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Wilson's Grassmannian and a Noncommutative Quadric

Vladimir Baranovsky, Victor Ginzburg,
and Alexander Kuznetsov

To Yuri Ivanovich Manin on the occasion of his 65th birthday

1 Introduction

Let the group $\boldsymbol{\mu}_m$ of mth roots of unity act on the complex line by multiplication. This gives a $\boldsymbol{\mu}_m$-action on Diff, the algebra of polynomial differential operators on the line. Following Crawley-Boevey and Holland [7], we introduce a multiparameter deformation D_τ of the smash-product Diff $\#\boldsymbol{\mu}_m$. Our main result provides natural bijections between (roughly speaking) the following spaces:

(1) $\boldsymbol{\mu}_m$-equivariant version of Wilson's *adelic Grassmannian* of rank r;
(2) rank r projective D_τ-modules (with generic trivialization data);
(3) rank r torsion-free sheaves on a "noncommutative quadric" $\mathbb{P}^1 \times_\tau \mathbb{P}^1$;
(4) disjoint union of Nakajima quiver varieties for the cyclic quiver with m vertices.

The bijection between (1) and (2) is provided by a version of Riemann-Hilbert correspondence between $\mathcal{D}$-modules and sheaves. The bijections between (2), (3), and (4) were motivated by our previous work [2]. The resulting bijection between (1) and (4) reduces, in the very special case $r = 1$ and $\boldsymbol{\mu}_m = \{1\}$, to the partition of (rank 1) adelic Grassmannian into a union of Calogero-Moser spaces discovered by Wilson. This gives, in particular, a natural and purely algebraic approach to Wilson's result [13].

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We proceed to more details. Recall that Nakajima *quiver varieties* can be viewed, according to our previous paper [2], as spaces parametrizing torsion-free sheaves on a "noncommutative plane." In the simplest case, this yields a relation, first observed by Berest and Wilson [3], between Calogero-Moser spaces and projective modules over the first Weyl algebra $\mathcal{D}(\mathbb{C})$ of polynomial differential operators on the line $\mathbb{C}$. The approach to this result (and to its quiver generalizations) used in [2] was purely algebraic and totally different from the approach in [3]. The latter involved a nonalgebraic *Baker function* and was based heavily on the earlier remarkable discovery by Wilson [13] of a connection between an *adelic Grassmannian* and Calogero-Moser spaces.

In this paper, we reverse the logic used by Berest and Wilson and *explain* (rather than *exploit*) the connection between adelic Grassmannians and quiver varieties by means of *noncommutative algebraic geometry*, using the strategy of [2].[1]

Our first key observation is that each point of adelic Grassmannian can be viewed as a "constructible sheaf" on the line built up from "infinite-rank" local systems. This way, the correspondence between projective (not *holonomic*) $\mathcal{D}(\mathbb{C})$-modules and points of the adelic Grassmannian becomes nothing but (a nonholonomic version of) the standard de Rham functor between $\mathcal{D}$-modules and constructible sheaves on the line.[2]

The de Rham correspondence works equally well in a more general context of equivariant $\mathcal{D}$-modules with respect to a natural action on the line $\mathbb{C}$ of the group $\boldsymbol{\mu}_m$ of mth roots of unity, by multiplication. Giving a $\boldsymbol{\mu}_m$-equivariant $\mathcal{D}$-module is clearly the same thing as giving a module over $\mathcal{D}(\mathbb{C})\#\boldsymbol{\mu}_m$, the smash product of $\mathcal{D}(\mathbb{C})$ with the group $\boldsymbol{\mu}_m$, acting on $\mathcal{D}(\mathbb{C})$ by algebra automorphisms. Note that in [2] *any*, not only cyclic, finite group $\Gamma \subset SL_2(\mathbb{C})$ of automorphisms of the Weyl algebra was considered. In order to have a de Rham functor, however, we need to specify a standard holonomic $\mathcal{D}$-module of *regular functions*. The choice of such a $\mathcal{D}$-module breaks the $SL_2(\mathbb{C})$-symmetry of the 2-plane formed by the generators of the first Weyl algebra. Thus, the group $\Gamma \subset SL_2(\mathbb{C})$ has to have an invariant line in $\mathbb{C}^2$. This leaves us with the only choice $\Gamma = \boldsymbol{\mu}_m$.

[1] During the preparation of this paper (which was first supposed to be part of [2]) another paper by Berest and Wilson appeared, see [4]. Our approach is similar to that of [4] (we treat more general case of "higher rank" and $\boldsymbol{\mu}_m$-equivariant projective modules). However, even in the rank 1 case, in [4], the authors do not provide an independent proof of the bijection between Calogero-Moser spaces and projective modules; instead, they construct a map inverse to the map constructed in [3], assuming that the latter is already known to be a bijection. An independent direct proof of the bijectivity in the rank 1 case was obtained in the Appendix to [4] by M. Van den Bergh who used some results of [2]. We emphasize that, for the reasons explained at the end of the introduction below, it seems to be impossible to extend the approach of [4] (connecting the adelic Grassmannian with rank 1 sheaves on a noncommutative $\mathbb{P}^2_\tau$) to the higher-rank case without replacing $\mathbb{P}^2_\tau$ by a noncommutative surface which fibers over $\mathbb{P}^1$, like the noncommutative quadric $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ that we are using in this paper.

[2] More generally, our construction of de Rham functor yields a similar correspondence between projective $\mathcal{D}$-modules on any smooth algebraic curve X and points of an appropriately defined adelic Grassmannian attached to the curve (in that case a noncommutative version of projective completion of T^*X should play the role of $\mathbb{P}^1 \times_\tau \mathbb{P}^1$). The case of an elliptic curve seems to be especially interesting; we hope to discuss it elsewhere.

Below, we will be working not only with the algebra $\mathcal{D}(\mathbb{C})\#\boldsymbol{\mu}_m$, but also with a multiparameter deformation

$$D_\tau = \mathbb{C}\langle x, y\rangle\#\boldsymbol{\mu}_m/\langle [y, x] = \tau\rangle \tag{1.1}$$

of the algebra introduced by Crawley-Boevey and Holland [7]. Here, τ (deformation parameter) is an arbitrary element in the group algebra $\mathbb{C}[\boldsymbol{\mu}_m]$ and $\mathbb{C}\langle x, y\rangle$ stands for the free $\mathbb{C}$-algebra of noncommutative polynomials in two variables x and y. Once a de Rham functor between projective D_τ-modules and points of an adelic Grassmannian is established, we can construct a Wilson-type connection between the adelic Grassmannian and quiver varieties as follows. First, view a projective D_τ-module as a vector bundle on an appropriate noncommutative plane $\mathbb{A}^2_\tau$. Next, extend (see [2]) this vector bundle to a (framed) torsion-free sheaf on a *completion* $X_\tau \supset \mathbb{A}^2_\tau$, a "noncommutative projective surface." Finally, we use a description of the framed torsion-free sheaves on X_τ in terms of *monads* (i.e., in terms of linear algebra data) developed in [2] to obtain a parametrisation of projective D_τ-modules by points of certain quiver varieties.

There are several possible choices for a "compactification" X_τ of the noncommutative plane $\mathbb{A}^2_\tau$. In [2], we used $X_\tau = \mathbb{P}^2_\tau$, a noncommutative version of projective plane. In this paper, we choose another "compactification" of $\mathbb{A}^2_\tau$, a noncommutative version $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ of two-dimensional quadric. This choice is essential for our present purposes. Our construction of the extension of a D_τ-module to a torsion-free sheaf on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ does not behave well enough in the case of $\mathbb{P}^2_\tau$. On the other hand, the relation of sheaves on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ to quiver varieties is a posteriori equivalent to the one used in [2] since the two noncommutative spaces $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ and $\mathbb{P}^2_\tau$ can be obtained from each other by "blowing up" and "blowing down" constructions. We will indicate the idea of such a construction in Remark 5.3 and it will be hinted there how a bijection between torsion-free sheaves on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ and on $\mathbb{P}^2_\tau$ can be established via a noncommutative version of Fourier-Mukai transform, see (5.11).

Our results generalize (and, hopefully, put in context) the results of Wilson [13] in two ways. First, we incorporate a $\boldsymbol{\mu}_m$-action. Second, Wilson only considered the rank 1 case that is the case of rank 1 sheaves on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ in the terminology of this paper. In Wilson's situation, the *whole* adelic Grassmannian gets partitioned into a disjoint union of Calogero-Moser spaces. In the more general setup of arbitrary rank $r \geq 1$, this is no longer true for two reasons. First, in our definition of the adelic Grassmannian, we drop the *index-zero* condition of [13, condition 2.1(ii)] (it has to do with replacing the group SL_r by GL_r). This makes our version of adelic Grassmannian somewhat larger than that of [13]. The geometric consequence of "index-zero" condition is (in our language) that the restriction of a coherent sheaf to the line $\mathbb{P}^1 \times \{\infty\} \subset \mathbb{P}^1 \times_\tau \mathbb{P}^1$ is a vector bundle with the vanishing first-Chern class: $c_1 = 0$. In the rank 1 case considered by Wilson, any such bundle is necessarily trivial, while this is clearly not true for higher ranks. Thus, our main result says that, for any $m \geq 1$ and $r \geq 1$, the part of ($\boldsymbol{\mu}_m$-equivariant) rank r adelic Grassmannian formed by sheaves, *trivial* on the line $\mathbb{P}^1 \times \{\infty\} \subset \mathbb{P}^1 \times_\tau \mathbb{P}^1$ can be partitioned into a disjoint union of quiver varieties of type $\mathbf{A}_{m-1}$.

