mathbb{P}_{\Gamma}^1$  is locally free (resp. Artin) if and only if it is locally free (resp. Artin) as a  $\Gamma$ -equivariant sheaf on  $\mathbb{P}^1$ . For any sheaf  $E \in \text{coh}(\mathbb{P}_{\Gamma}^1)$ , we denote by  $h_E(t)$ , resp.  $r(E)$ , the Hilbert polynomial, resp. the rank, of  $E$ , considered as a  $\Gamma$ -equivariant sheaf on  $\mathbb{P}^1$ .

**COROLLARY 3.2.8.** (1) *For any coherent sheaf  $E$  on  $\mathbb{P}_{\Gamma}^1$  we have  $E \cong F \oplus \mathcal{E}$ , where  $F$  is an Artin sheaf and  $\mathcal{E}$  is a locally free sheaf.*

(2) *Any locally free sheaf on  $\mathbb{P}_{\Gamma}^1$  has the form  $\mathcal{E} = \bigoplus_k (R_k \otimes_{\mathbb{C}\Gamma} \mathcal{O}(k))$ , for certain  $\Gamma$ -modules  $R_k$ .*

(3) *If  $\mathcal{E}$  is a locally free sheaf on  $\mathbb{P}_{\Gamma}^1$  and  $r(\mathcal{E}) = 1$ , then  $\mathcal{E} \cong R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(k)$ , where  $R$  is a one-dimensional  $\Gamma$ -module.*

### 3.3. THE FUNCTORS $i^*$ AND $i_*$

Recall that in Section 1.2 we have defined the functors

$$i_*: \text{coh}(\mathbb{P}_{\Gamma}^1) \rightarrow \text{coh}(\mathbb{P}_{\Gamma}^2) \quad \text{and} \quad i^*: \text{coh}(\mathbb{P}_{\Gamma}^2) \rightarrow \text{coh}(\mathbb{P}_{\Gamma}^1).$$

It is clear that  $i_*$  is the right adjoint of the functor  $i^*$ . It follows from the definition that  $i^*$  is right exact. We denote by  $L^p i^*$  the left derived functor.

PROPOSITION 3.3.9.

- (1) *The functor  $i_*$  is exact and faithful.*
- (2) *For any coherent sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  there is a canonical exact sequence:*  

$$0 \longrightarrow i_*L^1i^*E \longrightarrow E(-1) \xrightarrow{z} E \longrightarrow i_*L^0i^*E \longrightarrow 0$$
- (3) *For any  $E$  we have  $L^{>1}i^*E = 0$  and the functor  $L^1i^*$  is left exact.*
- (4) *If  $\mathcal{E}$  is locally free, then  $L^{>0}i^*\mathcal{E} = 0$ .*
- (5) *If  $E$  is torsion free, then  $L^{>0}i^*E = 0$ .*
- (6) *If  $\mathcal{E}$  is a locally free sheaf on  $\mathbb{P}_\Gamma^2$ , then  $i^*\mathcal{E}$  is locally free.*
- (7) *If  $E$  is torsion free, then  $r(i^*E) = r(E)$ .*
- (8) *For any sheaf  $E$  on  $\mathbb{P}_\Gamma^2$ , the adjunction morphism  $E \rightarrow i_*i^*E$  is an epimorphism.*
- (9) *We have  $i^*i_*E = E$ ,  $L^1i^*i_*E = E(-1)$ , for any sheaf  $E$  on  $\mathbb{P}_\Gamma^1$ .*
- (10) *If  $i^*E = 0$ , then  $E \cong E(1)$  and  $E$  is an Artin sheaf.*

*Proof.* (1) This claim becomes clear when translated into the module language.

(2) Since  $i_*$  is exact it suffices to check that  $i_*i^*E \cong E \otimes i_*\mathcal{O}_{\mathbb{P}_\Gamma^1}$  (which is clear from the point of view of modules), and to apply the resolution

$$0 \rightarrow \mathcal{O}(-1) \xrightarrow{z} \mathcal{O} \rightarrow i_*\mathcal{O}_{\mathbb{P}_\Gamma^1} \rightarrow 0.$$

(3) Follows from (2) and (1).

(4) It is clear that  $L^{>0}i^*\mathcal{O}(n) = 0$ . Choosing an embedding  $\mathcal{E} \rightarrow \mathcal{O}(n)^{\oplus m}$  like in (2.0.5) and applying (3) we obtain the claim.

(5) Given a torsion free sheaf  $E$ , choose an embedding  $E \hookrightarrow \mathcal{E}$ , into a locally free sheaf. Now, apply (3) and (4).

(6) Note that derived functor  $L^*i^*$  commutes with the derived functor  $\underline{\mathcal{E}xt}(-, \mathcal{O})$ . Applying (4) to the locally free sheaf  $\mathcal{E}$  we conclude that  $\underline{\mathcal{E}xt}^{>0}(i^*\mathcal{E}, \mathcal{O}_{\mathbb{P}_\Gamma^1}) = 0$ . Hence  $i^*\mathcal{E}$  is locally free.

(7) Since  $L^{>0}i^*E = 0$  by (5) it follows that  $h_{i^*E}(t) = h_E(t) - h_E(t - 1) = r(E)t + \dots$ . The claim follows.

(8) Follows from (2).

(9) Note that by (2) we have

$$i_*i^*i_*E = \mathbf{Coker}(i_*E(-1) \xrightarrow{z} i_*E) \quad \text{and} \quad i_*L^1i^*i_*E = \mathbf{Ker}(i_*E(-1) \xrightarrow{z} i_*E).$$

On the other hand,  $z$  vanishes on  $\mathbb{P}_\Gamma^1$ , hence the morphism  $i_*E(-1) \xrightarrow{z} i_*E$  vanishes. Thus  $i_*i^*i_*E = i_*E$  and  $i_*L^1i^*i_*E = i_*E(-1)$ . Hence, by (1) we have  $i^*i_*E = E$ ,  $L^1i^*i_*E = E(-1)$ .

(10) The equation  $i^*E = 0$  implies by (2) that the multiplication by  $z$  homomorphism:  $E \xrightarrow{z} E(1)$  is surjective. Let  $M$  be the graded  $A$ -module corresponding to  $E$ . It follows that, for  $k \gg 0$ , the  $z$ -multiplication  $M_k \xrightarrow{z} M_{k+1}$  is a surjection, by the ampleness. Since  $M$  is finitely generated it follows that  $M_k \cong M_{k+1}$  for  $k \gg 0$ . Hence  $E \cong E(1)$ . But this implies that for any  $k$  we have  $H^{>0}(\mathbb{P}_\Gamma^2, E(k)) = 0$ . Thus,  $E$  is an Artin sheaf. □

DEFINITION 3.3.10. A sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  is said to be *supported on*  $\mathbb{P}_\Gamma^1$  if it admits an increasing finite filtration:  $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ ,  $E_i \in \text{coh}(\mathbb{P}_\Gamma^2)$ , such that  $E_k/E_{k-1} = i_*F_k$ , for some  $F_k \in \text{coh}(\mathbb{P}_\Gamma^1)$ .

PROPOSITION 3.3.11. (1) *If  $E$  is supported on  $\mathbb{P}_\Gamma^1$ , then  $r(E) = 0$ .*

(2) *Assume that  $E$  is supported on  $\mathbb{P}_\Gamma^1$ . Let  $E^0 = E$  and, for  $k > 0$ , define  $E^k$  inductively by  $E^{k+1} := \mathbf{Ker}(E^k \rightarrow i_*i^*E^k)$ , then  $E^n = 0$ , for  $n \gg 0$ .*

(3) *If  $E$  is supported on  $\mathbb{P}_\Gamma^1$  and  $i^*E$  is Artin, then  $E$  is Artin.*

*Proof.* (1) It is clear that  $r(E) = \sum r(E_k/E_{k-1}) = \sum r(i_*F_k)$ . On the other hand, we have

$$h_{i_*F}(t) = h_F(t) = r(F)/t + \text{const},$$

for any sheaf  $F$  on  $\mathbb{P}_\Gamma^1$ . Hence  $r(i_*F) = 0$ .

(2) It suffices to prove the following claim: *Assume that we have an exact sequence*

$$0 \rightarrow E' \rightarrow E \rightarrow i_*F \rightarrow 0.$$

*Then  $\mathbf{Ker}(E \rightarrow i_*i^*E)$  is a subsheaf in  $E'$ .* Indeed, if the claim is true then it is easy to prove by induction that  $E^k \subset E_{n-k}$ , where  $E_\bullet$  is a filtration from the definition of a sheaf supported on  $\mathbb{P}_\Gamma^1$ . Hence  $E_n \subset E_0 = 0$ , and (2) follows.

We now prove the claim. Since the canonical map:  $i_*F \rightarrow i_*i^*i_*F$  is an isomorphism by 3.3.9 (9) it follows that the projection  $E \rightarrow i_*F$  factors through the morphism  $E \rightarrow i_*i^*E$ . Thus, the claim follows from the inclusion:

$$\mathbf{Ker}(E \rightarrow i_*i^*E) \subset \mathbf{Ker}(E \rightarrow i_*F) = E'.$$

(3) Assume that  $E$  is supported on  $\mathbb{P}_\Gamma^1$  and that  $i^*E$  is an Artin sheaf. We define inductively the sequence of sheaves  $E^0 = E$ ,  $E^{k+1} = \mathbf{Ker}(E^k \rightarrow i_*i^*E^k)$ ,  $k = 0, 1, 2, \dots$ , as in (2). According to part (2), there exists  $n > 0$  such that  $E^n = 0$ . We prove by descending induction on  $k$ , starting at  $k = n$ , that  $E^k$  is an Artin sheaf. Applying the functor  $i^*$  to the exact sequence

$$0 \rightarrow E^{k+1} \rightarrow E^k \rightarrow i_*i^*E^k \rightarrow 0$$

we get an exact sequence:

$$L^1i^*i_*i^*E^k \rightarrow i^*E^{k+1} \rightarrow i^*E^k \rightarrow i^*i_*i^*E^k \rightarrow 0.$$

Here, the morphism  $i^*E^k \rightarrow i^*i_*i^*E^k$  is an isomorphism by 3.3.9 (9). It follows that  $i^*E^{k+1}$  is a quotient of the Artin sheaf  $L^1i^*i_*i^*E^k \simeq E^k(-1)$ , see 3.3.9 (10). Since a quotient of an Artin sheaf on the commutative  $\mathbb{P}^1$  is obviously Artin again, it follows that  $i^*E^{k+1}$  is an Artin sheaf. The induction hypothesis implies that  $E^{k+1}$  is an Artin sheaf. Hence  $E$ , being an extension of Artin sheaves, is also Artin by 2.0.9 (5).  $\square$

DEFINITION 3.3.12. A sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  is called *z-torsion free* if either of the following equivalent conditions hold (equivalence is proved in 3.3.9(2))

- (1) the map  $z : E \rightarrow E(1)$  is a monomorphism;
- (2)  $L^1i^*E = 0$ .

LEMMA 3.3.13.

- (1) *If  $E$  is torsion free then  $E$  is  $z$ -torsion free.*
- (2) *For any coherent sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  there exists a unique subsheaf  $F \subset E$  supported on  $\mathbb{P}_\Gamma^1$  such that  $E/F$  is  $z$ -torsion free.*

*Proof.* (1) Follows from 3.3.9 (5).

(2) First we will check the existence. Consider the sequence  $F_k = \mathbf{Ker}(E \xrightarrow{z^k} E(k))$ ,  $k = 0, 1, 2, \dots$ , of subseaves in  $E$ . Since the algebra  $A^\tau$  is noetherian the sequence  $F_k$  stabilizes. Thus we have  $F_{n+k} = F_n$  for some  $n$  and for all  $k \geq 0$ . The sheaf  $F = F_n$  is clearly supported on  $\mathbb{P}_\Gamma^1$ . We claim that  $E/F$  is  $z$ -torsion free. To see this, note that the equality:  $F_n = F_{n+1}$  means that the composition of the embedding  $E/F \hookrightarrow E(n)$  with the morphism:  $E(n) \xrightarrow{z} E(n+1)$  is an embedding. But this composition can be factored through the morphism:  $E/F \xrightarrow{z} (E/F)(1)$ . Thus,  $E/F$  is  $z$ -torsion free.

It remains to prove the uniqueness. Assume that  $F'$  is a subsheaf of  $E$  supported on  $\mathbb{P}_\Gamma^1$  and such that  $E/F'$  is  $z$ -torsion free. Then for any  $k \in \mathbb{Z}$  we have an exact sequence

$$\mathbf{Ker}(F' \xrightarrow{z^{n+k}} F'(n+k)) \hookrightarrow \mathbf{Ker}(E \xrightarrow{z^{n+k}} E(n+k)) \rightarrow \mathbf{Ker}(E/F' \xrightarrow{z^{n+k}} (E/F')(n+k))$$

It is clear that for  $k \gg 0$  the first term in the above sequence coincides with  $F'$ , thesecond term coincides with  $F_{n+k} = F$  and the third term vanishes. Thus  $F' = F$ . □

### 3.4. PROPERTIES OF $\text{coh}(\mathbb{P}_\Gamma^2)$ FOR GENERIC $\tau$

The definition of ‘generic’ parameters  $\tau$ , due to [CBH], has been sketched in the Introduction. The only property that will be used below is that, for generic  $\tau$ , the algebra  $\mathcal{B}^\tau$  has no nontrivial finite-dimensional modules. In the context of deformed preprojective algebras, an equivalent property has been proved in [CBH, §7].

PROPOSITION 3.4.14. *Suppose that  $\tau$  is generic. Then,*

- (1) *if  $i^*E = 0$ , then  $E = 0$ .*
- (2) *If  $\phi \in \text{Hom}(E, F)$  and  $i^*\phi$  is an epimorphism, then  $\phi$  is an epimorphism.*
- (3) *If  $\phi \in \text{Hom}(E, F)$  and both  $i^*\phi$  and  $L^1i^*\phi$  are isomorphisms then so is  $\phi$ .*
- (4) *If  $\phi \in \text{Hom}(E, F)$ ,  $i^*\phi$  is a monomorphism and  $L^1i^*F = 0$  then  $\phi$  is a monomorphism.*
- (5) *A sheaf  $E$  is locally free if and only if  $L^{>0}i^*E = 0$  and  $i^*E$  is locally free.*
- (6) *If  $E \in \text{coh}(\mathbb{P}_\Gamma^2)$  is torsion free, and the sheaf  $i^*E$  is locally free, then  $E$  is locally free.*

*Proof.* (1) The statement translated into the language of  $A^\tau$ -modules reads: *If  $M$  is a finitely generated graded  $A^\tau$ -module such that  $\dim_{\mathbb{C}}(M/zM) < \infty$ , then  $\dim_{\mathbb{C}}(M) < \infty$ . To prove this, note first that, for any  $i \geq 0$ , the space  $M_i$  is finite-dimensional, since  $M$  is finitely generated. Now, assume  $M/zM$  is finite-dimensional.*



Hence, the multiplication map  $z: M_i \rightarrow M_{i+1}$  is surjective, for  $i \gg 0$ . It follows that, for  $i \gg 0$ , the sequence:  $\dim M_i \geq \dim M_{i+1} \geq \dots$ , stabilizes, hence the map  $z: M_i \rightarrow M_{i+1}$  is an isomorphism. But for such an  $i$ , the endomorphisms  $z^{-1}x, z^{-1}y: M_i \rightarrow M_i$  provide  $M_i$  with the structure of a finite-dimensional  $\mathcal{B}^\tau$ -module. For generic  $\tau$ , the algebra  $\mathcal{B}^\tau$  has no finite-dimensional representations (because it is Morita equivalent to the deformed preprojective algebra, and the latter has no finite-dimensional representations by Theorem 7.7 of [CBH]). Therefore  $M_i = 0$  for  $i \gg 0$ , hence  $M$  is finite dimensional.

(2) Assume that  $\phi \in \text{Hom}(E, F)$  is such that  $i^*\phi$  is an epimorphism. Since  $i^*$  is right exact it follows that  $i^* \mathbf{Coker} \phi = 0$ , hence  $\mathbf{Coker} \phi = 0$  by (1).

(3) It follows from (2) that  $\phi$  is an epimorphism. Hence, we have an exact sequence

$$L^1 i^* E \xrightarrow{L^1 i^* \phi} L^1 i^* F \longrightarrow i^* (\mathbf{Ker} \phi) \longrightarrow i^* E \xrightarrow{i^* \phi} i^* F \longrightarrow 0.$$

It follows that  $i^* \mathbf{Ker} \phi = 0$ , hence  $\mathbf{Ker} \phi = 0$  by (1), that is  $\phi$  is a monomorphism. Thus  $\phi$  is an isomorphism.

(4) If  $L^1 i^* F = 0$  then  $i^* \mathbf{Ker} \phi = \mathbf{Ker} i^* \phi = 0$ , hence  $\mathbf{Ker} \phi = 0$  by (1).

(5) If  $E$  is locally free then by 3.3.9(4) and 3.3.9(6) we have  $L^{>0} i^* E = 0$  and  $i^* E$  is locally free. Conversely, assume that  $L^{>0} i^* E = 0$ . Then we have a spectral sequence

$$E_2^{p,q} = L^{-q} i^* \underline{\mathcal{E}xt}^p(E, \mathcal{O}) \implies \underline{\mathcal{E}xt}^k(i^* E, \mathcal{O}_{\mathbb{P}^1}).$$

It follows from 3.3.9(3) that the spectral sequence degenerates in the second term. Hence, if  $i^* E$  is locally free, we get

$$E_2^{1,0} = i^* \underline{\mathcal{E}xt}^1(E, \mathcal{O}) = 0 \quad \text{and} \quad E_2^{2,0} = i^* \underline{\mathcal{E}xt}^2(E, \mathcal{O}) = 0$$

Thus,  $E$  is locally free by (1).

(6) Follows from (5) and 3.3.9(5) □

#### 4. Interpretation of Quiver Varieties

##### 4.1. FROM QUIVER DATA TO A SHEAF

Let  $L^*$  be the dual of the tautological two-dimensional  $\Gamma$ -module. We fix  $V$  and  $W$ , finite dimensional  $\Gamma$ -modules, and a triple

$$(B, I, J) \in \text{Hom}_\Gamma(V, V_\mathbb{C} \otimes L^*) \oplus \text{Hom}_\Gamma(W, V) \oplus \text{Hom}_\Gamma(V, W).$$

Let  $\{e_x, e_y\}$  be the basis of  $L^*$  dual to the basis  $\{x, y\}$  of  $L$ . Then we can consider  $e_x x + e_y y \in L^* \otimes_{\text{cr}} L$  as an element of

$$L^* \otimes_{\text{cr}} (L \oplus \text{triv}) = L^* \otimes_{\text{cr}} A_1 = H^0(\mathbb{P}_\Gamma^2, L^* \otimes_{\text{cr}} \mathcal{O}(1)) = \text{Hom}(\mathcal{O}(-1), L^* \otimes_{\text{cr}} \mathcal{O}),$$

where  $\text{triv}$  stands for the trivial one-dimensional  $\Gamma$ -module. Dually, viewing  $e_x x + e_y y$  as an element of  $L \otimes_{\text{cr}} L^*$ , we get a natural element in

$$(L \oplus \text{triv}) \otimes_{\text{cr}} L^* = A_1 \otimes_{\text{cr}} L^* = \text{Hom}(L \otimes_{\text{cr}} \mathcal{O}, \mathcal{O}(1)).$$

Thus, we can define canonical sheaf morphisms  $a = a_{B,I,J}$  and  $b = b_{B,I,J}$ , by:

$$a = \begin{pmatrix} B \cdot z - \text{Id}_V \otimes (e_x \cdot x + e_y \cdot y) \\ J \cdot z \end{pmatrix}: V \otimes_{\text{cr}} \mathcal{O}(-1) \longrightarrow ((V \otimes_{\text{c}} L) \oplus W) \otimes_{\text{cr}} \mathcal{O} \quad (4.1.1)$$

$$b = (B \cdot z - \text{Id}_V \otimes (e_x \cdot x + e_y \cdot y), I \cdot z): ((V \otimes_{\text{c}} L) \oplus W) \otimes_{\text{cr}} \mathcal{O} \longrightarrow V \otimes_{\text{cr}} \mathcal{O}(1).$$

From now on we use the canonical  $\Gamma$ -module isomorphism  $L^* \simeq L$ , to identify  $L^*$  with  $L$ . Thus, the maps (4.1.1) give the following morphisms of coherent sheaves on  $\mathbb{P}_{\Gamma}^2$ :

$$V \otimes_{\text{cr}} \mathcal{O}(-1) \xrightarrow{a_{B,I,J}} ((V \otimes_{\text{c}} L) \oplus W) \otimes_{\text{cr}} \mathcal{O} \xrightarrow{b_{B,I,J}} V \otimes_{\text{cr}} \mathcal{O}(1). \quad (4.1.2)$$

**DEFINITION 4.1.3.** A monad is a three-term complex,  $\mathcal{C}$ , concentrated in degrees  $-1, 0, 1$ , with the single nonzero cohomology group,  $H^0(\mathcal{C})$ , referred to as the cohomology of the monad.

**PROPOSITION 4.1.4.** *If  $a_{B,I,J}$  and  $b_{B,I,J}$  are given by (4.1.1) with  $(B, I, J) \in \mathbb{M}_{\Gamma}^{\text{c}}(V, W)$ , cf. (1.3.8.), then (4.1.2) is a monad and its middle cohomology sheaf  $E$  admits a canonical framing  $i^*E \cong W \otimes_{\text{cr}} \mathcal{O}$  and, moreover,  $H^1(\mathbb{P}_{\Gamma}^2, E(-1)) \cong V$ .*

The proof will take the rest of this subsection. It will be divided into a sequence of lemmas. We begin with an obvious

**LEMMA 4.1.5.** *The restriction of (4.1.2) to  $\mathbb{P}_{\Gamma}^1$  is a complex which is canonically quasi-isomorphic to  $W \otimes_{\text{cr}} \mathcal{O}$ .*

**LEMMA 4.1.6.** *The triple  $(B, I, J)$  satisfies the moment map equation if and only if  $b_{B,I,J} \circ a_{B,I,J} = 0$ .*

*Proof.* Straightforward computation shows that  $b_{B,I,J} \circ a_{B,I,J} = ([B, B] + IJ - \tau) \cdot z^2$ .  $\square$

From now on assume that  $(B, I, J)$  satisfies the moment map equation. Then (4.1.2) is a complex by Lemma 4.1.6. We choose the cohomological grading of this complex so that its middle term has degree zero. Let  $\mathcal{H}^p$ ,  $p = -1, 0, 1$ , denote the cohomology sheaves of this complex.

**LEMMA 4.1.7.** *We have  $i^*\mathcal{H}^{-1} = i^*\mathcal{H}^1 = 0$  and  $i^*\mathcal{H}^0$  is a subsheaf in  $W \otimes_{\text{cr}} \mathcal{O}$ . In particular,  $\mathcal{H}^{-1}$  and  $\mathcal{H}^1$  are Artin sheaves. If  $W = 0$  then  $\mathcal{H}^0$  is Artin as well.*

*Proof.* It follows from Lemma 4.1.5 that we have a spectral sequence with the second term

$$E_{p,q}^2 = L^{-q} i^* \mathcal{H}^p \implies E_i^{\infty} = \begin{cases} W \otimes_{\text{cr}} \mathcal{O}, & \text{if } i=0, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, 3.3.9 (3) implies that  $E_{p,q}^2 = 0$ , for  $q \neq 0, 1$ , hence this spectral sequence degenerates at the second term. Hence  $i^*\mathcal{H}^{-1} = i^*\mathcal{H}^1 = 0$  and  $i^*\mathcal{H}^0$  is a subsheaf in  $W \otimes_{\mathcal{C}_T} \mathcal{O}$ . But then 3.3.9 (10) implies that  $\mathcal{H}^{-1}$  and  $\mathcal{H}^1$  are Artin sheaves, and  $\mathcal{H}^0$  is also Artin whenever  $W = 0$ .  $\square$

LEMMA 4.1.8. *The map  $a_{B,I,J}$  is injective.*

*Proof.* We note that the sheaf  $\mathcal{H}^{-1} = \mathbf{Ker} a$  is simultaneously an Artin sheaf by 4.1.7 and locally free by 2.0.4 (1). Hence it vanishes by 2.0.9 (4).  $\square$

LEMMA 4.1.9. *If  $(B, I, J)$  is stable then  $b_{B,I,J}$  is surjective.*

*Proof.* The sheaf  $\mathcal{H}^1 = \mathbf{Coker} b$  is Artin by 4.1.7. Let  $\phi: V \otimes_{\mathcal{C}_T} \mathcal{O}(1) \twoheadrightarrow \mathcal{H}^1$  be the canonical projection and let

$$V' = \mathbf{Ker}[\phi(-1): H^0(\mathbb{P}_T^2, V \otimes_{\mathcal{C}_T} \mathcal{O}) \rightarrow H^0(\mathbb{P}_T^2, \mathcal{H}^1(-1))].$$

Note that  $z$ -multiplication gives an isomorphism  $\mathcal{H}^1(-1) \xrightarrow{\sim} \mathcal{H}^1$  (see 4.1.7 and 3.3.9 (10)). Hence the condition  $\phi \cdot b = 0$  implies:  $I(W) \subset V'$  and  $B(V') \subset V' \otimes L$ . Stability of  $(B, I, J)$  then yields:  $V' = V$ . Since  $\phi$  is surjective and  $\mathcal{H}^1$  is Artin it follows that  $\mathcal{H}^1 = 0$ , that is,  $b$  is surjective.  $\square$