Remark 1.1. We would like to end this introduction by bringing the reader's attention to a surprising correspondence resulting from comparing [2, Theorem 1.3.12] with [8, Theorem 1.13]. Specifically, let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup, and $D_\tau(\Gamma)$ the coordinate ring of the corresponding Γ-equivariant "noncommutative plane." Recall that, according to [2], there is a partition of the moduli space of projective $D_\tau(\Gamma)$-modules N with the fixed class $[N] \in K(D_\tau(\Gamma))$ corresponding to the class [triv] of the trivial 1-dimensional Γ-module (under the standard Grothendieck group isomorphism $K(D_\tau(\Gamma)) \cong K(\mathbb{C}\Gamma)$) into a disjoint union, according to the "second-Chern class" $c_2(N) := n \in \mathbb{Z}$. On the other hand, given $n \geq 1$, let $\boldsymbol{\Gamma}_n := S_n \ltimes (\Gamma \times \Gamma \times \cdots \times \Gamma) \subset Sp(\mathbb{C}^{2n})$ denote the corresponding wreath product and let $H_{0,\tau}(\boldsymbol{\Gamma}_n)$ be the *symplectic reflection algebra* attached to $\boldsymbol{\Gamma}_n$, see [8, page 249].[3] Further, [8, Theorem 1.13] shows that representation theories of the algebras $D_\tau(\Gamma)$ and $H_{0,\tau}(\boldsymbol{\Gamma}_n)$ are related by the following *mysterious bijection*:

$$\left\{\begin{matrix}\text{Isomorphism classes of finitely generated } \textit{projective} \\ D_\tau(\Gamma)\text{-modules } N \text{ such that } [N] = \text{triv and } c_2(N) = n\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{Isomorphism classes of} \\ \textit{simple } H_{0,\tau}(\boldsymbol{\Gamma}_n)\text{-modules}\end{matrix}\right\}. \tag{1.2}$$

Finding a direct conceptual explanation of the bijection above presents a challenging problem. We remark that even in the case of the trivial group Γ, where the moduli space on each side reduces, as a variety, to the Calogero-Moser space, the bijection is still completely unexplained. It may be analogous to *level-rank* type duality in representation theory of Kac-Moody algebras.

2 Statement of results

From now on, let $\Gamma = \boldsymbol{\mu}_m$ denote the group of mth roots of unity, and $\mathbb{C}\Gamma$ its group algebra. We fix an embedding $\Gamma = \boldsymbol{\mu}_m \hookrightarrow SL_2(\mathbb{C})$, and let L denote the tautological two-dimensional representation of Γ arising from the embedding. We have $L \cong \epsilon \oplus \epsilon^{-1}$, where

[3] In [2], we use the notation $\mathcal{B}_\tau$ instead of $D_\tau(\Gamma)$, and in [8], we write c for what we denote by τ in [2] (and in this paper).

ϵ is a primitive character of Γ. Let $\{x, y\}$ be a basis of L^*, compatible with the above direct sum decomposition, such that Γ acts on x by ϵ and on y by ϵ^{-1}. Write $\mathbb{C}[x]$ for the polynomial algebra on the line with coordinate x, and $\mathbb{C}(x)$ for the corresponding field of rational functions. We form the smash-product algebras

$$\mathbb{C}\Gamma[x] := \mathbb{C}[x]\#\Gamma, \qquad \mathbb{C}\Gamma(x) := \mathbb{C}(x)\#\Gamma. \tag{2.1}$$

The standard embedding $\mathbb{C}\Gamma \hookrightarrow \mathbb{C}[x]\#\Gamma$ makes $\mathbb{C}\Gamma[x]$ into a Γ-bimodule via left and right multiplication by Γ. There is a similar Γ-bimodule structure on $\mathbb{C}\Gamma(x)$.

Choose and fix a finite-dimensional Γ-module W. There is a natural $\mathbb{C}(x)^\Gamma$-action on $W \otimes_\Gamma \mathbb{C}\Gamma(x)$ given by $p : w \otimes f \mapsto p \cdot (w \otimes f) := w \otimes (p \cdot f)$.

Definition 2.1. A Γ-invariant vector subspace $\mathcal{U} \subset W \otimes_\Gamma \mathbb{C}\Gamma(x)$ is called *primary decomposable*[4] if the following two conditions hold:

(a) there exists a Γ-invariant polynomial $p = p(x)$ such that

$$p \cdot \big(W \otimes_\Gamma \mathbb{C}\Gamma[x]\big) \subset \mathcal{U} \subset \frac{1}{p} \cdot \big(W \otimes_\Gamma \mathbb{C}\Gamma[x]\big); \tag{2.2}$$

(b) if $p(x) = \prod_\mu (x-\mu)^{k_\mu}$, then the subspace on the left of (2.3) is compatible with the direct sum decomposition on the right (i.e., left-hand side = sum of its intersections with the direct summands on the right-hand side)

$$\frac{\mathcal{U}}{p \cdot \big(W \otimes_\Gamma \mathbb{C}\Gamma[x]\big)} \subset \frac{\frac{1}{p} \cdot \big(W \otimes_\Gamma \mathbb{C}\Gamma[x]\big)}{p \cdot \big(W \otimes_\Gamma \mathbb{C}\Gamma[x]\big)} = \bigoplus_\mu \frac{(x-\mu)^{-k_\mu} W \otimes_\Gamma \mathbb{C}\Gamma[x]}{(x-\mu)^{k_\mu} W \otimes_\Gamma \mathbb{C}\Gamma[x]}. \tag{2.3}$$

Define an *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}(W)$ to be the set of all primary decomposable subspaces $\mathcal{U} \subset W \otimes_\Gamma \mathbb{C}\Gamma(x)$.

Our first goal is to relate the adelic Grassmannian to modules over some noncommutative algebra. To that end, we fix an element $\tau \in \mathbb{C}\Gamma$ and consider the algebra D_τ, see (1.1), to be denoted as D in the future. Let D_{frac} be the localization of the algebra D with respect to the multiplicative system $\mathbb{C}[x]^\Gamma \setminus \{0\}$ of all nonzero Γ-invariant polynomials in x. This localization has a natural algebra structure extending that on D. Note further that we have a natural embedding $\mathbb{C}\Gamma(x) \hookrightarrow D_{\mathrm{frac}}$ and, moreover, this embedding yields a vector space isomorphism: $\mathbb{C}\Gamma(x) \xrightarrow{\sim} D_{\mathrm{frac}}/D_{\mathrm{frac}} \cdot y$. We make $\mathbb{C}\Gamma(x)$ into a left D_{frac}-module by transporting the obvious D_{frac}-module structure on $D_{\mathrm{frac}}/D_{\mathrm{frac}} \cdot y$ via the bijection above.

[4]This notion is due to Cannings and Holland [5].

For $\tau = 1$, this reduces essentially to the standard action on $\mathbb{C}\Gamma(x)$ by differential operators.

For any Γ-module W, the space $W \otimes_\Gamma D_{\mathrm{frac}}$ has an obvious structure of a projective right D_{frac}-module.

Definition 2.2. Let $G_W = GL_{D_{\mathrm{frac}}}(W \otimes_\Gamma D_{\mathrm{frac}})$ be the group of all (invertible) right D_{frac}-linear automorphisms of the D_{frac}-module $W \otimes_\Gamma D_{\mathrm{frac}}$.

We define a *left* G_W-action on the vector space $W \otimes_\Gamma \mathbb{C}\Gamma(x)$ and on the adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}(W)$ as follows. First, observe that the natural left $GL_{D_{\mathrm{frac}}}(W \otimes_\Gamma D_{\mathrm{frac}})$-action on $W \otimes_\Gamma D_{\mathrm{frac}}$ commutes with right multiplication by y, therefore, keeps the subspace $W \otimes_\Gamma D_{\mathrm{frac}} \cdot y \subset W \otimes_\Gamma D_{\mathrm{frac}}$ stable. Hence, there is a well-defined left G_W-action on the quotient $(W \otimes_\Gamma D_{\mathrm{frac}})/(W \otimes_\Gamma D_{\mathrm{frac}} \cdot y)$. Further, since left Γ-action commutes with right D_{frac}-action, and since W is a projective Γ-module, it follows that we have an isomorphism

$$\psi : W \otimes_\Gamma \mathbb{C}\Gamma(x) \xrightarrow{\sim} \big(W \otimes_\Gamma D_{\mathrm{frac}}\big)/\big(W \otimes_\Gamma D_{\mathrm{frac}} \cdot y\big), \tag{2.4}$$

induced by the isomorphism $\mathbb{C}\Gamma(x) \xrightarrow{\sim} D_{\mathrm{frac}}/D_{\mathrm{frac}} \cdot y$ considered two paragraphs above. We define the left G_W-action on $W \otimes_\Gamma \mathbb{C}\Gamma(x)$ by transporting the left G_W-action on the quotient $(W \otimes_\Gamma D_{\mathrm{frac}})/(W \otimes_\Gamma D_{\mathrm{frac}} \cdot y)$ via the isomorphism ψ. It is straightforward to verify that elements of G_W take primary decomposable subspaces of $W \otimes_\Gamma \mathbb{C}\Gamma(x)$ into primary decomposable subspaces of $W \otimes_\Gamma \mathbb{C}\Gamma(x)$. This gives a canonical left G_W-action on the adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}(W)$.

To formulate our first result, recall that there is a canonical isomorphism (due to Quillen) of Grothendieck K-groups: $K(\mathbb{C}\Gamma) \xrightarrow{\sim} K(D)$ induced by the functor $W \mapsto W \otimes_\Gamma D$. We write $[N] \in K(\mathbb{C}\Gamma)$ for the image of the class of a D-module N under the inverse isomorphism $K(D) \to K(\mathbb{C}\Gamma)$, and let $\dim : K(\mathbb{C}\Gamma) \to \mathbb{Z}$ denote the dimension homomorphism.

Further, there is a distinguished finite collection of codimension one hyperplanes in the vector space $\mathbb{C}\Gamma$, called the *root* hyperplanes. One way to define these hyperplanes is to use McKay correspondence. The latter associates to the cyclic group $\Gamma = \boldsymbol{\mu}_m \subset SL_2(\mathbb{C})$ an affine Dynkin graph of type $\widetilde{\mathbf{A}_{m-1}}$ such that the underlying vector space of the group algebra $\mathbb{C}\Gamma$ gets identified with the dual of the $\mathbb{C}$-vector space generated by simple roots of the corresponding affine root system. In particular, every root gives a hyperplane in the vector space $\mathbb{C}\Gamma$.

Definition 2.3. An element $\tau \in \mathbb{C}\Gamma$ is called *generic* if it does not belong to any root hyperplane in $\mathbb{C}\Gamma$.

Theorem 2.4 below is a noncommutative analogue of a well-known result due to Weil, providing a description of algebraic vector bundles on an algebraic curve in terms of an adelic double-coset construction.

Theorem 2.4. Assume that τ is generic. Let W be a Γ-module with $\dim W = r$. The set of isomorphism classes of projective (right) D-modules N such that $\dim[N] = r$ is in a canonical bijection with the coset space $G_W \backslash \mathrm{Gr}^{\mathrm{ad}}(W)$. □

To explain the main ideas involved in the proof of Theorem 2.4, we need the following definition.

Definition 2.5. A right D-submodule $N \subset W \otimes_\Gamma D_{\mathrm{frac}}$ is called *fat* if there exists a Γ-invariant polynomial $p(x)$ such that $p \cdot (W \otimes_\Gamma D) \subset N \subset (1/p) \cdot (W \otimes_\Gamma D)$.

Let $\mathrm{Gr}^D(W)$ be the set of all fat right D-submodules $N \subset W \otimes_\Gamma D_{\mathrm{frac}}$.