LEMMA 4.1.10. *The sheaf  $\mathcal{H}^0$  is torsion free.*

*Proof.* Let  $\mathcal{C}$  denote the complex (4.1.2) and let  $\mathcal{C}^*$  denote the dual complex (in the category  $\mathbf{qgr}_{\text{left}}(A^\tau)$  of sheaves of left modules). We have a spectral sequence:

$$E_{p,q}^2 = \underline{\mathcal{E}xt}^q(\mathcal{H}^{-p}(\mathcal{C}), \mathcal{O}) \implies E_i^\infty = \mathcal{H}^i(\mathcal{C}^*).$$

Since  $\mathcal{H}^{-1} = 0$  by 4.1.8, it follows that the spectral sequence degenerates at the third term. Moreover,  $\underline{\mathcal{E}xt}^2(\mathcal{H}^0(\mathcal{C}), \mathcal{O}) = 0$ , and  $\underline{\mathcal{E}xt}^1(\mathcal{H}^0(\mathcal{C}), \mathcal{O})$  is a quotient of the sheaf  $\mathcal{H}^1(\mathcal{C}^*)$ . On the other hand, the complex  $\mathcal{C}^*$  is the complex, corresponding to the dual quiver data  $(B^*, J^*, I^*)$  (in the category of sheaves of left modules), hence by 4.1.7 we have:  $i^*\mathcal{H}^1(\mathcal{C}^*) = 0$ . Since  $i^*$  is right exact it follows that  $i^*\underline{\mathcal{E}xt}^1(\mathcal{H}^0(\mathcal{C}), \mathcal{O}) = 0$ . Therefore,  $\underline{\mathcal{E}xt}^1(\mathcal{H}^0(\mathcal{C}), \mathcal{O})$  is Artin by 3.3.9 (10).

Now consider the spectral sequence

$$E_{p,q}^2 = \underline{\mathcal{E}xt}^q(\underline{\mathcal{E}xt}^{-p}(\mathcal{H}^0(\mathcal{C}), \mathcal{O}), \mathcal{O}) \implies E_i^\infty = \begin{cases} \mathcal{H}^0(\mathcal{C}), & \text{if } i=0, \\ 0, & \text{otherwise.} \end{cases}$$

We have already proved that  $\underline{\mathcal{E}xt}^2(\mathcal{H}^0(\mathcal{C}), \mathcal{O}) = 0$ , and that  $\underline{\mathcal{E}xt}^1(\mathcal{H}^0(\mathcal{C}), \mathcal{O})$  is Artin. It follows that the spectral sequence degenerates at the third term giving rise to a short exact sequence (see 2.0.9 (3)):

$$0 \rightarrow \mathcal{H}^0(\mathcal{C}) \rightarrow (\mathcal{H}^0(\mathcal{C}))^{**} \rightarrow \underline{\mathcal{E}xt}^2(\underline{\mathcal{E}xt}^1(\mathcal{H}^0(\mathcal{C}), \mathcal{O}), \mathcal{O}) \rightarrow 0.$$

The middle sheaf here is locally free by 2.0.4 (5), hence  $\mathcal{H}^0(\mathcal{C})$  is torsion free by definition.  $\square$

*Proof of Proposition 4.1.4.* If  $(B, I, J) \in \mathbb{M}_T^\tau(V, W)$  then (4.1.2) is a complex by 4.1.6, which is left exact by 4.1.8 and right exact by 4.1.9. Hence, it is a monad.

Moreover, its middle cohomology sheaf  $E$  is torsion free by 4.1.10 and admits a canonical framing by 4.1.5. Finally, it is easy to see that  $H^1(\mathbb{P}_r^2, E(-1)) \cong V$ .  $\square$

Associating to any quiver data  $(B, I, J) \in \mathbb{M}_r^\tau(V, W)$ , the middle cohomology sheaf of the corresponding monad (4.1.2) we obtain a map  $\mathbb{M}_r^\tau(V, W) \rightarrow \mathcal{M}_r^\tau(V, W)$ . It is clear that this map is  $G_r(V)$ -equivariant. Indeed, any element  $g \in G_r(V)$  gives an isomorphism between the complex corresponding to a quiver data  $(B, I, J)$  and the complex corresponding to the quiver data  $(gBg^{-1}, gI, Jg^{-1})$ . It follows that the corresponding middle cohomology sheaves are isomorphic. Thus we obtain a well-defined map:

$$\mathbb{M}_r^\tau(V, W) \longrightarrow \mathcal{M}_r^\tau(V, W), \quad (4.1.11)$$

We will show that this map provides the bijection claimed in Theorem 1.3.10.

#### 4.2. FROM A FRAMED SHEAF TO QUIVER DATA

We are going to study framed torsion free sheaves on  $\mathbb{P}_r^2$  using the Beilinson spectral sequence. We will need the following lemma, cf. [KKO].

LEMMA 4.2.12. *Let  $E$  be a framed torsion free sheaf on  $\mathbb{P}_r^2$ . We have*

- (1)  $H^0(\mathbb{P}_r^2, E(-1)) = H^0(\mathbb{P}_r^2, E(-2)) = 0$ ;
- (2)  $H^2(\mathbb{P}_r^2, E(-1)) = H^2(\mathbb{P}_r^2, E(-2)) = 0$ ;
- (3)  $\text{Hom}(T(-1), E(-1)) = \text{Ext}^2(T(-1), E(-1)) = 0$ ;
- (4)  $H^1(\mathbb{P}_r^2, E(-1)) \simeq H^1(\mathbb{P}_r^2, E(-2))$ .

*Proof.* (1) We have  $L^0i^*E = W \otimes_{\text{cr}} \mathcal{O}$ , and  $L^1i^*E = 0$ , by Proposition 3.3.9(5). Thus, the exact sequence of Proposition 3.3.9(2) reads

$$0 \longrightarrow E(k-1) \longrightarrow E(k) \longrightarrow i_*L^0i^*E(k) = i_*(W \otimes_{\text{cr}} \mathcal{O}(k)) \longrightarrow 0. \quad (4.2.13)$$

Since  $H^0(\mathbb{P}_r^2, i_*(W \otimes_{\text{cr}} \mathcal{O}(k))) = H^0(\mathbb{P}_r^1, W \otimes_{\text{cr}} \mathcal{O}(k)) = 0$ , for all  $k < 0$ , we get

$$H^0(\mathbb{P}_r^2, E(-1)) = H^0(\mathbb{P}_r^2, E(-2)) = \dots = H^0(\mathbb{P}_r^2, E(-k)), \quad \forall k > 0.$$

Since  $E$  is torsion free it can be embedded into a sheaf  $\mathcal{O}(n)^{\oplus m}$ , by 2.0.4 (4). Hence

$$H^0(\mathbb{P}_r^2, E(-k)) \subset H^0(\mathbb{P}_r^2, \mathcal{O}(n-k)^{\oplus m}) = 0, \quad \forall k > n.$$

It follows that  $H^0(\mathbb{P}_r^2, E(-1)) = H^0(\mathbb{P}_r^2, E(-2)) = 0$ .

(2) Similarly, since

$$H^1(\mathbb{P}_r^2, i_*(W \otimes_{\text{cr}} \mathcal{O}(k))) = H^1(\mathbb{P}_r^1, W \otimes_{\text{cr}} \mathcal{O}(k)) = 0, \quad \text{for all } k \geq -1,$$

we get

$$H^2(\mathbb{P}_r^2, E(-2)) = H^2(\mathbb{P}_r^2, E(-1)) = \dots = H^2(\mathbb{P}_r^2, E(k)), \quad k > 0.$$

But the ampleness implies that, for  $k$  sufficiently large, one has  $H^2(\mathbb{P}_r^2, E(k)) = 0$ . Thus,  $H^2(\mathbb{P}_r^2, E(-2)) = H^2(\mathbb{P}_r^2, E(-1)) = 0$ .

(3) The first of the sequences (3.1.2) and part (1) imply  $\text{Hom}(T(-1), E(-1)) = 0$ . Similarly, the second of the sequences (3.1.2) and part (2) imply  $\text{Ext}^2(T(-1), E(-1)) = 0$ .

(4) Follows from (4.2.13) for  $k = -1$ .  $\square$

**COROLLARY 4.2.14.** *Any framed torsion free sheaf  $E$  can be represented as the middle cohomology sheaf of a monad of the form*

$$0 \rightarrow \tilde{V} \otimes_{\text{cr}} \mathcal{O}(-1) \rightarrow \tilde{V}' \otimes_{\text{cr}} \mathcal{O} \rightarrow \tilde{V} \otimes_{\text{cr}} \mathcal{O}(-1) \rightarrow 0,$$

where  $\tilde{V} = H^1(\mathbb{P}_r^2, E(-1))$  and  $\tilde{V}' = \text{Ext}^1(T, E)$ .

*Proof.* Follows from Beilinson's spectral sequence, see Section 3.1, applied to  $E(-1)$ , and the vanishing results of Lemma 4.2.12.  $\square$

Now, fix  $E \in \mathcal{M}_r^{\tau}(V, W)$ , and put  $\tilde{V} = H^1(\mathbb{P}_r^2, E(-1))$  and  $\tilde{V}' = \text{Ext}^1(T, E)$ , and fix a monad as in Lemma 4.2.14. Choose a triple  $(B, I, J) \in \mathcal{M}_r^{\tau}(V, W)$ , and consider the corresponding monad (4.1.2)

**LEMMA 4.2.15.** *Any vector space isomorphism  $\varphi: \tilde{V} \xrightarrow{\sim} V$  can be uniquely extended to an isomorphism of monads:*

$$\begin{array}{ccccc} \text{monad 4.2.14: } & \tilde{V} \otimes_{\text{cr}} \mathcal{O}(-1) & \hookrightarrow & \tilde{V}' \otimes_{\text{cr}} \mathcal{O} & \twoheadrightarrow & \tilde{V} \otimes_{\text{cr}} \mathcal{O}(-1) \\ & \downarrow \varphi & & \downarrow & & \downarrow \\ \text{monad 4.1.2: } & V \otimes_{\text{cr}} \mathcal{O}(-1) & \xrightarrow{a_{B,I,J}} & ((V \otimes_{\text{c}} L) \oplus W) \otimes_{\text{cr}} \mathcal{O} & \xrightarrow{b_{B,I,J}} & V \otimes_{\text{cr}} \mathcal{O}(1), \end{array}$$

which is compatible with framings.

*Proof.* A repetition of the proof in [KKO], Theorem 6.7.  $\square$

**LEMMA 4.2.16.** *If the monad in Corollary 4.2.14 is isomorphic to a complex (4.1.2) for some  $(B, I, J)$ , then the data  $(B, I, J)$  satisfy both the Moment Map Equation and the Stability Condition.*

*Proof.* The first part follows immediately from 4.1.6. Thus we have to check the stability. Assume that the triple  $(B, I, J)$  is not stable and let  $V' \subset V$  be a  $\Gamma$ -submodule such that  $V' \neq V$ ,  $I(W) \subset V'$ , and  $B(V') \subset V' \otimes L$ . Let  $V'' = V/V'$  and let  $\mathcal{C}(V, W)$ ,  $\mathcal{C}(V', W)$  and  $\mathcal{C}(V'', 0)$  be the complexes of the form (4.1.2) given by the triples, induced by  $(B, I, J)$ . Then we have an exact sequence of complexes

$$0 \rightarrow \mathcal{C}(V', W) \rightarrow \mathcal{C}(V, W) \rightarrow \mathcal{C}(V'', 0) \rightarrow 0,$$

which gives rise to a long exact sequence of cohomology sheaves

$$\dots \rightarrow \mathcal{H}^0 \mathcal{C}(V'', 0) \rightarrow \mathcal{H}^1 \mathcal{C}(V', W) \rightarrow \mathcal{H}^1 \mathcal{C}(V, W) \rightarrow \mathcal{H}^1 \mathcal{C}(V'', 0) \rightarrow 0. \quad (4.2.17)$$

Note that  $\mathcal{H}^{-1} \mathcal{C}(V'', 0) = 0$  by 4.1.8. The cohomology  $\mathcal{H}^0 \mathcal{C}(V'', 0)$  is both an Artin sheaf by 4.1.7, and a torsion free sheaf, by 4.1.10. It follows that  $\mathcal{H}^0 \mathcal{C}(V'', 0) = 0$ , by 2.0.6 and 2.0.9(4). Assume  $\mathcal{H}^1 \mathcal{C}(V'', 0) = 0$ . Then the complex  $\mathcal{C}(V'', 0)(-1)$  is

quasi-isomorphic to zero, hence the hypercohomology spectral sequence with  $E_2^{p,q}$ -term:  $E_2^{p,q} = H^p(\mathbb{P}_\Gamma^2, \mathcal{H}^q \mathcal{C}(V'', 0)(-1))$  would converge to zero. This sequence, however, clearly converges to the vector space  $V''$ . Therefore,  $\mathcal{H}^1 \mathcal{C}(V'', 0) \neq 0$ . But then the long exact sequence (4.2.17) would force:  $\mathcal{H}^1 \mathcal{C}(V, W) \neq 0$ , which is a contradiction, because  $\mathcal{C}(V, W)$  is a monad. The contradiction implies that  $V'' = 0$ .  $\square$

**PROPOSITION 4.2.18.** *The map:  $\mathfrak{M}_\Gamma^\tau(V, W) \rightarrow \mathcal{M}_\Gamma^\tau(V, W)$  defined in (4.1.11.) is a bijection.*

*Proof.* Surjectivity of the map follows from 4.2.14, 4.2.15 and 4.2.16. To check injectivity, note that it follows from 4.2.15 that the quiver data  $(B, I, J)$  are uniquely determined by a torsion free framed sheaf  $E$  up to isomorphism  $H^1(\mathbb{P}_\Gamma^2, E(-1)) \cong V$ , that is up to the action of the group  $G_\Gamma(V)$ .  $\square$

Proposition 4.2.18 completes the proof of Theorem 1.3.10.  
We finish this section with a few remarks.