Now, the proof goes as follows. First, we check that for generic τ any projective right D-module can be embedded into $W \otimes_\Gamma D_{\mathrm{frac}}$ as a fat D-submodule. The embedding is unique up to a G_W-action. Then, it remains to relate the Grassmanians $\mathrm{Gr}^D(W)$ and $\mathrm{Gr}^{\mathrm{ad}}(W)$. To this end, recall that we have equipped the space $\mathbb{C}\Gamma[x]$ with a canonical structure of left D-module, that clearly commutes with right Γ-action by multiplication.

We introduce a de Rham functor $\mathcal{DR}$ from the category of *right* D-modules to the category of right Γ-modules as follows:

$$N \longmapsto \mathcal{DR}(N) := N \otimes_D \mathbb{C}\Gamma[x]. \tag{2.5}$$

Given a nonzero polynomial $p \in \mathbb{C}[x]^\Gamma$, we write $p \cdot (W \otimes_\Gamma D) := (W \otimes_\Gamma (p \cdot D))$. The space $p \cdot (W \otimes_\Gamma D)$ has an obvious right D-module structure, and we have

$$\mathcal{DR}\big(p \cdot (W \otimes_\Gamma D)\big) = p \cdot \big(W \otimes_\Gamma D\big) \otimes_D \mathbb{C}\Gamma[x] = p \cdot \big(W \otimes_\Gamma \mathbb{C}\Gamma[x]\big). \tag{2.6}$$

Hence, the de Rham functor takes any fat D-submodule of $W \otimes_\Gamma D_{\mathrm{frac}}$ to a Γ-invariant vector subspace $\mathcal{U} \subset W \otimes_\Gamma \mathbb{C}\Gamma(x)$ such that $p \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x]) \subset \mathcal{U} \subset (1/p) \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])$. Moreover, we can check that the latter vector space is primary decomposable.

Example 2.6. Assume that $\Gamma = \{1\}$ is trivial, that is, $m = 1$. Then $\mathbb{C}\Gamma = \mathbb{C}$ and $\tau \in \mathbb{C}\Gamma = \mathbb{C}$ is generic if and only if $\tau \neq 0$. In this case, the algebra D is isomorphic to $\mathcal{D}(\mathbb{C})$, the algebra of differential operators on the line $\mathbb{C}$, and the algebra D_{frac} is isomorphic to the algebra of differential operators with rational coefficients. The functor $\mathcal{DR}$ becomes the standard de Rham functor.

We have seen that the de Rham functor yields a map $\mathcal{DR} : \mathrm{Gr}^{\mathrm{D}}(W) \to \mathrm{Gr}^{\mathrm{ad}}(W)$. We also define a map $\mathrm{Gr}^{\mathrm{ad}}(W) \to \mathrm{Gr}^{\mathrm{D}}(W)$ (in the opposite direction) as follows. Let $\mathcal{U} \subset W \otimes_\Gamma \mathbb{C}\Gamma(x)$ be a primary decomposable subspace. Then the left $\mathrm{D}_{\mathrm{frac}}$-action map $\mathrm{D}_{\mathrm{frac}} \otimes \mathbb{C}\Gamma(x) \to \mathbb{C}\Gamma(x)$ induces, after tensoring with W and restricting to $\mathbb{C}\Gamma[x]$, a linear map $a : W \otimes_\Gamma \mathrm{D}_{\mathrm{frac}} \otimes_{\mathbb{C}} \mathbb{C}\Gamma[x] \to W \otimes_\Gamma \mathbb{C}\Gamma(x)$. Let $\mathrm{Diff}_\tau(\mathcal{U})$ denote the set of all elements $u \in W \otimes_\Gamma \mathrm{D}_{\mathrm{frac}}$ such that $a(u \otimes \mathbb{C}\Gamma[x]) \subset \mathcal{U}$. It is easy to show that $\mathrm{Diff}_\tau(\mathcal{U}) \subset W \otimes_\Gamma \mathrm{D}_{\mathrm{frac}}$ is a fat D-submodule. Thus, we obtain a map $\mathrm{Diff}_\tau : \mathrm{Gr}^{\mathrm{ad}}(W) \to \mathrm{Gr}^{\mathrm{D}}(W)$.

Theorem 2.7. For generic τ, the maps $\mathcal{DR}$ and Diff_τ give mutually inverse bijections $\mathrm{Gr}^{\mathrm{D}}(W) \cong \mathrm{Gr}^{\mathrm{ad}}(W)$. □

Our next step is to interpret the space $\mathrm{Gr}^{\mathrm{D}}(W)$, using the formalism of noncommutative geometry. To this end, we introduce the algebra

$$Q = \mathbb{C}\langle x, z, y, w\rangle \# \Gamma \big/ \big\langle [x, z] = [y, z] = [z, w] = [y, w] = [x, w] = 0,\ [y, x] = \tau \cdot zw \big\rangle, \tag{2.7}$$

where for any $\gamma \in \Gamma$, we put

$$\gamma \cdot x \cdot \gamma^{-1} = \epsilon(\gamma) \cdot x, \qquad \gamma \cdot y \cdot \gamma^{-1} = \epsilon^{-1}(\gamma) \cdot y, \qquad \gamma \cdot z \cdot \gamma^{-1} = z, \qquad \gamma \cdot w \cdot \gamma^{-1} = w. \tag{2.8}$$

Define a bigrading $Q = \oplus_{i,j \geq 0} Q_{i,j}$ on the algebra Q by letting $\deg x = \deg z = (1, 0)$, $\deg y = \deg w = (0, 1)$, and $\deg \gamma = (0, 0)$ for any $\gamma \in \Gamma$. Thus, $Q_{0,0} = \mathbb{C}\Gamma$.

When Γ is trivial and $\tau = 0$, the algebra Q reduces to the standard bigraded algebra associated with the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ and a pair of line bundles $L_1 = \mathcal{O}(1, 0)$ and $L_2 = \mathcal{O}(0, 1)$, that is, for any $i, j \geq 0$, we have $Q_{i,j} = H^0(\mathbb{P}^1 \times \mathbb{P}^1, L_1^{\otimes i} \otimes L_2^{\otimes j})$. In this case, the category of coherent sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ can be described as a quotient category of the category of bigraded Q-modules (see Section 4).

In the general case of a nontrivial group Γ and arbitrary τ, a similar quotient category construction may still be applied formally to the bigraded ring Q. Following [1], see also [2] and Appendices A and B below, we will view objects of the resulting quotient category as coherent sheaves on a "noncommutative quadric" $\mathbb{P}^1 \times_\tau \mathbb{P}^1$.

Note that, in the commutative case $\tau = 0$, the equations $z = 0$ and $w = 0$ give rise to two embeddings $i_z : \mathbb{P}^1_z \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $i_w : \mathbb{P}^1_w \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of the corresponding factors $\mathbb{P}^1$. Thus, we may consider the restriction functor i_z^*, taking coherent sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ to coherent sheaves on $\mathbb{P}^1_z$ (and also consider coherent sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ which are trivialized in some formal neighbourhood of $\mathbb{P}^1_z$). In Section 4 and Appendix A, we show how to extend all relevant concepts to the noncommutative case. The homogeneous coordinate rings of $\mathbb{P}^1_z$ and $\mathbb{P}^1_w$ will be replaced by $\mathbb{C}[y, w]\#\Gamma$ and $\mathbb{C}[x, z]\#\Gamma$, respectively.

The latter algebras are only slightly noncommutative in the sense that the corresponding quotient categories of "noncommutative coherent sheaves" are nothing but the categories of Γ-equivariant coherent sheaves on $\mathbb{P}^1$, the ordinary (commutative) projective line. This leads to the following provisional definition (see Definitions 4.3 and 4.5 for details).

Definition 2.8. Let $\mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ be the set of all (equivalence classes of) coherent sheaves E on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ trivialized in a formal neighbourhood of $\mathbb{P}^1_z$ and such that $i_z^*E \simeq W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_z}$.

Note that in the commutative situation we have, $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (\mathbb{P}^1_z \cup \mathbb{P}^1_w) = \mathbb{A}^2$ is an affine plane with coordinates x and y. In the noncommutative case we have an algebra isomorphism

$$D \simeq Q/\big((z-1)Q + (w-1)Q\big). \tag{2.9}$$

Therefore, the algebra D may be viewed as coordinate ring of a noncommutative affine plane $j : \mathbb{A}^2_\tau \hookrightarrow \mathbb{P}^1 \times_\tau \mathbb{P}^1$. This gives rise to a restriction functor $j^* : \mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1) \to \mathrm{mod}(D)$, taking the category of coherent sheaves on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ to D-modules. It is easy to see that j^* takes any sheaf trivialized in a neighborhood of $\mathbb{P}^1_z$ to a fat D-submodule of $W \otimes_\Gamma D_{\mathrm{frac}}$.

Theorem 2.9. The "restriction" functor j^* induces a bijection $\mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \xrightarrow{\sim} \mathrm{Gr}^D(W)$. □

We write $j_{!*}$ for an inverse of the bijection j^*.

The fourth (and the last) infinite Grassmannian considered in this paper is an *affine Grassmannian* $\mathrm{Gr}^{\mathrm{aff}}(W)$ introduced below.

Definition 2.10. A right $\mathbb{C}\Gamma[x]$-submodule $\mathcal{W} \subset W \otimes_\Gamma \mathbb{C}\Gamma(x)$ is called *fat* if there exists a Γ-invariant polynomial $p(x)$ such that $p \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x]) \subset \mathcal{W} \subset (1/p) \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])$.

Define $\mathrm{Gr}^{\mathrm{aff}}(W)$ to be the set of all fat $\mathbb{C}\Gamma[x]$-submodules in $W \otimes_\Gamma \mathbb{C}\Gamma(x)$.

It is clear that a (right) Γ-stable subspace $\mathcal{W} \subset W \otimes_\Gamma \mathbb{C}\Gamma(x)$ is a fat $\mathbb{C}\Gamma[x]$-submodule if and only if $\mathcal{W}$ is a finitely-generated $\mathbb{C}[x]$-submodule such that $\mathcal{W} \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = W \otimes_\Gamma \mathbb{C}\Gamma(x)$. Thus, a fat $\mathbb{C}\Gamma[x]$-submodule may be viewed as a Γ-stable *lattice* in the $\mathbb{C}(x)$-vector space $W \otimes_\Gamma \mathbb{C}\Gamma(x)$. For this reason, we refer to $\mathrm{Gr}^{\mathrm{aff}}(W)$ as the *affine Grassmannian*. The standard relation between loop-Grassmannians and vector bundles on the Riemann sphere (see, e.g., [12]) shows that the space $\mathrm{Gr}^{\mathrm{aff}}(W)$ can also be interpreted as the set of all Γ-equivariant vector bundles on $\mathbb{P}^1$ trivialized in a Zariski neighbourhood of the point $\infty \in \mathbb{P}^1$ (see Definition 4.3) with W being a fiber at ∞. Thus, the restriction functor i_w^* takes $\mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ to $\mathrm{Gr}^{\mathrm{aff}}(W)$.