**DEFINITION 4.2.19.** A triple  $(B, I, J)$  is costable if for any  $\Gamma$ -submodule  $V' \subset V$  such that  $J(V') = 0$  and  $B(V') \subset V' \otimes L$  we have  $V' = 0$ .

Note that a triple  $(B, I, J)$  is costable if and only if the dual triple  $(B^*, J^*, I^*)$  is stable.

**PROPOSITION 4.2.20 ([VV]).** *The framed sheaf corresponding to a (stable) quiver data  $(B, I, J) \in \mathfrak{M}_\Gamma^\tau(V, W)$  is locally free if and only if the triple  $(B, I, J)$  is costable.*

*Proof.* Repeating the arguments in the proof of 4.1.10 we see that for the middle cohomology sheaf  $\mathcal{H}^0$  of the complex (4.1.2) we have  $\underline{\mathcal{E}xt}^2(\mathcal{H}^0, \mathcal{O}) = 0$ , while  $\underline{\mathcal{E}xt}^1(\mathcal{H}^0, \mathcal{O}) = 0$  if and only if the complex corresponding to the dual triple is exact at the right term. But 4.1.9 and 4.2.16 imply that this holds if and only if the dual triple is stable.  $\square$

*Remark.* In general, locally-free framed sheaves form an open dense subset in  $\mathcal{M}_\Gamma^\tau(V, W)$  since the costability condition is open. However, for a generic value of the parameter  $\tau$  every stable quiver data is automatically costable, i.e. all torsion free framed sheaves are automatically locally free, cf. 3.4.14(6).

### 5. Proof of the Crawley-Boevey and Holland Conjecture

Throughout this section we assume  $\tau$  to be generic, though some results remain true for an arbitrary  $\tau$ .

Recall that we have defined in the Introduction the algebra  $\mathcal{B}^\tau = A^\tau / (z - 1) \cdot A^\tau$ , where  $z$  is the degree one central variable in the algebra  $A^\tau$ . Explicitly, we have  $\mathcal{B}^\tau = \mathbb{C}\langle x, y \rangle \# \Gamma / \langle [y, x] - \tau \rangle$ . The standard grading on the algebra  $A^\tau$  induces a canonical increasing filtration:  $\mathbb{C}\Gamma = \mathcal{B}_0^\tau \subset \mathcal{B}_1^\tau \subset \mathcal{B}_2^\tau \subset \dots$ , on  $\mathcal{B}^\tau$  such that  $\text{gr}(\mathcal{B}^\tau)$ , the associated graded algebra, has finite homological dimension. Thus, it follows

from a general result due to Quillen [Q], that the assignment  $R \mapsto R \otimes_{\text{cr}} \mathcal{B}^\tau$  induces an isomorphism:  $K(\Gamma) \xrightarrow{\sim} K(\mathcal{B}^\tau)$  of the corresponding Grothendieck groups of projective modules. Let  $[N]$  denote the class of a  $\mathcal{B}^\tau$ -module  $N$  in  $K(\mathcal{B}^\tau) \cong K(\Gamma)$ .

Recall that by Proposition 1.3.11, to any one-dimensional class  $R \in K(\Gamma)$ , one can canonically attach (isomorphism classes of)  $\Gamma$ -modules  $W$  and  $V$ , such that in  $K(\Gamma)$  we have:  $R = [W] + [V] \cdot ([L] - 2[\text{triv}])$  and, moreover,  $\dim W = 1$  and  $V$  does not contain the regular representation as a submodule. The goal of this section is to prove the following

**THEOREM 5.0.1.** *Given a class  $R \in K(\Gamma)$  such that  $\dim R = 1$ , let  $V, W$  be  $\Gamma$ -modules attached to  $R$  in Proposition 1.3.11. Then, there exists a natural bijection:*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of finitely generated projective} \\ \mathcal{B}^\tau\text{-modules } N \text{ such that } [N] = R \text{ in } K(\mathcal{B}^\tau) = K(\Gamma) \end{array} \right\} \\ \simeq \bigsqcup_{k=0}^{\infty} \mathcal{M}_\Gamma^\tau(V \oplus \mathbb{C}\Gamma^{\oplus k}, W).$$

This theorem together with Theorem 1.3.10 yields Theorem 1.3.12.

5.1. FROM SHEAVES ON  $\mathbb{P}_\Gamma^2$  TO PROJECTIVE  $\mathcal{B}^\tau$ -MODULES

Let  $\text{mod}(\mathcal{B}^\tau)$  denote the category of finitely generated (right)  $\mathcal{B}^\tau$ -modules. There is a natural ‘open restriction’ functor  $j^*: \text{gr}(A^\tau) \rightarrow \text{mod}(\mathcal{B}^\tau)$ ,  $M \mapsto M/(z-1) \cdot M$ .

It will be convenient to use an equivalent definition of the algebra  $\mathcal{B}^\tau$  that will make the open restriction functor  $j^*$  manifestly *exact*. Namely, let  $A^\tau[z^{-1}]$  denote the localization of the algebra  $A^\tau = \bigoplus_k A_k^\tau$  with respect to  $z$ , and  $A^\tau[z^{-1}]_0$ , the degree zero component of the localized algebra. We have:

$$\mathcal{B}^\tau \simeq A^\tau[z^{-1}]_0 = \varinjlim A_k^\tau,$$

where the direct limit is taken with respect to the embeddings  $A_k^\tau \xrightarrow{z} A_{k+1}^\tau$ , induced by multiplication by  $z$ . Using this formula one can rewrite the functor  $j^*$  in the form:

$$j^*: M = \bigoplus_k M_k \mapsto j^*M = \varinjlim M_k,$$

where the limit is taken with respect to the embeddings  $M_k \xrightarrow{z} M_{k+1}$ , induced by the  $z$ -action.

**LEMMA 5.1.2.**

- (1) *The functor  $j^*$  factors through the category  $\text{qgr}(A^\tau) = \text{coh}(\mathbb{P}_\Gamma^2)$ .*
- (2) *The functor  $j^*$  is exact.*
- (3) *The functor  $j^*: \text{coh}(\mathbb{P}_\Gamma^2) \rightarrow \text{mod}(\mathcal{B}^\tau)$  commutes with the dualization, i.e. for any coherent sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  we have  $j^*(E^*) = \text{Hom}_{\text{mod}(\mathcal{B}^\tau)}(j^*E, \mathcal{B}^\tau)$ .*
- (4) *For any  $\Gamma$ -module  $R$  we have:  $j^*(R \otimes_{\text{cr}} \mathcal{O}(k)) = R \otimes_{\text{cr}} \mathcal{B}^\tau$ . In particular, we have  $[j^*(R \otimes_{\text{cr}} \mathcal{O}(k))] = [R] \in K(\Gamma)$ .*
- (5) *If  $j^*E = 0$  then the sheaf  $E$  is supported on  $\mathbb{P}_\Gamma^1$ . In particular,  $r(E) = 0$ .*

*Proof.* (1) It suffices to check that, for any finite-dimensional graded  $A^\tau$ -module  $M$ , we have:  $j^*M = 0$ . But this is clear, because in this case, for  $k \gg 0$ , one has  $M_k = 0$ , hence  $\varinjlim M_k = 0$ .

(2) It follows from the exactness of the direct limit.

(3) Let  $M$  be the graded  $A^\tau$ -module corresponding to a sheaf  $E$ . Then we have

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{B}^\tau)}(j^*E, \mathcal{B}^\tau) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{B}^\tau)}\left(\varinjlim_k M_k, \mathcal{B}^\tau\right).$$

The dual sheaf  $E^*$  corresponds to the  $A^\tau$ -module  $\bigoplus_k \mathrm{Hom}_{\mathrm{coh}(\mathbb{P}_\Gamma^2)}(E, \mathcal{O}(k))$ , by definition. Hence, we obtain

$$\begin{aligned} j^*(E^*) &= \varinjlim_k \mathrm{Hom}_{\mathrm{coh}(\mathbb{P}_\Gamma^2)}(E, \mathcal{O}(k)) \\ &= \varinjlim_k \mathrm{Hom}_{\mathrm{qgr}(A^\tau)}\left(\bigoplus_{n=0}^\infty M_n, A^\tau(k)\right) \\ &= \varinjlim_{m,k} \mathrm{Hom}_{\mathrm{gr}(A^\tau)}\left(\bigoplus_{n=m}^\infty M_n, A^\tau(k)\right) \\ &= \varinjlim_m \mathrm{Hom}_{\mathrm{gr}(A^\tau)}\left(\bigoplus_{n=m}^\infty M_n, \varinjlim_k A^\tau(k)\right) \\ &= \mathrm{Hom}_{\mathrm{mod}(\mathcal{B}^\tau)}(\varinjlim_n M_n, \mathcal{B}^\tau). \end{aligned}$$

This is precisely what we need.

(4) We have

$$j^*(M \otimes_{\mathrm{cr}} \mathcal{O}(k)) = \varinjlim_l (M \otimes_{\mathrm{cr}} A_{k+l}^\tau) = M \otimes_{\mathrm{cr}} \left(\varinjlim_l A_{k+l}^\tau\right) = M \otimes_{\mathrm{cr}} \mathcal{B}^\tau.$$

(5) Let  $M$  be the graded  $A^\tau$ -module corresponding to a sheaf  $E$ . Then  $j^*E = 0$  implies that for any  $m \in M$ , there exists  $n \gg 0$  such that  $mz^n = 0$ . Since  $M$  is finitely generated we conclude that  $Mz^n = 0$  for some  $n \gg 0$ . Let  $M_k = \mathbf{Ker} z^k \subset M$  and put  $E_k = \pi(M_k)$ . This gives a filtration on  $E$ . Finally, for each  $k \in \mathbb{Z}_+$ , the element  $z$  annihilates the quotient  $M_k/M_{k-1}$ , hence  $E_k/E_{k-1} = i_*F_k$  for some coherent sheaves  $F_k$  on  $\mathbb{P}_\Gamma^1$ .  $\square$

**PROPOSITION 5.1.3.** *If  $E \in \mathcal{M}_\Gamma^\tau(V, W)$  is a framed torsion free sheaf then  $N := j^*E$  is a projective  $\mathcal{B}^\tau$ -module with*

$$[N] = [W] + [V \otimes L] - 2[V] \in K(\mathcal{B}^\tau) = K(\Gamma). \tag{5.1.4}$$

*Proof.* If  $E$  is a torsion free sheaf, one can find an embedding  $E \hookrightarrow \mathcal{O}(n)^{\oplus m}$ , for some  $n$  and  $m$ . Applying  $j^*$  we obtain an embedding  $N \hookrightarrow (\mathcal{B}^\tau)^{\oplus m}$ . On the other hand, by [CBH] Theorem 0.4, for generic  $\tau$ , the global homological dimension of the algebra  $\mathcal{B}^\tau$  equals 1. It follows that any submodule of a free  $\mathcal{B}^\tau$ -module is projective. Thus  $N$  is projective.



It remains to compute the class of  $N$  in  $K(\mathcal{B}^\tau)$ . To this end we use the monadic description of torsion free sheaves provided by Corollary 4.2.14 and Lemma 4.2.15. Writing  $E$  as the cohomology of the monad corresponding to a triple  $(B, I, J) \in \mathcal{M}_\Gamma^\tau(V, W)$  and using 5.1.2 (1), (4), we find,

$$\begin{aligned} [N] &= [j^*E] = [j^*((V \otimes L \oplus W) \otimes_{\mathbb{C}\Gamma} \mathcal{O})] - \\ &\quad - [j^*(V \otimes_{\mathbb{C}\Gamma} \mathcal{O}(-1))] - [j^*(V \otimes_{\mathbb{C}\Gamma} \mathcal{O}(1))] = [W] + [V \otimes L] - 2[V]. \quad \square \end{aligned}$$

Note, that if  $R = \mathbb{C}\Gamma^{\oplus k}$  is a multiple of the regular representation of  $\Gamma$  then we have an isomorphism of  $\Gamma$ -modules  $R \otimes L \cong R \oplus R$ , hence,  $[R \otimes L] - 2[R] = 0$ . Therefore, given a one-dimensional  $\Gamma$ -module  $W$  and a  $\Gamma$ -module  $V$  that *does not contain the regular representation as a submodule*, we see from Proposition 5.1.3. that the assignment:  $E \mapsto j^*E$  gives a map

$$\bigsqcup_{k=0}^{\infty} \mathcal{M}_\Gamma^\tau(V \oplus \mathbb{C}\Gamma^{\oplus k}, W) \xrightarrow{j^*} \left\{ \begin{array}{l} \text{projective } \mathcal{B}^\tau\text{-modules } N \text{ such that} \\ [N] = [W] + [V] \cdot ([L] - 2[\text{triv}]) \end{array} \right\}. \quad (5.1.5)$$

(here  $[N] \in K(\mathcal{B}^\tau)$  is treated as a class in  $K(\Gamma)$  via the isomorphism  $K(\mathcal{B}^\tau) \simeq K(\Gamma)$ , as above). We will prove below that (5.1.5) is a bijection. This will imply Theorem 5.0.1.

## 5.2. EXTENDING $\mathcal{B}^\tau$ -MODULES TO SHEAVES ON $\mathbb{P}_\Gamma^2$

In this subsection we show that, given a projective  $\mathcal{B}^\tau$ -module  $N$ , there exists an essentially unique (up to isomorphism) way to extend  $N$  to a framed torsion free sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  such that  $j^*E \cong N$ .

Recall first that the standard grading on the algebra  $A^\tau$  induces a canonical increasing filtration:  $\mathbb{C}\Gamma = \mathcal{B}_0^\tau \subset \mathcal{B}_1^\tau \subset \mathcal{B}_2^\tau \subset \dots$ , on the algebra  $\mathcal{B}^\tau = A^\tau / (z-1)A^\tau$ . Given a  $\mathcal{B}^\tau$ -module  $N$ , we say that an increasing filtration  $\{N_k\}$  on  $N$  is *compatible* with the canonical filtration on  $\mathcal{B}^\tau$  if, for all  $k, l$ , we have  $N_k \cdot \mathcal{B}_l^\tau \subset N_{k+l}$ . The filtration  $\{N_k\}$  is said to be *finitely generated* if  $\dim N_k < \infty$ ,  $\forall i$ , and there exists  $k$  such that  $N_k \cdot \mathcal{B}_l^\tau = N_{k+l}$  for all  $l \geq 0$ . The filtration is called *exhausting* if  $N = \bigcup_k N_k$ . Finally, two filtrations  $\{N_k\}$  and  $\{N'_k\}$  on  $N$  are called *equivalent*, for all  $k \gg 0$ , we have  $N_k = N'_k$ .

**PROPOSITION 5.2.6.** *The set of  $z$ -torsion free coherent sheaves  $E$  on  $\mathbb{P}_\Gamma^2$  such that  $j^*E \cong N$  is in bijection with the set of equivalence classes of finitely generated increasing exhausting filtrations  $\{N_k\}$  on  $N$ , compatible with the canonical filtration of the algebra  $\mathcal{B}^\tau$ .*

*Proof.* (1) If  $M = \bigoplus M_k$  is the graded  $A^\tau$ -module corresponding to a  $z$ -torsion free sheaf  $E$  then, for  $k \gg 0$ , the  $z$ -multiplication map  $M_k \xrightarrow{z} M_{k+1}$  is injective. Hence, the images of  $\{M_k\}_{k \geq 0}$  form an increasing filtration on  $\lim_{\rightarrow} M_k = j^*E$ . This filtration is clearly finitely generated, exhausting and compatible with the canonical filtration of  $\mathcal{B}^\tau$ .

Conversely, assume that  $N$  is a  $\mathcal{B}^\tau$ -module with a finitely generated increasing exhausting filtration  $\{N_k\}$  compatible with the canonical filtration of  $\mathcal{B}^\tau$ . Then the graded vector space  $\bigoplus_k N_k$  admits the structure of a graded  $A^\tau$ -module by means of the standard Rees construction. Specifically, we let  $x, y \in A^\tau$  act on  $N = \bigoplus_k N_k$  via the maps  $x, y: N_k \rightarrow N_{k+1}$  induced by the same named elements of  $\mathcal{B}^\tau$ , and we let  $z$  act as the tautological embedding:  $N_k \hookrightarrow N_{k+1}$ . It is clear that the graded  $A^\tau$ -module thus defined is finitely generated and the corresponding coherent sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  is  $z$ -torsion free. Moreover, it is easy to show that equivalent filtrations give rise to isomorphic coherent sheaves. Finally, it is clear that the construction of this paragraph is inverse to that of the preceding one.  $\square$

**PROPOSITION 5.2.7.** *If  $E$  and  $E'$  are  $z$ -torsion free sheaves on  $\mathbb{P}_\Gamma^2$ , and  $\phi: j^*E \rightarrow j^*E'$  is a morphism of  $\mathcal{B}^\tau$ -modules then there exists  $n \geq 0$  and a morphism  $\tilde{\phi}: E \rightarrow E'(n)$ , such that  $j^*\tilde{\phi} = \phi$ .*

*Proof.* Let  $N = j^*E$ ,  $N' = j^*E'$  and let  $\{N_k\}$ ,  $\{N'_k\}$  be the corresponding finitely generated filtrations on  $N$  and  $N'$ . Since the filtration  $\{N_k\}$  is finitely generated it follows that there exists  $n \geq 0$  such that  $\phi(N_k) \subset N'_{k+n}$ , for all  $k \gg 0$ . Hence  $\phi$  gives a morphism of graded  $A^\tau$ -modules  $\bigoplus_k N_k \rightarrow \bigoplus_k N'_{k+n}$  or, equivalently, a morphism of coherent sheaves  $\tilde{\phi}: E \rightarrow E'(n)$ . It is clear that  $j^*\tilde{\phi} = \phi$ .  $\square$

**LEMMA 5.2.8.** *For any coherent sheaf  $E$  on  $\mathbb{P}_\Gamma^2$  we have  $r(E) = \dim_{\mathbb{C}}[j^*E]$ , where  $\dim_{\mathbb{C}}: K(\Gamma) \rightarrow \mathbb{Z}$  is the linear function given by  $\dim_{\mathbb{C}}([R]) = \dim_{\mathbb{C}} R$ .*

*Proof.* Note that both right-hand side and left-hand side are linear functions on  $K(\text{coh}(\mathbb{P}_\Gamma^2))$ , see 3.1.6 and 5.1.2(2). Thus it suffices to verify the equality only for  $E = R \otimes_{\mathbb{C}\Gamma} \mathcal{O}(i)$ , see 3.1.3. This has been done in 3.1.6 and 5.1.2(4).  $\square$

### 5.3. PROOF OF BIJECTIVITY

From now until the end of this section we fix a class  $R \in K(\Gamma)$  such that  $\dim R = 1$ . By Proposition 1.3.11, in  $K(\Gamma)$  we can write:  $R = [W] + [V] \cdot ([L] - 2[\text{triv}])$ , for certain uniquely determined (isomorphism classes of)  $\Gamma$ -modules  $W$  and  $V$ , such that  $\dim W = 1$  and such that  $V$  does not contain the regular representation as a submodule. With  $W$  and  $V$  as above, we have

**PROPOSITION 5.3.9.** *If  $N$  is a projective finitely generated  $\mathcal{B}^\tau$ -module such that  $[N] = R$ , then there exists a framed locally free sheaf  $\mathcal{E} \in \mathcal{M}_\Gamma^\tau(V \oplus \mathbb{C}\Gamma^{\oplus k}, W)$  such that  $j^*\mathcal{E} \cong N$ .*

*Proof.* Choose an arbitrary finitely generated increasing exhausting filtration on  $N$  compatible with canonical filtration of  $\mathcal{B}^\tau$  and let  $E$  be the corresponding  $z$ -torsion free sheaf on  $\mathbb{P}_\Gamma^2$  such that  $j^*E \cong N$  (see 5.2.6). Now, for the sheaf  $E^{**}$ , by 5.1.2 (3), we have

$$j^*(E^{**}) = \text{Hom}_{\mathcal{B}^\tau}(j^*(E^*), \mathcal{B}^\tau) = \text{Hom}_{\mathcal{B}^\tau}(\text{Hom}_{\mathcal{B}^\tau}(N, \mathcal{B}^\tau), \mathcal{B}^\tau) = N,$$

since  $N$  is projective. On the other hand,  $E^{**}$  is a locally free sheaf by 2.0.4 (5). Hence,  $i^*E^{**}$  is locally free by 3.3.9 (6). Moreover, by 3.3.9 (7) and 5.2.8, we get

$$r(i^*E^{**}) = r(E^{**}) = \dim_{\mathbb{C}}[j^*E^{**}] = \dim_{\mathbb{C}}[N] = \dim_{\mathbb{C}}[R] = 1$$

Hence by 3.2.8(3) we have:  $i^*E^{**} \cong W \otimes_{\mathbb{C}\Gamma} \mathcal{O}(n)$ , for a one-dimensional  $\Gamma$ -module  $W$  and some  $n \in \mathbb{Z}$ .

Let  $\mathcal{E} = E^{**}(-n)$ . Then  $\mathcal{E}$  is a locally free framed sheaf on  $\mathbb{P}_{\Gamma}^2$ , hence  $\mathcal{E} \in \mathcal{M}_{\Gamma}(V', W)^{\tau}$ , for a certain  $\Gamma$ -module  $V'$ . On the other hand, it is clear that

$$j^*\mathcal{E} = j^*(E^{**}(-n)) \cong j^*E^{**} \cong N.$$

Hence by 5.1.3 we have  $[R] = [W] + [V \otimes L] - 2[V]$ ; hence, Lemma 1.3.11 yields:  $V' \cong V \oplus \mathbb{C}\Gamma^{\oplus k}$ , moreover,  $V$  and  $W$  are  $\Gamma$ -modules corresponding to the class  $[R]$  in the sense of Lemma 1.3.11.  $\square$

**PROPOSITION 5.3.10.** *Let  $\mathcal{E} \in \mathcal{M}_{\Gamma}^{\tau}(V \oplus \mathbb{C}\Gamma^{\oplus k}, W)$  and  $\mathcal{E}' \in \mathcal{M}_{\Gamma}^{\tau}(V \oplus \mathbb{C}\Gamma^{\oplus k'}, W)$  be locally free sheaves such that  $j^*\mathcal{E} \cong j^*\mathcal{E}'$ . Then,  $k = k'$ , and  $\mathcal{E} \cong \mathcal{E}'$ .*

*Proof.* An isomorphism  $j^*\mathcal{E} \cong j^*\mathcal{E}'$  gives by 5.2.7 a morphism  $\phi: \mathcal{E} \rightarrow \mathcal{E}'(n)$  such that  $j^*\phi$  is an isomorphism. Let  $K = \mathbf{Ker} \phi$ ,  $C = \mathbf{Coker} \phi$ . Then by 5.1.2(2) we have  $j^*K = j^*C = 0$ , hence by 5.1.2(5) both  $K$  and  $C$  are supported on  $\mathbb{P}_{\Gamma}^1$ . On the other hand, a locally free sheaf contains no sheaves supported on  $\mathbb{P}_{\Gamma}^1$ , by 3.3.13(2) and (1). It follows that  $K = 0$ . Thus we have a short exact sequence:

$$0 \longrightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{E}'(n) \longrightarrow C \longrightarrow 0. \quad (5.3.11)$$

Applying the functor  $i^*$  we get a short exact sequence:

$$0 \longrightarrow L^1 i^*C \longrightarrow W \otimes_{\mathbb{C}\Gamma} \mathcal{O} \xrightarrow{i^*\phi} W \otimes_{\mathbb{C}\Gamma} \mathcal{O}(n) \longrightarrow i^*C \longrightarrow 0.$$

Since  $\dim_{\mathbb{C}} W = 1$  it follows that either  $i^*\phi = 0$  or  $i^*C$  is an Artin sheaf.