We observe that the group G_W, see Definition 2.2, acts naturally on each of the Grassmannians $\mathrm{Gr}^{\mathrm{ad}}(W)$, $\mathrm{Gr}^{D}(W)$, and $\mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$. Specifically, the action on $\mathrm{Gr}^{\mathrm{ad}}(W)$ has been defined earlier, and the action on $\mathrm{Gr}^{D}(W)$ is induced by the corresponding G_W-action on $W\otimes_\Gamma D_{\mathrm{frac}}$. The action on $\mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ arises from G_W-action on the direct system formed by the sheaves $W\otimes_\Gamma \mathcal{O}(n,0)$, $n=1,2,\ldots$. Observe further that the affine Grassmannian $\mathrm{Gr}^{\mathrm{aff}}(W)$ has an action of the subgroup $GL_{\mathbb{C}\Gamma(x)}(W\otimes_\Gamma \mathbb{C}\Gamma(x))\subset G_W$ (the group G_W itself does not act on $\mathrm{Gr}^{\mathrm{aff}}(W)$ since it does not preserve the condition to be a $\mathbb{C}\Gamma[x]$-submodule).

All the objects and the maps we have introduced so far are incorporated in the following diagram:

$$\mathrm{Gr}^{\mathrm{ad}}(W) \underset{\mathcal{DR}}{\overset{\mathrm{Diff}_\tau}{\rightleftarrows}} \mathrm{Gr}^{D}(W) \underset{j^*}{\overset{j_{!*}}{\rightleftarrows}} \mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \xrightarrow{\;i_w^*\;} \mathrm{Gr}^{\mathrm{aff}}(W). \tag{2.10}$$

*Principal symbol map "*Symb*".* The algebra D has a natural increasing filtration $\mathbb{C}\Gamma[x]=F_0D\subset F_1D\subset F_2D\subset\cdots$, where F_kD is the $\mathbb{C}\Gamma[x]$-submodule generated by $\{1,y,\ldots,y^k\}$. This filtration, by the *order* of differential operator, extends canonically to a similar filtration $\mathbb{C}\Gamma(x)=F_0D_{\mathrm{frac}}\subset F_1D_{\mathrm{frac}}\subset F_2D_{\mathrm{frac}}\subset\cdots$ on the algebra D_{frac}, and for the corresponding associated graded algebras, we have $\mathrm{gr}^F D\simeq\mathbb{C}\Gamma[x,y]$ and $\mathrm{gr}^F D_{\mathrm{frac}}\simeq\mathbb{C}\Gamma(x)[y]$. The filtration on D_{frac} also induces an increasing filtration, $F_k(W\otimes_\Gamma D_{\mathrm{frac}}):=W\otimes_\Gamma F_kD_{\mathrm{frac}}$, on the D_{frac}-module $W\otimes_\Gamma D_{\mathrm{frac}}$ such that $\mathrm{gr}^F(W\otimes_\Gamma D_{\mathrm{frac}})\simeq(W\otimes_\Gamma\mathbb{C}\Gamma(x))[y]$.

Now, given a D-submodule $N\subset W\otimes_\Gamma D_{\mathrm{frac}}$, we put

$$\begin{aligned}\mathrm{Symb}(N):=\big\{f\in W\otimes_\Gamma\mathbb{C}\Gamma(x)\mid f\cdot y^k+u_{k-1}\in N,\\ \text{for some } k\in\mathbb{Z}\text{ and some } u_{k-1}\in F_{k-1}\big(W\otimes_\Gamma D_{\mathrm{frac}}\big)\big\}.\end{aligned} \tag{2.11}$$

This is a $\mathbb{C}\Gamma(x)$-submodule in $W\otimes_\Gamma\mathbb{C}\Gamma(x)$ that can be equivalently defined as follows.

Right multiplication by y gives rise to a direct system of bijective maps $\mathbb{C}\Gamma(x)\xrightarrow{\sim}\mathbb{C}\Gamma(x)\cdot y\xrightarrow{\sim}\mathbb{C}\Gamma(x)\cdot y^2\xrightarrow{\sim}\cdots$, and it is clear that this yields isomorphisms

$$\begin{aligned}&\mathbb{C}\Gamma(x)=\mathbb{C}\Gamma(x)\cdot y^0\xrightarrow{\;\sim\;}\varinjlim_k\mathbb{C}\Gamma(x)\cdot y^k,\\ &W\otimes_\Gamma\mathbb{C}\Gamma(x)=W\otimes_\Gamma\mathbb{C}\Gamma(x)\cdot y^0\xrightarrow{\;\sim\;}\varinjlim_k\big(W\otimes_\Gamma\mathbb{C}\Gamma(x)\big)\cdot y^k.\end{aligned} \tag{2.12}$$

Let $\mathrm{gr}^F N\subset\mathrm{gr}^F(W\otimes_\Gamma D_{\mathrm{frac}})\simeq W\otimes_\Gamma\mathbb{C}\Gamma(x)[y]$ denote the associated graded module of N with respect to the induced filtration $F_\bullet N:=N\cap F_\bullet(W\otimes_\Gamma D_{\mathrm{frac}})$. Form the direct system $\mathrm{gr}_0^F N\to\mathrm{gr}_1^F N\to\mathrm{gr}_2^F N\to\cdots$ induced by the y-action on $\mathrm{gr}^F N$ (which is not necessarily bijective). Using the identification provided by (2.12), we have $\mathrm{Symb}(N)=\varinjlim\mathrm{gr}_k^F N$.

It is clear that the right-hand side is a $\mathbb{C}\Gamma[x]$-submodule in $\mathbb{C}\Gamma(x)$. The assignment $N \mapsto \mathrm{Symb}(N)$ gives a (discontinuous) map $\mathrm{Symb} : \mathrm{Gr}^{D}(W) \to \mathrm{Gr}^{\mathrm{aff}}(W)$.

Let σ denote the composite map, see (2.10),

$$\sigma : \mathrm{Gr}^{\mathrm{ad}}(W) \xrightarrow{\mathrm{Diff}_\tau} \mathrm{Gr}^{D}(W) \xrightarrow{j_{!*}} \mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \xrightarrow{i^*_{\mathcal{W}}} \mathrm{Gr}^{\mathrm{aff}}(W). \tag{2.13}$$

The following is an enriched version of diagram (2.10).

Theorem 2.11. Assume that τ is generic. Then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Gr}^{\mathrm{ad}}(W) \arrow[r, shift left, "\mathrm{Diff}_\tau"] \arrow[dr, "\sigma"'] & \mathrm{Gr}^{D}(W) \arrow[l, shift left, "\mathcal{DR}"] \arrow[r, shift left, "j_{!*}"] \arrow[d, "\mathrm{Symb}"] & \mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \arrow[l, shift left, "j^*"] \arrow[dl, "i^*_{\mathcal{W}}"] \\ & \mathrm{Gr}^{\mathrm{aff}}(W), & \end{tikzcd} \tag{2.14}$$

where the maps $\mathcal{DR}$, Diff_τ, $j_{!*}$, and j^* are G_W-equivariant bijections, and the maps $i^*_{\mathcal{W}}$, Symb, and σ are $GL_{\mathbb{C}\Gamma(x)}(W \otimes_\Gamma \mathbb{C}\Gamma(x))$-equivariant. $\square$

Finally, we explain how quiver varieties enter the picture. Given a pair of finite-dimensional Γ-modules W, V, and an element $\tau \in \mathbb{C}\Gamma$ define, following Nakajima, a locally closed subvariety of *quiver data*

$$\mathbb{M}^\tau_\Gamma(V,W) := \Big\{(B,I,J) \in \mathrm{Hom}_\Gamma\,\big(V, V\otimes_{\mathbb{C}} L\big) \bigoplus \mathrm{Hom}_\Gamma(W,V) \bigoplus \mathrm{Hom}_\Gamma(V,W)\Big\} \tag{2.15}$$

formed by the triples (B,I,J) satisfying the following two conditions:

(a) *moment map equation*: $[B,B] + IJ = \tau|_V$; and

(b) *stability condition*: if $V' \subset V$ is a Γ-submodule such that $B(V') \subset V' \otimes L$ and $I(W) \subset V'$, then $V' = V$.

The group $G_\Gamma(V) = GL(V)^\Gamma$ of Γ-equivariant automorphisms of V acts on $\mathbb{M}^\tau_\Gamma(V,W)$ by the formula $g(B,I,J) = (gBg^{-1}, gI, Jg^{-1})$. Note that this $G_\Gamma(V)$-action is free due to the stability condition.

Definition 2.12. Let $\mathfrak{M}^\tau_\Gamma(V,W) = \mathbb{M}^\tau_\Gamma(V,W)/G_\Gamma(V)$ be the geometric invariant theory quotient, called a (Nakajima) quiver variety.

The affine Grassmannian $\mathrm{Gr}^{\mathrm{aff}}(W)$ has a marked point $\mathcal{W}_0 = W \otimes_\Gamma \mathbb{C}\Gamma[x]$ corresponding to the Γ-equivariant sheaf $W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_{\mathcal{W}}}$ with its natural trivialization in the Zariski neighbourhood of infinity.

Theorem 2.13. Let τ be generic. Then there is a canonical bijection

$$\bigsqcup_{V} \mathfrak{M}_{\Gamma}^{\tau}(V, W) \cong \sigma^{-1}(\mathcal{W}_0) \subset \mathrm{Gr}^{\mathrm{ad}}(W), \tag{2.16}$$

where V runs through the set of isomorphism classes of all finite dimensional Γ-modules.

□

Example 2.14. Let the group Γ be trivial and $\tau \neq 0$. If $W = \mathbb{C}$ and $V = \mathbb{C}^n$, then the corresponding quiver variety is isomorphic to the Calogero-Moser space CM_n. Further, the affine Grassmannian reduces to the coset space $\mathbb{C}(x)^\times/\mathbb{C}^\times$. Moreover, the subset $\mathrm{Gr}_0^{\mathrm{ad}}(\mathbb{C}) := \sigma^{-1}(\mathcal{W}_0) \subset \mathrm{Gr}^{\mathrm{ad}}(\mathbb{C})$ consists of primary decomposable subspaces of "index zero" (in the sense of [13, condition 2(ii)]) at every point. Thus, Theorem 2.13 implies in this case Wilson's theorem saying that $\mathrm{Gr}_0^{\mathrm{ad}}(\mathbb{C}) = \bigsqcup_{n\geq 0} \mathrm{CM}_n$ is a union of the Calogero-Moser spaces. Note that our proof is purely algebraic (as opposed to [13]) and totally different from that in [13].