If  $i^*C$  is Artin then by 3.3.11(3) the sheaf  $C$  is Artin as well. On the other hand, applying  $\underline{\mathcal{H}om}(-, \mathcal{O})$  to (5.3.11) we get an exact sequence

$$\underline{\mathcal{E}xt}^1(\mathcal{E}, \mathcal{O}) \longrightarrow \underline{\mathcal{E}xt}^2(C, \mathcal{O}) \longrightarrow \underline{\mathcal{E}xt}^2(\mathcal{E}'(n), \mathcal{O}),$$

but  $\mathcal{E}$  and  $\mathcal{E}'$  are locally free, hence  $\underline{\mathcal{E}xt}^2(C, \mathcal{O}) = 0$ , hence  $C = 0$  by 2.0.9 (4). Thus  $\phi$  is an isomorphism. It follows that  $i^*\phi$  is an isomorphism, hence  $W \otimes_{\mathbb{C}\Gamma} \mathcal{O} \cong W \otimes_{\mathbb{C}\Gamma} \mathcal{O}(n)$ , hence  $n = 0$  and  $\mathcal{E} \cong \mathcal{E}'$ .

If  $i^*\phi = 0$  it follows that  $\phi$  factors through the embedding  $\mathcal{E}'(n-1) \xrightarrow{\cong} \mathcal{E}'(n)$ . Repeating this argument for  $n$  being replaced by  $n-1, n-2, \dots$ , we obtain that either  $\mathcal{E} \cong \mathcal{E}'$  or there exists an embedding:  $\mathcal{E} \hookrightarrow \mathcal{E}'(n)$ , for arbitrarily small  $n \in \mathbb{Z}$ . The latter is impossible by 2.0.2 and 2.0.4 (4).  $\square$

Now, Proposition 5.3.9 gives surjectivity of the map (5.1.5), and Proposition 5.3.10 gives injectivity of (5.1.5). Hence, this map is bijective, and Theorem 5.0.1 follows.

## 6. Appendix A: Graded Preprojective Algebra

In this section we define a graded version  $\mathbf{P}$  of the deformed preprojective algebra introduced in [CBH]. Let  $Q$  be a quiver, i.e. an oriented graph with vertex set  $\mathcal{V}$ . For any (oriented) edge  $a \in Q$ , we write  $\text{in}(a) = j$ ,  $\text{out}(a) = i$ , if  $a: i \rightarrow j$ . Let  $\bar{Q}$  be the double of  $Q$ , obtained by adding a reverse edge  $a^*: j \rightarrow i$  for every edge  $a: i \rightarrow j$  in  $Q$ .

Let  $\mathbf{P}_0 = \bigoplus_{v \in \mathcal{V}} \mathbb{C}$  be the direct sum of  $|\mathcal{V}|$  copies of the field  $\mathbb{C}$ , a commutative semisimple  $\mathbb{C}$ -algebra. For  $v \in \mathcal{V}$ , we write  $e_v \in \mathbf{P}_0$  for the projector on the  $v$ -th copy (an idempotent). We define a  $\mathbf{P}_0$ -bimodule  $\mathbf{P}_1$  by the formula

$$\mathbf{P}_1 = \left( \bigoplus_{a \in \bar{Q}} \mathbb{C} \cdot a \right) \oplus \mathbf{P}_0.$$

Here, in the first summand, for  $a \in \bar{Q}$ , with  $\text{in}(a) = j$ ,  $\text{out}(a) = i$ , we put  $e_i a = a e_j = a$ , and all other products:  $e_k a$ ,  $a e_s$  are set equal to zero. Let  $f$  denote the canonical generator of the second summand in  $\mathbf{P}_1$  corresponding to the element  $1 \in \mathbf{P}_0$ . We put  $f_i = e_i \cdot f = f \cdot e_i$ , so that  $f = \sum f_i$ .

The  $\mathbf{P}_0$ -bimodule  $\mathbf{P}_1$  gives rise to the tensor algebra  $T_{\mathbf{P}_0}^{\bullet}(\mathbf{P}_1) = \bigoplus_{n \geq 0} T_{\mathbf{P}_0}^n \mathbf{P}_1$ , where  $T_{\mathbf{P}_0}^n \mathbf{P}_1 = \mathbf{P}_1 \otimes_{\mathbf{P}_0} \cdots \otimes_{\mathbf{P}_0} \mathbf{P}_1$  is the  $n$ -fold tensor product. Note that, since the product is taken over  $\mathbf{P}_0$ , for any two arrows  $a, a' \in \bar{Q}$ , in  $T_{\mathbf{P}_0}^{\bullet}(\mathbf{P}_1)$  we have:  $a \cdot a' = 0$  unless  $\text{in}(a') = \text{out}(a)$ .

**DEFINITION. 1** Choose an element  $\tau \in \mathbf{P}_0$ ,  $\tau = \sum_{i=0}^n \tau_i e_i$ . The graded deformed preprojective algebra,  $\mathbf{P}^{\tau} = \mathbf{P}^{\tau}(Q)$ , is defined as  $\mathbf{P}^{\tau} = T_{\mathbf{P}_0}^{\bullet}(\mathbf{P}_1) / \langle\langle R \rangle\rangle$ , a quotient of the tensor algebra  $T_{\mathbf{P}_0}^{\bullet}(\mathbf{P}_1)$  by the two-sided ideal generated by the  $\mathbf{P}_0$ -bimodule  $R \subset \mathbf{P}_1 \otimes_{\mathbf{P}_0} \mathbf{P}_1$  formed by the following quadratic relations:

- (a)  $f_i \cdot a = a \cdot f_j$ , if  $a: i \rightarrow j$  is an arrow in  $\bar{Q}$
- (b)  $\sum_{\{a \in Q | \text{out}(a) = i\}} a \cdot a^* - \sum_{\{a \in Q | \text{in}(a) = i\}} a^* \cdot a = \tau_i \cdot f_i^2$ ,  $\forall i \in \mathcal{V}$ .

*Koszul complex.* By the definition the algebra  $\mathbf{P}^{\tau}$  is quadratic (see Appendix B). Therefore one can write its right Koszul complex  $\mathcal{K}^{\bullet} \mathbf{P}^{\tau}$  (see Appendix B for the definition, and also [BGS]). In our particular case it boils down to

$$0 \rightarrow \mathcal{K}^3 \otimes_{\mathbf{P}_0} \mathbf{P}^{\tau}(-3) \rightarrow \mathcal{K}^2 \otimes_{\mathbf{P}_0} \mathbf{P}^{\tau}(-2) \rightarrow \mathcal{K}^1 \otimes_{\mathbf{P}_0} \mathbf{P}^{\tau}(-1) \rightarrow \mathbf{P}^{\tau} \rightarrow \mathbf{P}_0 \rightarrow 0$$

where  $\mathcal{K}^i = \mathcal{K}^i \mathbf{P}^{\tau}$  are  $\mathbf{P}_0$ -bimodules given by

- (a)  $\mathcal{K}^1 \mathbf{P}^{\tau} = \mathbf{P}_1$ ;
- (b)  $\mathcal{K}^2 \mathbf{P}^{\tau} = R \subset \mathbf{P}_1 \otimes_{\mathbf{P}_0} \mathbf{P}_1$ , is the submodule of generating relations;
- (c)  $\mathcal{K}^3 \mathbf{P}^{\tau} = (\mathcal{K}^2 \otimes_{\mathbf{P}_0} \mathbf{P}_1) \cap (\mathbf{P}_1 \otimes_{\mathbf{P}_0} \mathcal{K}^2) \subset T_{\mathbf{P}_0}^3 \mathbf{P}_1$  is a  $\mathbf{P}_0$ -bimodule that can be shown to have a single generator:

$$\begin{aligned} & \tau \cdot f \otimes f \otimes f + \sum_{a \in \bar{Q}} a \otimes f \otimes a^* - f \otimes a \otimes a^* - a^* \otimes f \otimes a + \\ & + f \otimes a^* \otimes a - a \otimes a^* \otimes f + a^* \otimes a \otimes f \end{aligned}$$

The differentials of the Koszul complex are given by restricting the map:

$$T_{\mathbf{P}_0}^n \mathbf{P}_1 \otimes_{\mathbf{P}_0} \mathbf{P}^{\tau}(-1) \rightarrow T_{\mathbf{P}_0}^{n-1} \mathbf{P}_1 \otimes_{\mathbf{P}_0} \mathbf{P}^{\tau}; \quad (v_1 \otimes \cdots \otimes v_n) \otimes x \mapsto (v_1 \otimes \cdots \otimes v_{n-1}) \otimes v_n x.$$

The algebra  ${}^1\mathbf{P}^\tau$ , see Appendix B for the general definition of the dual quadratic algebra, is generated by the  $\mathbf{P}_0$ -bimodule  ${}^1\mathbf{P}_1$  which is spanned over  $\mathbb{C}$  by two collections of elements:  $\{b\}_{b \in \bar{Q}}$ , and  $\{r_i\}_{i \in \mathcal{V}}$ , subject to the following relations

- (a)  $b \cdot r_i + r_j \cdot b = 0$ , if  $b : i \rightarrow j$  is an arrow in  $\bar{Q}$ ;
- (b)  $b_1 \cdot b_2 = 0$ , unless  $b_1 \in Q$  &  $b_2 = b_1^*$ , or  $b_2 \in Q$  &  $b_1 = b_2^*$ ;
- (c)  $\tau_i \cdot b \cdot b^* = r_i^2$ , if  $b \in Q$  and  $\text{in}(b) = i$
- (d)  $\tau_i \cdot b^* \cdot b = r_i^2$ , if  $b \in Q$  and  $\text{out}(b) = i$
- (e)  $b_1^* \cdot b_1 = b_2 \cdot b_2^*$ , if  $b_1, b_2 \in Q$  and  $\text{in}(b_2) = i = \text{out}(b_1)$ .

(relation (e) does not follow from (c) and (d) if and only if  $\tau_i = 0$ ). One can check that the relations above imply  ${}^1\mathbf{P}_i^\tau = 0$ , for all  $i \geq 4$ .

It is known, see [GMT], that, for any quiver  $Q$ , the corresponding algebra  $\mathbf{P}^\tau = \mathbf{P}^\tau(Q)$  is *Koszul*. However,  $\mathbf{P}^\tau$  is Noetherian and has polynomial growth if and only if the underlying graph  $Q$  is either of affine or of finite Dynkin ADE-type.

We assume that  $Q$  is an affine Dynkin graph, hence it is associated, by means of McKay correspondence, to a finite subgroup  $\Gamma \subset \text{SL}_2(\mathbb{C})$ . We write  $Q = Q(\Gamma)$ , and let  $R_i$  be the simple  $\Gamma$ -module corresponding to a vertex  $i \in \mathcal{V}$ . Given  $\tau \in \mathbb{Z}(\text{C}\Gamma)$ , let  $\tau_i \in \mathbb{C}$  be the complex number such that  $\tau$  acts in  $R_i$  as the scalar operator:  $\tau_i \cdot \text{Id}_{R_i}$ . This way we identify  $\tau$  with the element  $\sum_i \tau_i \cdot e_i \in \mathbf{P}_0$ , still to be denoted by  $\tau$ . With this understood, one proves as in [CBH]:

**PROPOSITION 6.0.1.** *Let  $Q(\Gamma)$  be the graph of affine ADE-type arising from a finite group  $\Gamma \subset \text{SL}_2(\mathbb{C})$  via the MacKay correspondence. Then*

- (i) *The algebras  $A^\tau$  and  $\mathbf{P}^\tau = \mathbf{P}^\tau(Q)$  are Morita equivalent. In particular,*
- (ii) *The category  $\text{gr}(A^\tau)$  is equivalent to the category  $\text{gr}(\mathbf{P}^\tau)$ , resp. category  $\text{qgr}(A^\tau)$ , is equivalent to  $\text{qgr}(\mathbf{P}^\tau)$ .*

**DEFINITION 6.0.2.** The deformed preprojective algebra of the quiver  $Q$  is defined as the quotient algebra

$$\Pi^\tau := \mathbf{P}^\tau(Q) / \langle\langle f - \tau \rangle\rangle = \mathbf{P}^\tau(Q) / \langle\langle f_i - \tau_i \rangle\rangle_{i \in \mathcal{V}}.$$

A vertex  $v \in \mathcal{V}$  of the graph  $Q = Q(\Gamma)$  is said to be an extended vertex if the  $\Gamma$ -module corresponding to ‘ $v$ ’ is one-dimensional. In such a case, removing ‘ $v$ ’ from the graph  $Q$  one obtains a Dynkin graph  $Q^{\text{fin}}$  of finite type, moreover, the graph  $Q$  is the extended affine graph for  $Q^{\text{fin}}$ . From Theorem 1.3.12 we deduce the following generalization of the *Crawley-Boevey and Holland conjecture*, (cf. [BLB, Example 5.7]).

**COROLLARY 6.0.3.** *Let  $Q(\Gamma)$  be the McKay graph of  $\Gamma$ , and  $v \in \mathcal{V}$  an extended vertex of  $Q(\Gamma)$ . Let  $\tau$  be generic, and  $R$  be the one-dimensional (simple) representation of  $\Gamma$  corresponding to the vertex  $v$ . Then, there exists a natural bijection*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of finitely generated projective} \\ \Pi^\tau\text{-modules } N \text{ such that } [N] = [R] \text{ in } K(\Pi^\tau) = K(\Gamma) \end{array} \right\} \\ \simeq \bigsqcup_{k=0}^{\infty} \mathfrak{M}_\tau^{\text{pr}}(V \oplus \mathbb{C}\Gamma^{\oplus k}, W). \quad \square$$

Here we have used the natural isomorphism  $K(\Pi^\tau) \simeq K(\Gamma)$ , see [Q].

### 7. Appendix B: Algebraic Generalities

#### 7.1. LINEAR ALGEBRA OVER A SEMISIMPLE ALGEBRA [BGS, §2.7]

Let  $A_0$  be a finite-dimensional semisimple  $\mathbb{C}$ -algebra.

Recall that for any left  $A_0$ -module  $V$  the space  $V^* = \text{Hom}_{A_0\text{-mod}}(V, A_0)$  can be given the structure of a right  $A_0$ -module via the assignment  $(fa)(v) = (f(v))a$ . Similarly, for any right  $A_0$ -module  $W$  the space  ${}^*W = \text{Hom}_{\text{-mod } A_0}(W, A_0)$  can be given the structure of a left  $A_0$ -module via the assignment  $(ag)(w) = a(g(w))$ . For finitely generated left  $A_0$ -modules  $V, W$ , the canonical evaluation maps:  $V \rightarrow {}^*(V^*)$  and  $W \rightarrow ({}^*W)^*$  are isomorphisms.

For an  $A_0$ -bimodule  $V$ , both  $V^*$  and  ${}^*V$  are bimodules defined as follows:

$$(af)(v) = f(va) \quad \text{and} \quad (ga)(v) = g(av), \quad \forall g \in {}^*V, f \in V^*, v \in V, a \in A_0.$$

All standard results of linear algebra over a field can be generalized in an appropriate way to  $A_0$ -modules (e.g.  $W^* \otimes V^* \simeq (V \otimes W)^*$ ) if we take all tensor products over  $A_0$ . We will use these generalizations freely, referring the interested reader to [BGS, §2.7].

Fix an algebra  $A = \bigoplus_{n \geq 0} A_n$ , and put  $X = \text{Proj } A$ .

**PROPOSITION 7.1.1.** *If  $A$  is strongly regular of dimension  $d$ , then one has:*

$$H^p(X, \mathcal{O}(i)) = \begin{cases} A_i, & \text{if } p = 0 \text{ and } i \geq 0, \\ {}^*A_{-i-d-1}, & \text{if } p = d \text{ and } i \leq -d - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  ${}^*A_{-i-d-1}$  is the dual of the  $A_0$ -bimodule  $A_{-i-d-1}$  in the sense explained above.

This result has been proved in [AZ, Theorem 8.1(3)] in the special case  $A_0 = \mathbb{C}$ . The proof sketched below shows that the Proposition remains valid in the general case of an arbitrary semisimple finite-dimensional algebra  $A_0$ .

*Sketch of Proof.* Let  $M$  be a graded module and  $F$  the corresponding sheaf. For any fixed  $i \geq 0$ , consider the right  $A$ -module  $\bigoplus_{n=-\infty}^{\infty} H^i(F(n))$ . It was shown in [AZ, Proposition 7.2(2)] that, for  $i \geq 1$ , one has

$$\bigoplus_{n=-\infty}^{\infty} H^i(F(n)) \simeq \lim_{m \rightarrow \infty} \left( \bigoplus_{n=-\infty}^{\infty} \text{Ext}^i(A/A_{\geq m}, M(n)) \right), \quad A_{\geq m} = \bigoplus_{k \geq m} A_k.$$

Now take  $M = A$  and apply the above formula. Since  $A/A_{\geq m}$  is finite dimensional, the Gorenstein condition implies that the LHS above vanishes for  $i = 1, \dots, d - 1$ .

This yields the cohomology vanishing part of the formula of the Proposition. Other claims follow easily using Serre duality.  $\square$

### 7.2. KOSZUL AND CO-KOSZUL ALGEBRAS

From now on we assume that  $A = \bigoplus_{n \geq 0} A_n$  is a positively graded  $\mathbb{C}$ -algebra such that all graded components  $A_n$  are finite-dimensional over  $\mathbb{C}$ .

**DEFINITION.** An algebra  $A = \bigoplus_{n \geq 0} A_n$  is called *quadratic* if

- $A_0$  is a finite-dimensional semisimple  $\mathbb{C}$ -algebra;
- the  $A_0$ -bimodule  $A_1$  generates  $A$  over  $A_0$ ;
- the relations ideal is generated by the subspace of quadratic relations  $R \subset A_1 \otimes_{A_0} A_1$ .

Given a quadratic algebra  $A = \bigoplus_{n \geq 0} A_n$ , we can represent  $A$  as  $T_{A_0} A_1 / \langle\langle R \rangle\rangle$ , the quotient of the tensor algebra by the ideal  $\langle\langle R \rangle\rangle$  generated by the space of quadratic relations  $R \subset A_1 \otimes_{A_0} A_1$ .

Define  $A^!$ , the *left dual* of  $A$ , to be the quadratic algebra:  $T_{A_0}(A_1^*) / \langle\langle R^\perp \rangle\rangle$ , with  $R^\perp \subset A_1^* \otimes_{A_0} A_1^* = (A_1 \otimes_{A_0} A_1)^*$  being the annihilator of  $R$ . Analogously, the *right dual*  ${}^!A$  is defined as the algebra:  $T_{A_0}({}^*A_1) / \langle\langle {}^\perp R \rangle\rangle$ , with  ${}^\perp R \subset {}^*A_1 \otimes_{A_0} {}^*A_1 = {}^*(A_1 \otimes_{A_0} A_1)$ .

The *right Koszul complex*  $\mathcal{K}^\bullet A$  is a complex of the form, see, e.g., [BGS], [Ma]:

$$\dots \xrightarrow{d} ({}^!A_3)^* \otimes_{A_0} A(-3) \xrightarrow{d} ({}^!A_2)^* \otimes_{A_0} A(-2) \xrightarrow{d} ({}^!A_1)^* \otimes_{A_0} A(-1) \rightarrow A \rightarrow A_0 \rightarrow 0$$

where the differential ‘d’ is defined as follows. Observe that:  $({}^!A_i)^* \otimes_{A_0} A = \text{Hom}_{A_0\text{-mod}}({}^!A_i, A)$ . Under the canonical isomorphism  $\text{Hom}_{\text{mod-}A_0}(A_1, A_1) = A_1 \otimes_{A_0} {}^*A_1$ , let  $\text{Id}_{A_1} = \sum v_x \otimes \check{v}_x$ . Then for  $f \in \text{Hom}_{A_0\text{-mod}}({}^!A_{i+1}, A)$  and  $a \in {}^!A_i$  we set  $df(a) = \sum v_x \cdot f(\check{v}_x \cdot a)$ . One can check that this formula indeed defines a complex. One can also define a left Koszul complex. It is known, cf. for example [BGS], that the exactness of the right Koszul complex is equivalent to the exactness of the left Koszul complex.

Similarly, there is a natural differential on the space  $\mathcal{K}_\bullet A = {}^!A \otimes_{A_0} A$ , see [Ma], making it into a complex, called the (right) *co-Koszul complex* of  $A$ .

**DEFINITION 7.2.2.** (i) A quadratic ring  $A$  is called Koszul if its (right) Koszul complex,  $\mathcal{K}^\bullet A$  has the only nontrivial cohomology in degree zero.

(ii) A quadratic ring  $A$  is called co-Koszul of degree  $d$ , if its (right) co-Koszul complex,  $\mathcal{K}_\bullet A$  has the only nontrivial cohomology in degree  $d$  and, moreover,  $H^d(\mathcal{K}_\bullet A) \simeq A_0(d)$ .

The conditions in the Proposition below are the basic conditions that allow us to start out with the noncommutative geometry as discussed in Section 2.

**PROPOSITION 7.2.3.** *Let  $A = \bigoplus_{n \geq 0} A_n$  be a quadratic algebra with  ${}^1A = \bigoplus_k {}^1A_k$ . Assume that:*

- *${}^1A$  has no nonzero graded components in degrees  $k > d$ , and  ${}^1A_d \simeq A_0$ ;*
- *The algebra  $A$  is Noetherian, and has polynomial growth;*
- *The algebra  $A$  is both Koszul and co-Koszul (of degree  $d$ ).*

*Then  $A$  is strongly regular of dimension  $d$  in the sense of Definition 1.1.1.*

*Proof.* To prove that the global dimension,  $\text{gl.dim}(A)$ , is finite we apply [Hu] and conclude that  $\text{gl.dim}(A)$  equals the minimal length of projective resolution for  $A_0$ . The Koszul complex, if exact, provides such a minimal resolution. Thus, for a Koszul algebra  $A$ , the global dimension equals the number of nonzero graded components of the algebra  ${}^1A$ . Since  $\dim_c({}^1A)$  is finite, we conclude that  $\text{gl.dim}(A) < \infty$ .

Notice next that an obvious canonical isomorphism  ${}^1A \otimes_{A_0} A \simeq \text{Hom}_A(({}^1A)^* \otimes_{A_0} A, A)$ , gives an isomorphism of complexes:  $\mathcal{K}_\bullet A \simeq \text{Hom}_A(\mathcal{K}^\bullet A, A)$ . It follows that, for a Koszul algebra  $A$ , the complex  $\mathcal{K}_\bullet A$  computes the Ext-groups:  $\text{Ext}_A^\bullet(A_0, A)$ . Thus,  $A$  is co-Koszul of degree  $d$  if and only if it is Gorenstein with parameters  $(d, d)$ . □

*Remark.* One shows similarly that if  $A$  is Koszul of global dimension  $d$  and Gorenstein, then it is Gorenstein with parameters  $(d, d)$ , co-Koszul of degree  $d$  and, moreover, the dual algebra  ${}^1A$  is Frobenius of index  $d$ .

Fix  $A = \bigoplus_{n \geq 0} A_n$ , an algebra satisfying the conditions of Proposition 7.2.3, and put  $X = \text{Proj } A$ . A key role in our study of sheaves on  $X$  is played by

*Beilinson spectral sequence ([KKO]):* For any sheaf  $E$  on  $X$  there is a spectral sequence with the first term

$$E_1^{p,q} = \text{Ext}^q(Q_{-p}(p), E) \otimes_{A_0} \mathcal{O}(-p) \implies E_\infty^i = \begin{cases} E, & \text{for } i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $p = -d, \dots, 0$ , and  $Q_{-p}$  is the sheaf on  $X$  corresponding to the cohomology  $\tilde{Q}_{-p}$  of the truncated Koszul complex:

$$0 \rightarrow A \rightarrow A_1^! \otimes_{A_0} A(1) \rightarrow \dots \rightarrow A_{1-p}^! \otimes_{A_0} A(-p) \rightarrow \tilde{Q}_{-p} \rightarrow 0$$

(here  $A(n)$  stands for the algebra  $A$  with the grading being shifted by  $n$ ). Equivalently,  $\tilde{Q}_{-p}$  can be described as follows

$$0 \rightarrow \tilde{Q}_{-p} \rightarrow A_{1-p}^! \otimes_{A_0} A(1-p) \rightarrow \dots \rightarrow A_{d+1}^! \otimes_{A_0} A(d+1) \rightarrow A_0(d+1) \rightarrow 0.$$

Note that all the sheaves  $Q_p$  are naturally  $A_0$ -bimodules, hence the tensor product in the  $E_1$ -term makes sense.



7.3. SPECIAL CASE  $\mathbf{A} = \mathbf{A}^\tau$

Recall that:  $A^\tau = ((TL[z])\#\Gamma)/\langle\langle u \cdot v - v \cdot u - \omega(u, v) \cdot \tau z^2 \rangle\rangle_{u, v \in L}$ , see Definition 1.2.5. Write  $\Lambda L^*$  for the exterior algebra of the two-dimensional vector space  $L^*$ , and let  $\Lambda L^* \dot{\otimes} \mathbb{C}[\zeta]$  be the *super*-tensor product of  $\Lambda L^*$  with the polynomial algebra in an *odd* variable  $\zeta$  of degree 1. Thus, by definition, for any  $v \in L^* \subset \Lambda L^*$ , in  $\Lambda L^* \dot{\otimes} \mathbb{C}[\zeta]$  we have:  $v \cdot \zeta = -\zeta \cdot v$ . We will view  $\Lambda L^*$  as a subalgebra in  $\Lambda L^* \dot{\otimes} \mathbb{C}[\zeta]$ . Further, let  $\omega \in \Lambda^2 L^*$  be the element corresponding to the symplectic form on  $L$ .