This paper is organized as follows. In Section 3, we prove Theorem 2.7 by means of a D-module version of Kashiwara's theorem describing $\mathcal{D}$-modules concentrated on a point. In Section 4, we reinterpret D-modules in terms of noncommutative geometry, and prove Theorems 2.9 and 2.11. Sections 5 and 6 contain proofs of Theorems 2.13 and 2.4, respectively. Appendix A deals with modifications that we have to introduce in the formalism of [1] in order to be able to work with polygraded algebras. Finally, in Appendix B, we prove some technical results on the "noncommutative surface" $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ including Serre duality and Beilinson spectral sequence.

3 Kashiwara theorem and de Rham functor

In this section, we prove Theorem 2.7 by reducing it to a deformed version of Kashiwara's theorem on $\mathcal{D}$-modules supported on a single point.

To begin the proof of Theorem 2.7, observe first that any primary decomposable subspace $p \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x]) \subset \mathcal{U} \subset (1/p) \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])$ is determined by the subspace $p \cdot \mathcal{U}/p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x]) \subset W \otimes_\Gamma \mathbb{C}\Gamma[x]/p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])$. Similarly, any fat D-submodule $p \cdot (W \otimes_\Gamma D) \subset N \subset (1/p) \cdot (W \otimes_\Gamma D)$ is determined by the D-submodule $p \cdot N/p^2 \cdot (W \otimes_\Gamma D) \subset W \otimes_\Gamma D/p^2 \cdot (W \otimes_\Gamma D)$. Observe further that the de Rham functor $\mathcal{DR}$ is right exact and the homological dimension of the category of D-modules equals 1 (see [7]). Moreover, since $W \otimes_\Gamma D$ and $p^2 \cdot (W \otimes_\Gamma D)$ are projective D-modules, we get

$$\mathcal{DR}\left(\frac{p \cdot N}{p^2 \cdot (W \otimes_\Gamma D)}\right) = \frac{p \cdot \mathcal{DR}(N)}{p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])} \subset \frac{W \otimes_\Gamma \mathbb{C}\Gamma[x]}{p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])}. \tag{3.1}$$

Therefore, to prove Theorem 2.7, it suffices, according to the definitions of $\mathrm{Gr}^{D}(W)$ and $\mathrm{Gr}^{\mathrm{ad}}(W)$, to show that the functor $\mathcal{DR}$ induces a bijection between the following sets:

(a) the set of D-submodules of $(W \otimes_\Gamma D)/p^2 \cdot (W \otimes_\Gamma D)$; and

(b) the set of vector subspaces in $(W \otimes_\Gamma \mathbb{C}\Gamma[x])/p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])$ which are compatible with the direct sum decomposition

$$\frac{W \otimes_\Gamma \mathbb{C}\Gamma[x]}{p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])} = \bigoplus_{\mu} \frac{W \otimes_\Gamma \mathbb{C}\Gamma[x]}{(x-\mu)^{2k_\mu} W \otimes_\Gamma \mathbb{C}\Gamma[x]}. \tag{3.2}$$

Here $p(x) = \prod_\mu (x-\mu)^{k_\mu}$ is a fixed Γ-invariant polynomial, and we have used an identification

$$\frac{(1/p) \cdot (W \otimes_\Gamma D)}{p \cdot (W \otimes_\Gamma D)} \xrightarrow{\sim} \frac{(W \otimes_\Gamma D)}{p^2 \cdot (W \otimes_\Gamma D)} \tag{3.3}$$

provided by multiplication by p (and similarly for $\mathbb{C}\Gamma[x]$).

Next, equip the vector space $(W \otimes_\Gamma \mathbb{C}\Gamma[x])/p^2 \cdot (W \otimes_\Gamma \mathbb{C}\Gamma[x])$ with a new $\mathbb{C}\Gamma[x]$-module structure by requiring that $x \in \mathbb{C}\Gamma[x]$ acts on $(W \otimes_\Gamma \mathbb{C}\Gamma[x])/((x-\mu)^{2k_\mu} \cdot W \otimes_\Gamma \mathbb{C}\Gamma[x])$ as multiplication by μ, and Γ acts as before. In other words, we replace the natural x-action by its semisimple part. Let $S(W, p^2)$ denote the result of such a semisimplification. The key observation is that a vector subspace $\mathcal{U} \subset (W \otimes_\Gamma \mathbb{C}\Gamma[x])/p^2(W \otimes_\Gamma \mathbb{C}\Gamma[x])$ is compatible with the direct sum decomposition as in (b) above if and only if $\mathcal{U}$ is a $\mathbb{C}\Gamma[x]$-submodule in $S(W, p^2)$. Thus, Theorem 2.7 reduces to the assertion that $\mathcal{DR}$ induces a bijection between

(a) the set of D-submodules of $(W \otimes_\Gamma D)/p^2 \cdot (W \otimes_\Gamma D)$; and

(b) the set of $\mathbb{C}\Gamma[x]$-submodules in $S(W, p^2)$.

Our next step is to show that the polynomial $p^2 = p(x)^2$ above can be replaced by a simpler polynomial. For any $\mu \in \mathbb{C}$, let Γ_μ be the stabilizer of μ in Γ, m_μ the order of the group Γ/Γ_μ, and p_μ the minimal Γ-semi-invariant polynomial vanishing on the orbit $\Gamma \cdot \mu$. In other words, we put

$$\begin{aligned} &\Gamma_\mu := \{1\}, \quad m_\mu := m, \quad p_\mu(x) := x^m - \mu^m, \quad \text{if } \mu \neq 0; \\ &\Gamma_\mu := \Gamma, \quad m_\mu := 1, \quad p_\mu(x) := x, \quad \text{if } \mu = 0. \end{aligned} \tag{3.4}$$

Then any Γ-invariant polynomial $p(x)$ can be factored as $p(x) = \prod_{\mu \in \mathbb{C}/\Gamma} p_\mu(x)^{s_\mu}$. This factorization induces direct sum decompositions

$$\frac{W \otimes_\Gamma D}{p^2 \cdot (W \otimes_\Gamma D)} = \bigoplus_{\mu \in \mathbb{C}/\Gamma} \frac{W \otimes_\Gamma D}{p_\mu^{2s_\mu} \cdot (W \otimes_\Gamma D)}; \qquad S(W, p^2) = \bigoplus_{\mu \in \mathbb{C}/\Gamma} S(W, p_\mu^{2s_\mu}). \tag{3.5}$$

The following lemma is clear.

Lemma 3.1. For any D-submodule $N \subset \bigoplus_{\mu\in\mathbb{C}/\Gamma}(W\otimes_\Gamma D/p_\mu^{2s_\mu}\cdot(W\otimes_\Gamma D))$, there is a direct sum decomposition

$$N = \bigoplus_{\mu\in\mathbb{C}/\Gamma}\left(N \cap \frac{W\otimes_\Gamma D}{p_\mu^{2s_\mu}\cdot(W\otimes_\Gamma D)}\right). \tag{3.6}$$

□

Due to the above lemma, we may (and will) assume, without any loss of generality, that $p(x) = p_\mu(x)^{2s_\mu}$, for some fixed $\mu \in \mathbb{C}$ and some $s_\mu = 1, 2, \ldots$. Further, we have $S(W, p_\mu^{2s_\mu}) \simeq S(W, p_\mu)^{\oplus 2s_\mu}$, and this space is, in effect, a module over the quotient algebra $\mathbb{C}\Gamma[x]/p_\mu\mathbb{C}\Gamma[x]$. The set of submodules in $S(W, p_\mu^{2s_\mu})$ may be therefore described by the following result, which is proved by a straightforward computation.

Lemma 3.2. (a) The correspondence $\mathcal{U} \mapsto \mathcal{U}\otimes_{\Gamma_\mu}(\mathbb{C}\Gamma[x]/(x-\mu)\mathbb{C}\Gamma[x])$ establishes a Morita equivalence between the category $\mathbf{Rep}(\Gamma_\mu)$ of finite-dimensional representations of Γ_μ and the category of finite-dimensional $\mathbb{C}\Gamma[x]/p_\mu\mathbb{C}\Gamma[x]$-modules.

(b) The Γ_μ-module $\mathcal{U}(W, p_\mu^{2s_\mu})$ corresponding to $S(W, p_\mu^{2s_\mu})$ via this equivalence is equal to $W^{\oplus 2s_\mu}$, viewed as a vector space ($=$ module over $\Gamma_\mu = \{1\}$) if $\mu \neq 0$ and as a Γ-module $W \otimes_{\mathbb{C}} (\mathbb{C}[x]/x^{2s_0}\mathbb{C}[x])$ if $\mu = 0$. □

Thus, to prove Theorem 2.7, we have to establish a bijection between the following sets:

(a) the set of all D-submodules of $W\otimes_\Gamma D/p_\mu^{2s_\mu}\cdot(W\otimes_\Gamma D)$; and

(b) the set of all Γ_μ-submodules of $\mathcal{U}(W, p_\mu^{2s_\mu})$.

To that end, we introduce the following definition.

Definition 3.3. Denote by $\mathrm{mod}_{\Gamma\cdot\mu}(D)$ the category of all finitely generated D-modules $\mathcal{M}$ such that $p_\mu(x)$ acts locally nilpotently on $\mathcal{M}$.

If $\mathcal{M}$ is an object in $\mathrm{mod}_{\Gamma\cdot\mu}(D)$, then

$$K_\mu(\mathcal{M}) := \mathbf{Ker}(x-\mu) \subset \mathcal{M} \tag{3.7}$$

is a Γ_μ-module. Moreover, it is clear that the assignment $\mathcal{M} \mapsto K_\mu(\mathcal{M})$ gives a functor $K_\mu : \mathrm{mod}_{\Gamma\cdot\mu}(D) \to \mathbf{Rep}(\Gamma_\mu)$. Further, consider the induction functor

$$I_\mu : \mathbf{Rep}\,(\Gamma_\mu) \longrightarrow \mathrm{mod}_{\Gamma\cdot\mu}(D), \qquad \mathcal{U} \longmapsto \mathcal{U}\otimes_{\mathbb{C}\Gamma_\mu[x]} D, \tag{3.8}$$

where $\mathbb{C}\Gamma_\mu[x] = \mathbb{C}\Gamma_\mu \otimes \mathbb{C}[x]$ and where the $\mathbb{C}\Gamma_\mu[x]$-module structure on $\mathcal{U} \in \mathbf{Rep}(\Gamma_\mu)$ is given by the standard action of Γ_μ and the action of x by the μ-multiplication.

The following theorem is a deformed (and equivariant) analogue of a well-known result of Kashiwara saying that any $\mathcal{D}$-module concentrated at a point is the $\mathcal{D}$-module direct image of a vector space (= $\mathcal{D}$-module on that point).

Theorem 3.4 (Kashiwara theorem)**.** Assume that the element $\tau \in \mathbb{C}\Gamma$, involved in the definition of D, is generic in the sense of Definition 2.3. Then the functors K_μ and I_μ give mutually inverse equivalences between the categories $\mathrm{mod}_{\Gamma\cdot\mu}(D)$ and $\mathbf{Rep}(\Gamma_\mu)$. □

Before we prove this theorem, we record a few consequences of the condition: τ *is generic*. For any $k = 1, 2, \dots$ and any integers $0 \le a \le b$, we define elements $\tau^{(k)}$, $\tau_{[a,b]} \in \mathbb{C}\Gamma$ by the equations

$$y^k \cdot \tau = \tau^{(k)} \cdot y^k, \qquad \tau_{[a,b]} = \sum_{k=a}^{b} \tau^{(k)}. \tag{3.9}$$

The definition yields

$$y \cdot \tau_{[a,b]} = \tau_{[a+1,b+1]} \cdot y, \qquad x \cdot \tau_{[a,b]} = \tau_{[a-1,b-1]} \cdot x. \tag{3.10}$$

Lemma 3.5. (a) The element τ is generic if and only if for all $a \le b$, the element $\tau_{[a,b]} \in \mathbb{C}\Gamma$ is invertible. Furthermore, in this case, for any $a \in \mathbb{Z}$, the element $\tau_{[a,a+m-1]}$ acts by a constant (independent of a) in any representation of Γ.

(b) If τ is generic, then for any $\mu \in \mathbb{C}$ and all $a \le b$, the element $\sum_{i=a}^{b} \tau_{[i,i+m_\mu-1]} \in \mathbb{C}\Gamma$ is invertible.

(c) The following identity holds: $[y, p_\mu(x)] = (\tau_{[0,m_\mu-1]}/m_\mu) \cdot p'_\mu(x)$. □

Proof. To prove (a), recall (cf., e.g., [7]) that McKay correspondence associates to the cyclic group $\mathbb{Z}/m\mathbb{Z} = \boldsymbol{\mu}_m$, the affine Dynkin graph $\widetilde{A}_{m-1}$. Using an explicit expression for the roots, it is easy to deduce assertion (a) of the lemma.

To prove (b), note that if $\mu \neq 0$, then $m_\mu = m$ and the sum in question equals $|\tau| \cdot (b - a + 1)$, where $|\tau|$ is the constant of part (a). Hence, this sum is invertible. If $\mu = 0$, then $m_\mu = 1$ and we have $\sum_{i=a}^{b} \tau_{[i,i+m_\mu-1]} = \tau_{[a,b]}$. Hence, this element is also invertible.

Part (c) is proved by a direct computation. ■

Proof of Theorem 3.4. It follows from the definitions of K_μ and I_μ that the functor K_μ is right adjoint to I_μ. This gives canonical adjunction morphisms $\mathcal{U} \to K_\mu(I_\mu(\mathcal{U}))$ and $I_\mu(K_\mu(\mathcal{M})) \to \mathcal{M}$ for any $\mathcal{M} \in \mathrm{Ob}(\mathrm{mod}_{\Gamma\cdot\mu} D)$ and $\mathcal{U} \in \mathrm{Ob}(\mathbf{Rep}\, \Gamma_\mu)$.

To show that $I_\mu(K_\mu(\mathcal{M})) \to \mathcal{M}$ is an isomorphism, set $\mathcal{M}_k := \mathbf{Ker}\, p_\mu(x)^{k+1} \subset \mathcal{M}$ and write $M_k := \mathcal{M}_k/\mathcal{M}_{k-1}$, for short. On $\oplus_k M_k$, we have the following structure.

First, it is clear that the increasing filtration $\{\mathcal{M}_k\}_{k=0,1,\ldots}$ is stable under the action of the subalgebra $\Gamma[x] \subset D$; hence, each M_k is a $\Gamma[x]$-module. Further, multiplication by $p_\mu(x)$ takes $\mathcal{M}_k$ to $\mathcal{M}_{k-1}$, and thus induces a map $p : M_k \to M_{k-1}$. Moreover, the action of y moves $\mathcal{M}_k$ to $\mathcal{M}_{k+1}$, and thus induces a map $M_k \to M_{k+1}$. Finally, it is clear that the map p is an embedding of $\Gamma[x]$-modules, while y is a morphism of $\mathbb{C}[x]$-modules, and $p'_\mu(x) : M_k \to M_k$ is an isomorphism (because $p_\mu(x)$ and $p'_\mu(x)$ are coprime). We prove, by induction in k, that

$$\begin{aligned}
(p \cdot y)|_{M_k} &= -\sum_{i=0}^{k} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu(x), \\
(y \cdot p)|_{M_{k+1}} &= -\sum_{i=1}^{k+1} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu(x).
\end{aligned} \tag{3.11}$$

The base of induction ($k = -1$) is clear. Assume that we have verified (3.11) for $k-1$. Then for any $a \in M_k$, applying Lemma 3.5(c) and the induction hypothesis, we get

$$\begin{aligned}
p \cdot y \cdot a &= y \cdot p \cdot a - [y, p] \cdot a \\
&= -\sum_{i=1}^{k} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu a - \frac{\tau_{[0,m_\mu-1]}}{m_\mu} \cdot p'_\mu a \\
&= -\sum_{i=0}^{k} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu a.
\end{aligned} \tag{3.12}$$

Note that since τ is generic, it follows from Lemma 3.5(b) that the map $p \cdot y : M_k \to M_k$ is a bijection. On the other hand, p is injective by definition. Hence, y gives an isomorphism $M_k \xrightarrow{\sim} M_{k+1}$. It follows that, for any $b \in M_{k+1}$, there exists an $a \in M_k$ such that $b = y \cdot a$. Applying Lemma 3.5(b), we calculate

$$\begin{aligned}
y \cdot p \cdot b = y \cdot p \cdot y \cdot a = -y \cdot \sum_{i=0}^{k} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu \cdot a &= -\sum_{i=0}^{k} \frac{\tau_{[i+1,i+m_\mu]}}{m_\mu} \cdot y \cdot p'_\mu \cdot a \\
= -\sum_{i=1}^{k+1} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu \cdot y \cdot a = -\sum_{i=1}^{k+1} \frac{\tau_{[i,i+m_\mu-1]}}{m_\mu} \cdot p'_\mu \cdot b.&
\end{aligned} \tag{3.13}$$

(In the third equality, we use the fact that the operators $y : M_k \to M_{k+1}$ and $p'_\mu(x) : M_k \to M_k$ commute, because their commutator on $\mathcal{M}$ takes $\mathcal{M}_k$ to $\mathcal{M}_k$, hence induces the zero map $M_k \to M_{k+1}$.) Thus, we have proved (3.11) for any k and, moreover, we have shown along the way that for any k, the map $y : M_k \to M_{k+1}$ is an isomorphism. This means that the action of the subalgebra $\mathbb{C}[y] \subset D$ gives an isomorphism $\mathcal{M} \cong \mathcal{M}_0 \otimes \mathbb{C}[y]$. Further, we have $\mathcal{M}_0 = \mathbf{Ker}\, p_\mu(x)$ and $K_\mu(\mathcal{M}) = \mathbf{Ker}(x-\mu)$. We use the equalities

$$\mathbf{Ker}\, p_\mu(x) = \bigoplus_{\gamma\in\Gamma/\Gamma_\mu} \mathbf{Ker}\left(\gamma(x-\mu)\gamma^{-1}\right) = \bigoplus_{\gamma\in\Gamma/\Gamma_\mu} \left(\mathbf{Ker}(x-\mu)\right)\gamma^{-1} = \mathbf{Ker}(x-\mu) \otimes_{\mathbb{C}[\Gamma_\mu]} \mathbb{C}\Gamma \tag{3.14}$$

to conclude that $I_\mu(K_\mu(\mathcal{M})) \to \mathcal{M}$ is an isomorphism.

To show that the canonical morphism $f : \mathcal{U} \to K_\mu(I_\mu(\mathcal{U}))$ is an isomorphism, first note that it is clearly injective. Hence, we have an exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow K_\mu\big(I_\mu(\mathcal{U})\big) \longrightarrow \mathcal{U}' \longrightarrow 0, \quad \text{where } \mathcal{U}' := \mathbf{Coker}(f). \tag{3.15}$$

On the other hand, the functor I_μ is exact since D is a flat $\Gamma_\mu[x]$-module. Hence, applying the functor $I_\mu(-)$, we obtain an exact sequence

$$0 \longrightarrow I_\mu(\mathcal{U}) \xrightarrow{\ \alpha\ } I_\mu\big(K_\mu\big(I_\mu(\mathcal{U})\big)\big) \longrightarrow I_\mu(\mathcal{U}') \longrightarrow 0. \tag{3.16}$$

The argument of the first part of the proof, applied to the D-module $I_\mu(\mathcal{U})$, shows that the morphism α above is an isomorphism. Hence, $I_\mu(\mathcal{U}') = 0$. But this clearly yields $\mathcal{U}' = 0$. Thus, the map $\mathcal{U} \to K_\mu(I_\mu(\mathcal{U}))$ is an isomorphism, and Theorem 3.4 follows. ■

End of proof of Theorem 2.7. The de Rham functor $\mathcal{DR}$, restricted to the set of submodules in $(W \otimes_\Gamma D)/p_\mu^{2s_\mu} \cdot (W \otimes_\Gamma D)$, can be factored as a composition of the equivalence $K_\mu : \mathrm{mod}_{\Gamma\cdot\mu} D \to \mathbf{Rep}\,\Gamma_\mu$ and the Morita equivalence $\mathbf{Rep}\,\Gamma_\mu \to \mathrm{mod}(\mathbb{C}\Gamma[x]/p_\mu \cdot \mathbb{C}\Gamma[x])$ of Lemma 3.2(b). Hence, it is an equivalence as well. By a straightforward (but a bit tedious) computation, we deduce that

$$K_\mu\big((W \otimes_\Gamma D)/p_\mu^{2s_\mu} \cdot (W \otimes_\Gamma D)\big) \simeq \mathcal{U}(W, p_\mu^{2s_\mu}), \tag{3.17}$$

where $\mathcal{U}(W, p_\mu^{2s_\mu})$ is given by Lemma 3.2(b). This implies Theorem 2.7, as we have seen in the first half of this section.