**PROPOSITION 7.3.4.** *The algebra  $A^\tau$  is a Noetherian algebra of polynomial growth. Moreover,  $A^\tau$  is both Koszul and co-Koszul (of degree 3), and we have*

$${}^1A^\tau = ((\Lambda L^* \dot{\otimes} \mathbb{C}[\zeta])\#\Gamma)/\langle\langle \omega - \zeta^2 \cdot \tau \rangle\rangle.$$

*Proof.* For  $\tau = 0$  we have:  $A^\tau = \mathbb{C}[x, y, z]\#\Gamma$ . In this case all the claims are easy and follow e.g., from [GMT]. Next, one checks that the relations defining  $A^\tau$  are 3-self-concordant in the sense of [Dr]. Hence the graded components of  $A^\tau$  are isomorphic to those of  $A^0$  as vector spaces (cf. [Dr]). This implies that  $A^\tau$  has polynomial growth, for any  $\tau$ . Furthermore, it follows from the Drinfeld’s result that we may view the family of Koszul complexes  $\mathcal{K}^\bullet A^\tau$ , resp.  $\mathcal{K}_\bullet A^\tau$ , as a family of varying (with  $\tau$ ) differentials on the Koszul complex for  $A^0$ . Since the differential for  $\tau = 0$  has a single non-trivial cohomology, the same is true for all values of  $\tau$  close enough to zero. However, the algebras  $A^\tau$  and  $A^{\alpha \cdot \tau}$  are isomorphic for any  $\alpha \in \mathbb{C}^*$ . Thus,  $A^\tau$  is both Koszul and co-Koszul (of degree 3), for any  $\tau$ .

The expression for  ${}^1A^\tau$  is obtained by a direct calculation. It shows, in particular, that the algebra  ${}^1A^\tau$  has nonvanishing graded components in degrees  $i = 0, 1, 2, 3$  only. Thus,  $d = \text{gl.dim}(A^\tau) = 3$ . In particular, formula (3.1.1) follows from Proposition 7.1.1.  $\square$

Let  $R_i, i \in \mathcal{V}$ , be a complete collection of the isomorphism classes of simple modules over  $A_0 = \mathbb{C}\Gamma$ , and  $\mathcal{R}_i = \pi(R_i \otimes_{\mathbb{C}\Gamma} A^\tau)$  the coherent sheaf corresponding to the graded right  $A^\tau$ -module  $R_i \otimes_{\mathbb{C}\Gamma} A^\tau$ . Since  $R_i$  is a direct summand in  $A_0$ , the sheaf  $\mathcal{R}_i$  is a direct summand of  $\mathcal{O}$ . In particular, for any  $i \in \mathcal{V}$ , the sheaf  $\mathcal{R}_i$  is locally free in the sense of Definition 1.1.4.

*Remark.* Consider collection  $\{\mathcal{R}_0, \dots, \mathcal{R}_n, \dots, \mathcal{R}_0(d-1), \dots, \mathcal{R}_n(d-1)\}$  of sheaves on  $\mathbb{P}^2_\Gamma = \mathbf{Proj} A^\tau$ . It follows from the explicit form of the cohomology of the sheaves  $\mathcal{O}(n)$  that the above collection is a *strong exceptional collection*, see [Ru]. The Beilinson spectral sequence of a sheaf  $E$  can be considered as a decomposition of  $E$  with respect to the above exceptional collection.

**PROPOSITION 7.3.5.** *Any coherent sheaf  $E$  on  $\mathbb{P}^2_\Gamma$  admits a resolution of the form*

$$0 \longrightarrow \bigoplus_{i \in \mathcal{V}} V_i^d \otimes_{\mathbb{C}} \mathcal{R}_i(k-d) \longrightarrow \dots \longrightarrow \bigoplus_{i \in \mathcal{V}} V_i^0 \otimes_{\mathbb{C}} \mathcal{R}_i(k) \longrightarrow E \longrightarrow 0$$

where  $V_i^d, \dots, V_i^0$  are certain complex vector spaces.

*Proof.* Consider the Beilinson spectral sequence of the sheaf  $E(n)$ . By ampleness, for  $n \gg 0$  all higher Ext-groups in the spectral sequence vanish and only the  $q = 0$  row of it will be non-trivial. This gives a resolution of type (2.0.3) for the sheaf  $E(n)$ . Now tensor it with  $\mathcal{O}(-n)$ .  $\square$

### 8. Appendix C: Minuscule Classes

The goal of this section is to prove Proposition 1.3.11. Thus, we fix  $\Gamma$ , a finite subgroup in  $SL_2$  and let  $Q(\Gamma)$  denote the corresponding affine Dynkin graph. We identify the set  $I$  of vertices of  $Q(\Gamma)$  with simple roots of the affine root system associated to  $Q(\Gamma)$ , and write  $\omega_i$ , resp.  $R_i$ , for the fundamental weight, resp. irreducible  $\Gamma$ -module, corresponding to the vertex  $i \in I$ . Thus, the  $\omega_i$ 's form a basis of the weight lattice  $\hat{P}$  (of the affine root system), and the  $R_i$ 's form a basis of  $K(\Gamma)$ . The correspondence  $\omega_i \leftrightarrow R_i$  yields an isomorphism of lattices  $\hat{P} \cong K(\Gamma)$ .

Let  $L$  denote the tautological two dimensional representation of  $\Gamma$  and let  $\text{triv}$  be the trivial (one-dimensional) representation of  $\Gamma$ . The map  $K(\Gamma) \rightarrow K(\Gamma), [V] \mapsto [V] \otimes ([L] - 2[\text{triv}])$  gets identified under the isomorphism  $\hat{P} \cong K(\Gamma)$  with the Cartan operator  $\hat{C}: \hat{P} \rightarrow \hat{P}$ , i.e., the linear map given (in the basis  $\{\omega_i\}$ ) by the Cartan matrix.

Let  $\hat{P}^* = \text{Hom}(\hat{P}, \mathbb{Z})$  be the coroot lattice, and  $\hat{C}^*: \hat{P}^* \rightarrow \hat{P}^*$  the dual Cartan operator. Note that  $\dim \mathbf{Ker} \hat{C} = 1$  and  $\dim \mathbf{Ker} \hat{C}^* = 1$ , since our root system is of affine type. Let  $\hat{\theta} \in \hat{P}$  and  $\hat{\theta}^\vee \in \hat{P}^*$  denote the minimal positive elements in  $\mathbf{Ker} \hat{C}$  and in  $\mathbf{Ker} \hat{C}^*$ , respectively. In the other words,  $\hat{\theta}$  and  $\hat{\theta}^\vee$  are the minimal positive imaginary root and coroot, respectively. The class in  $K(\Gamma)$  of the regular representation of  $\Gamma$  gets identified with  $\hat{\theta} \in \hat{P}$ , while the dimension function  $\dim: K(\Gamma) \rightarrow \mathbb{Z}$  gets identified with the element  $\hat{\theta}^\vee \in \hat{P}^*$  considered as a function  $\hat{P} \rightarrow \mathbb{Z}$ .

We see that the isomorphism classes of one-dimensional  $\Gamma$ -modules are in bijection with *extended* vertices of  $Q(\Gamma)$ , i.e., the vertices  $i \in I$  such that  $\dim[R_i] = 1$ , or equivalently  $\hat{\theta}^\vee(\omega_i) = 1$ . A vertex  $i \in I$  is known to be extended if and only if the graph  $Q(\Gamma)$  is obtained from a finite Dynkin graph  $Q^{\text{fin}}$  with vertex set  $I \setminus \{i\}$  by adding the vertex  $i$ .

Now Proposition 1.3.11 can be reformulated as follows.

**LEMMA 8.0.1.** *For any  $\omega \in \hat{P}$  such that  $\hat{\theta}^\vee(\omega) = 1$  there exists a uniquely determined pair  $(i, \omega_0)$ , where  $i \in I$  is an extended vertex and  $\omega_0 \in \hat{P}$  is a dominant weight, such that the weight  $\omega_0 - \theta$  is not dominant and  $\omega = \omega_i + \hat{C}(\omega_0)$ .*

Since  $\mathbf{Ker} \hat{C} = \mathbb{Z}\hat{\theta}$  it follows that for any  $\omega \in \mathbf{Im} \hat{C}$  there exists a unique  $\omega_0 \in \hat{P}$  such that  $\omega_0$  is dominant,  $\omega_0 - \theta$  is not dominant and  $\omega = \hat{C}(\omega_0)$ . Hence, Lemma 8.0.1 is equivalent to

**LEMMA 8.0.2.** *For any  $\omega \in \hat{P}$  such that  $\hat{\theta}^\vee(\omega) = 1$  there exists a uniquely determined extended vertex  $i$  such that  $\omega - \omega_i \in \mathbf{Im} \hat{C}$ .*

From now on we fix some extended vertex  $v \in I$  and let  $I^{\text{fin}} = I \setminus \{v\}$  be the vertex set of the corresponding Dynkin graph  $Q^{\text{fin}}$  of finite type. Let  $\mathbf{P}$  be the weight lattice of  $Q^{\text{fin}}$ , let  $C: \mathbf{P} \rightarrow \mathbf{P}$  be its Cartan operator, and let  $\theta \in \mathbf{P}$  and  $\theta^\vee \in \mathbf{P}^*$  denote the maximal root and coroot in the root and coroot system of  $Q^{\text{fin}}$  respectively.

The decomposition  $I = I^{\text{fin}} \sqcup \{v\}$  gives rise to the direct sum decompositions  $\hat{\mathbf{P}} = \mathbf{P} \oplus \mathbb{Z} \cdot \omega_v$  and  $\hat{\mathbf{P}}^* = \mathbf{P}^* \oplus \mathbb{Z} \cdot \alpha_v^\vee$ , where  $\alpha_v^\vee$  is the simple coroot of  $Q(\Gamma)$  corresponding to the vertex  $v$ . It is well known that we have

$$\hat{\theta}^\vee = (\theta^\vee, \alpha_v^\vee), \quad \hat{\theta} = (\theta, \omega_v), \quad \text{and} \quad \hat{C} = \begin{pmatrix} C & & \\ & \cdot & \\ & & \cdot \end{pmatrix}.$$

It follows that the projection  $\text{pr}: \hat{\mathbf{P}} = \mathbf{P} \oplus \mathbb{Z} \cdot \omega_v \rightarrow \mathbf{P}$  gives rise to the following isomorphisms

$$\begin{aligned} \{\omega \in \hat{\mathbf{P}} \mid \hat{\theta}^\vee(\omega) = 1\} &\cong \mathbf{P}, \\ \{i \in I \mid \hat{\theta}^\vee(\omega_i) = 1\} &\cong \{v\} \sqcup \{i \in I^{\text{fin}} \mid \theta^\vee(\omega_i) = 1\}. \end{aligned} \tag{8.0.3}$$

It follows from (8.0.3) that if  $\hat{\theta}^\vee(\omega) = 1$ , then the condition

$$\omega = \omega_i + \hat{C}(\omega'), \quad \text{where } i \text{ is an extended vertex in } I \tag{8.0.4}$$

is equivalent to the condition  $\text{pr}(\omega) = \text{pr}(\omega_i) + \text{pr}(\hat{C}(\omega'))$ . On the other hand, it is clear that  $\omega' = \omega'' + \alpha_v^\vee(\omega_v) \cdot \omega_v$ , for some  $\omega'' \in \mathbf{P}$ . Hence  $\text{pr}(\hat{C}(\omega')) = \text{pr}(\hat{C}(\omega'')) = C(\omega'')$ . Thus condition (8.0.4) is equivalent to

$$\text{pr}(\omega) = \text{pr}(\omega_i) + C(\omega''), \quad \text{where } i \text{ is an extended vertex in } I^{\text{fin}}, \text{ or } \omega_i = 0.$$

Thus we obtain the following reformulation of Lemma 8.0.2.

**LEMMA 8.0.5.** *For any  $\omega \in \mathbf{P}$  there exists a uniquely determined pair  $(i, \omega'')$ , where  $i$  is either  $v$  or an extended vertex of  $I^{\text{fin}}$  and  $\omega'' \in \mathbf{P}$  is a weight, such that  $\omega = \omega_i + C(\omega'')$ .*

*Proof of Lemma 8.0.5.* The image of the Cartan operator  $C$  is the root sublattice  $\mathbf{Q} \subset \mathbf{P}$ . On the other hand, it is well known (see [Bou], §2, Ex. 5) that each coset in  $\mathbf{P}/\mathbf{Q}$  contains a unique minuscule weight or zero. Finally, according to [Bou], §1, Ex. 24 the set of minuscule weights coincides with the set of fundamental weights of extended vertices of  $I^{\text{fin}}$ .  $\square$

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