Finally, we can check that the map Diff_τ, defined just above the statement of Theorem 2.7, is in effect the inverse bijection $\mathrm{Gr}^{\mathrm{ad}}(W) \to \mathrm{Gr}^D(W)$. ■

4 D-module Grassmannian and sheaves on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$

Recall the bigraded algebra Q defined in (2.7). Let $\mathrm{gr}^2(Q)$ be the category of finitely generated bigraded right Q-modules $M = \oplus M_{i,j}$. Let $\mathrm{tor}^2(Q)$ denote its Serre subcategory formed by all modules M such that there exists a pair (i_0, j_0), such that for any $i > i_0$ and $j > j_0$, we have $M_{i,j} = 0$. Consider the quotient category $\mathrm{qgr}^2(Q) = \mathrm{gr}^2(Q)/\mathrm{tor}^2(Q)$. The category $\mathrm{qgr}^2(Q)$ will be viewed as the category of coherent sheaves on a noncommutative scheme $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ (see Appendix A for details). Thus, *by definition*, we put

$$\mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right) := \mathrm{qgr}^2(Q), \tag{4.1}$$

and we write $\pi : \mathrm{gr}^2(Q) \to \mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1)$ for the canonical projection functor.

The isomorphism $D \cong Q/((z-1)Q + (w-1)Q)$ gives rise to a "restriction" functor

$$\begin{aligned} j^* &: \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right) \longrightarrow \mathrm{mod}(D), \\ E = \pi(M) &\longmapsto \varinjlim_{k,l} M_{k,l} \cong M/\big((z-1)M + (w-1)M\big), \end{aligned} \tag{4.2}$$

where the direct limit is taken with respect to the maps $M_{k,l} \to M_{k+1,l}$ and $M_{k,l} \to M_{k,l+1}$, induced by multiplication by z and w, respectively.

There are canonical isomorphisms

$$Q/zQ \cong \big(\mathbb{C}[x] \otimes \mathbb{C}[y, w]\big)\#\Gamma, \qquad Q/wQ \cong \big(\mathbb{C}[x, z] \otimes \mathbb{C}[y]\big)\#\Gamma. \tag{4.3}$$

Thus, we obtain the following equivalences of categories $\mathrm{qgr}^2(Q/zQ) \simeq \mathrm{qgr}(\mathbb{C}[xy, xw]\#\Gamma)$ and $\mathrm{qgr}^2(Q/wQ) \simeq \mathrm{qgr}(\mathbb{C}[xy, yz]\#\Gamma)$ (see Corollary A.8). The two categories on the right can be viewed as the categories of Γ-equivariant coherent sheaves on the ordinary projective line $\mathbb{P}^1$. We denote the corresponding copies of $\mathbb{P}^1$ by $\mathbb{P}^1_z$ and $\mathbb{P}^1_w$, respectively. We have the corresponding push-forward and pullback functors

$$\begin{aligned} (i_z)_* &: \mathrm{coh}\left(\mathbb{P}^1_z\right) \longrightarrow \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right), \qquad i_z^* : \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right) \longrightarrow \mathrm{coh}\left(\mathbb{P}^1_z\right), \\ (i_w)_* &: \mathrm{coh}\left(\mathbb{P}^1_w\right) \longrightarrow \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right), \qquad i_w^* : \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right) \longrightarrow \mathrm{coh}\left(\mathbb{P}^1_w\right). \end{aligned} \tag{4.4}$$

Let $L^1 i^*$ stand for the first derive functor.

Given a D-module $\mathcal{M}$, we let its *support* be the support of $\mathcal{M}$, viewed as a module over the subalgebra $\mathbb{C}[x] \subset D$ (more precisely, the union of supports of all $\mathbb{C}[x]$-finitely generated submodules in $\mathcal{M}$).

Definition 4.1. (i) Let $\mathrm{Quot}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W)$ be the category of all surjections $W \otimes_\Gamma \mathcal{O} \twoheadrightarrow F$ in the category $\mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1)$ such that

$$i_z^* F = L^1 i_z^* F = 0, \qquad L^1 i_w^* F = 0. \tag{4.5}$$

(ii) Let $\mathrm{Quot}^D(W)$ be the category of all D-module surjections $W \otimes_\Gamma D \twoheadrightarrow \mathcal{M}$ such that $\mathcal{M}$ has zero-dimensional support.

Theorem 4.2. The functor j^* takes any object of the category $\mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ to an object of $\mathrm{Quot}^{D}(W)$ and, moreover, gives an equivalence $\mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \xrightarrow{\sim} \mathrm{Quot}^{D}(W)$. □

Proof. First, note that j^* is exact, being a direct limit functor. Thus, to show that j^* takes $\mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ to $\mathrm{Quot}^{D}(W)$, it suffices to show that for any object of $\mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ of the form $W\otimes\mathcal{O} \twoheadrightarrow F$, the D-module j^*F has zero-dimensional support. Indeed, let $F = \pi(M)$ where $M = \oplus M_{k,l}$. Then conditions $i_z^*F = L^1 i_z^*F = 0$ imply that z-multiplication gives an isomorphism $M_{k,l} \xrightarrow{\sim} M_{k+1,l}$ for $k, l \gg 0$. Since M is finitely generated, we can choose k_0 and l_0 such that $\oplus_{k\geq k_0,\, l\geq l_0} M_{k,l}$ is generated by M_{k_0,l_0} and such that $z : M_{k,l} \to M_{k+1,l}$ is an isomorphism for all $k \geq k_0$, $l \geq l_0$. Let p be the characteristic polynomial of the operator $z^{-1}x : M_{k_0,l_0} \to M_{k_0,l_0}$. Then it is easy to see that $p(x)$ acts locally nilpotently on $j^*F = \varinjlim M_{k,l}$, and hence j^* indeed defines a functor $j^* : \mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \to \mathrm{Quot}^{D}(W)$.

The assertion that j^* is an equivalence will be proved by constructing a quasi-inverse functor $j_{!*}$. To that end, let $\psi : W \otimes_\Gamma D \twoheadrightarrow \mathcal{M}$ be an object of $\mathrm{Quot}^{D}(W)$. Since $\mathcal{M}$ has zero-dimensional support, there exists a Γ-invariant polynomial $p(x)$ that acts by zero on the subspace $\psi(W \otimes_\Gamma \mathbb{C}\Gamma) \subset \mathcal{M}$. It is clear that we have $\psi(W \otimes p(x)) = 0$; hence,

$$\psi\big(p \cdot (W \otimes_\Gamma D)\big) = 0. \tag{4.6}$$

Let $D_{k,l}$ be the natural increasing bifiltration of D (induced by the bigrading of Q) and

$$\mathcal{M}_{k,l} = \psi\big(W \otimes_\Gamma D_{k,l}\big) \subset \mathcal{M}, \tag{4.7}$$

the induced bifiltration of $\mathcal{M}$. It follows from (4.6) that this bifiltration stabilizes with respect to the first index when $k \geq d = \deg p(x)$, that is, we have

$$\mathcal{M}_{k,l} = \psi\big(W \otimes_\Gamma D_{k,l}\big) = \psi\big(W \otimes_\Gamma D_{d,l}\big) = \mathcal{M}_{d,l} \subset \mathcal{M}, \quad \text{for } k \geq d \text{ and all } l. \tag{4.8}$$

It follows from the definition that the bifiltration $\mathcal{M}_{k,l}$ is compatible with the bifiltration on D. Moreover, it is clearly increasing, finitely generated and exhaustive (because ψ is surjective). Hence, $M = \oplus_{k,l}\mathcal{M}_{k,l}$ is a finitely generated Q-module, where the action on M of x-generators and y-generators of Q is given by the x and y multiplication maps $\mathcal{M}_{k,l} \to \mathcal{M}_{k+1,l}$ and $\mathcal{M}_{k,l} \to \mathcal{M}_{k,l+1}$, respectively, and the action of z and w generators is given by the natural embeddings $\mathcal{M}_{k,l} \hookrightarrow \mathcal{M}_{k+1,l}$ and $\mathcal{M}_{k,l} \hookrightarrow \mathcal{M}_{k,l+1}$, respectively.

Consider $F = \pi(M)$, a coherent sheaf on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$. It follows from the definition of F that the z-multiplication map $F \to F(1,0)$ is an isomorphism (since $\mathcal{M}_{k,l} = \mathcal{M}_{k+1,l}$ for $k \geq d$), hence $i_z^*F = L^1 i_z^*F = 0$. On the other hand, the w-multiplication map $F \to F(0,1)$ is an embedding (because $\mathcal{M}_{k,l} \subset \mathcal{M}_{k,l+1}$), hence $L^1 i_w^*F = 0$. Finally note that the map

ψ is compatible with the bifiltrations on $W \otimes_\Gamma D$ and $\mathcal{M}$, hence it gives rise to a map $\tilde{\psi} : W \otimes_\Gamma \mathcal{O} \to F$ of coherent sheaves on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$. Moreover, this map is surjective by definition, hence it gives an object of $\mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$. Finally, it is easy to show that this way we obtain a functor $j_{!*} : \mathrm{Quot}^D(W) \to \mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$.

We show that $j_{!*}$ and j^* are quasi-inverse. Let $W \otimes_\Gamma \mathcal{O} \xrightarrow{\psi} F$ be an object of the category $\mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ and let $M = \oplus M_{k,l}$, where $M_{k,l} = H^0(\mathbb{P}^1 \times_\tau \mathbb{P}^1, F(k,l))$. Then M is a bigraded Q-module and it is clear that $\pi(M) = F$. Note that we have exact sequences

$$\begin{aligned}
&0 \longrightarrow (i_z)_* L^1 i_z^* F(k+1,l) \longrightarrow F(k,l) \longrightarrow F(k+1,l) \longrightarrow (i_z)_* i_z^* F(k+1,l) \longrightarrow 0,\\
&0 \longrightarrow (i_w)_* L^1 i_w^* F(k,l+1) \longrightarrow F(k,l) \longrightarrow F(k,l+1) \longrightarrow (i_w)_* i_w^* F(k,l+1) \longrightarrow 0.
\end{aligned} \tag{4.9}$$

Moreover, applying (4.5), we get

$$\begin{aligned}
&L^1 i_z^* F(k+1,l) \cong \big(L^1 i_z^* F\big)(l) = 0, \qquad i_z^* F(k+1,l) \cong \big(i_z^* F\big)(l) = 0,\\
&L^1 i_w^* F(k,l+1) \cong \big(L^1 i_w^* F\big)(k) = 0.
\end{aligned} \tag{4.10}$$

Combining these isomorphisms with the above exact sequences and with the definition of $M_{k,l}$, we see that the maps $M_{k,l} \xrightarrow{z} M_{k+1,l}$ and $M_{k,l} \xrightarrow{w} M_{k,l+1}$ are an isomorphism and an embedding, respectively. Therefore, $j^*F = \bigcup_l M_{k,l}$ for any k. On the other hand, we have $D_{k,l} = H^0(\mathbb{P}^1 \times_\tau \mathbb{P}^1, \mathcal{O}(k,l))$ by definition, and it is clear that the map $j^*\psi$ sends $W \otimes_\Gamma D_{k,l}$ to $M_{k,l}$ and coincides there with the map $H^0(\psi(k,l))$. Thus, to show that $j_{!*}(j^*F) \cong F$, it suffices to show that this map is surjective for all k and l sufficiently large. The latter is nothing but the definition of the map ψ being a surjection in the category $\mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1)$.

Now assume that $W \otimes_\Gamma D \to \mathcal{M}$ is an object of $\mathrm{Quot}^D(W)$. Then by definition of $j_{!*}$, we have $\mathcal{M}_{k+1,l} = \mathcal{M}_{k,l}$ for $k \geq d$ and all l, and when $k \geq d$ is fixed, the filtration $\mathcal{M}_{k,l}$ of $\mathcal{M}$ is exhaustive. Hence, $\mathcal{M} = \varinjlim \mathcal{M}_{k,l}$, that is, $j^* j_{!*}(\mathcal{M}) = \mathcal{M}$. ■

Now we give a more rigorous version of Definition 2.8. Let E be a coherent sheaf on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ such that $i_z^* E \cong W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_z}$. Recall that ϵ denotes a fixed primitive character $\Gamma = \boldsymbol{\mu}_m \hookrightarrow \mathbb{C}^*$.

Definition 4.3. (i) We say that the sheaf E is *trivialized in a neighborhood of* $\mathbb{P}^1_z$ if we are given embeddings

$$\big(W \otimes \epsilon^{-n}\big) \otimes_\Gamma \mathcal{O}(-n,0) \overset{\phi}{\hookrightarrow} E \overset{\psi}{\hookrightarrow} \big(W \otimes \epsilon^{n}\big) \otimes_\Gamma \mathcal{O}(n,0), \tag{4.11}$$

such that the composite

$$\begin{aligned}\psi \circ \phi &\in \operatorname{Hom}\left(\left(W \otimes \epsilon^{-n}\right) \otimes_\Gamma \mathcal{O}(-n,0), \left(W \otimes \epsilon^{n}\right) \otimes_\Gamma \mathcal{O}(n,0)\right)\\ &\cong \operatorname{Hom}_\Gamma\left(W, \left(W \otimes \epsilon^{2n}\right) \otimes_\Gamma Q_{2n,0}\right)\end{aligned} \tag{4.12}$$

equals multiplication by $P(x,z)^2$, where $P(x,z) \in \mathbb{C}[x,z]$ is a Γ-semi-invariant homogeneous polynomial of degree n such that $P(1,0) = 1$.

(ii) We call two trivializations (ϕ,ψ) and (ϕ',ψ') of the sheaf E *equivalent* if there exists a pair of Γ-semi-invariant homogeneous polynomials $\tilde{q}(x,z)$ and $\tilde{q}'(x,z)$ such that $\tilde{q}(1,0) = \tilde{q}'(1,0) = 1$, and the following diagram commutes:

$$\begin{tikzcd}
\left(W \otimes \epsilon^{-n}\right) \otimes_\Gamma \mathcal{O}(-n,0) \arrow[r, "\phi"'] & E \arrow[r, "\psi"'] \arrow[dd, equal] & \left(W \otimes \epsilon^{n}\right) \otimes_\Gamma \mathcal{O}(n,0) \arrow[d, "\tilde{q}"'] \\
\left(W \otimes \epsilon^{-n''}\right) \otimes_\Gamma \mathcal{O}(-n'',0) \arrow[u, "\tilde{q}"'] \arrow[d, "\tilde{q}'"'] & & \left(W \otimes \epsilon^{n''}\right) \otimes_\Gamma \mathcal{O}(n'',0) \\
\left(W \otimes \epsilon^{-n'}\right) \otimes_\Gamma \mathcal{O}(-n',0) \arrow[r, "\phi'"] & E \arrow[r, "\psi'"] & \left(W \otimes \epsilon^{n'}\right) \otimes_\Gamma \mathcal{O}(n',0). \arrow[u, "\tilde{q}'"']
\end{tikzcd} \tag{4.13}$$

Remark 4.4. We can always replace a trivialization by an equivalent one with $n \equiv 0 \bmod m$, thus getting rid of ϵ^n and ϵ^{-n} factors in the definition and making the polynomial $P(x,z)$ Γ-invariant.

Definition 4.5. Let $\mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W)$ be the set of all equivalence classes of trivializations in a neighborhood of $\mathbb{P}^1_z$ of coherent sheaves E on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ such that $i_z^* E \cong W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_z}$.

Proof of Theorem 2.9 (bijection between $\mathrm{Gr}^D(W)$ and $\mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W)$). For any Γ-invariant polynomial $p(x) = \sum_{k=0}^d a_k x^k$, let $\mathrm{Gr}_p^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W) \subset \mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W)$ denote the set of all sheaves admitting a trivialization (ϕ,ψ) with $\psi \circ \phi = P(x,z)^2$, where $P(x,z)$ is the homogenization of $p(x)$, that is, $P(x,z) = \sum_{k=0}^d a_k x^k z^{d-k}$. Then we have

$$\mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W) = \bigcup_{p(x)} \mathrm{Gr}_p^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W). \tag{4.14}$$

We will show that the functor j^* induces a bijection between $\mathrm{Gr}_p^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W) \subset \mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W)$ and the subset $\mathrm{Gr}_p^D(W) \subset \mathrm{Gr}^D(W)$ formed by all D-submodules (or, equivalently, quotient modules) of $(1/p) \cdot (W \otimes_\Gamma D)/p \cdot (W \otimes_\Gamma D)$.

Let $\mathrm{Quot}_p^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \subset \mathrm{Quot}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ be the subset formed by surjections $W\otimes_\Gamma \mathcal{O} \twoheadrightarrow F$, which send the image of the map $W\otimes_\Gamma \mathcal{O}(-2n,0) \xrightarrow{P(x,z)^2} W\otimes_\Gamma \mathcal{O}$ to zero in F. We may identify the set $\mathrm{Gr}_p^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ with $\mathrm{Quot}_p^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ via the assignment

$$\begin{aligned}&\{W\otimes_\Gamma \mathcal{O}(-n,0) \xrightarrow{\phi} E \xrightarrow{\psi} W\otimes_\Gamma \mathcal{O}(n,0)\}\\ &\qquad\longmapsto \{W\otimes_\Gamma \mathcal{O} \longrightarrow \mathbf{Coker}\left(E(-n,0) \xrightarrow{\psi(-n,0)} W\otimes_\Gamma \mathcal{O}\right)\}.\end{aligned} \tag{4.15}$$

Hence, Theorem 4.2 implies that the functor j^* provides an identification of the set $\mathrm{Quot}_p^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ with the subset $\mathrm{Quot}_p^D(W) \subset \mathrm{Quot}^D(W)$ formed by all surjections $W\otimes_\Gamma D \twoheadrightarrow \mathcal{M}$ which send $p^2\cdot(W\otimes_\Gamma D)$ to zero (in $\mathcal{M}$). On the other hand, to any object

$$p\cdot(W\otimes_\Gamma D) \xrightarrow{\phi} N \xrightarrow{\psi} \frac{1}{p}\cdot(W\otimes_\Gamma D) \tag{4.16}$$

in $\mathrm{Gr}_p^D(W)$, we associate the quotient

$$W\otimes_\Gamma D \longrightarrow\!\!\!\!\!\rightarrow \mathbf{Coker}\left(p\cdot N \xrightarrow{p\psi} W\otimes_\Gamma D\right). \tag{4.17}$$

This yields an identification of $\mathrm{Quot}_p^D(W)$ with $\mathrm{Gr}_p^D(W)$. Therefore, we get a bijection $\mathrm{Gr}_p^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \xrightarrow{\sim} \mathrm{Gr}_p^D(W)$.

Note that, for any polynomial $p(x)$ dividing $q(x)$, the map j^* commutes with the natural embeddings $\mathrm{Gr}_p^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \hookrightarrow \mathrm{Gr}_q^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W)$ and $\mathrm{Gr}_p^D(W) \hookrightarrow \mathrm{Gr}_q^D(W)$. The assertion of Theorem 2.9 follows. ∎

Finally, we prove Theorem 2.11. Recall that the pull back functor i_w^* takes any sheaf trivialized in a neighborhood of $\mathbb{P}_z^1$ to a sheaf on $\mathbb{P}_w^1$ trivialized in a neighborhood of the point $P = \mathbb{P}_z^1 \cap \mathbb{P}_w^1$, which is the same as a Γ-equivariant sheaf on the ordinary projective line $\mathbb{P}^1$ trivialized in a Zariski neighborhood of the infinity. Thus, i_w^* induces a map $\mathrm{Gr}^{\mathbb{P}^1\times_\tau\mathbb{P}^1}(W) \to \mathrm{Gr}^{\mathrm{aff}}(W)$.

Proof of Theorem 2.11. The claim easily follows from the definitions of the maps involved. In more details, let $N \subset p\cdot(W\otimes_\Gamma D)$ be a fat D-submodule and $M = p\cdot(W\otimes_\Gamma D)$. Then it follows that $j_{!*}(N) = \pi(\oplus N_{k,l})$, where $N_{k,l} = N\cap(W\otimes_\Gamma D_{k,l})$ and $j_{!*}$ is the quasi-inverse to j^* defined in the proof of Theorem 4.2. Hence, $i_w^*(j_{!*}(N)) = \pi(\oplus N_{k,l}/N_{k,l-1})$ and the restriction of this sheaf to $\mathbb{P}_w^1 \setminus \{\infty\}$ is isomorphic to $\varinjlim_k N_{k,l}/N_{k,l-1} = \mathrm{Symb}(N)$. This shows that diagram (2.14) is commutative. Further, the maps in the upper row of the diagram are bijections by Theorems 2.7 and 2.9. Further, the maps $\mathcal{DR}$ and j^* are G_W-equivariant, hence Diff_τ and $j_{!*}$ are G_W-equivariant as well. And finally, the map Symb is $GL_{\mathbb{C}\Gamma(x)}(W\otimes_\Gamma \mathbb{C}\Gamma(x))$-equivariant, hence the maps i_w^* and σ are $GL_{\mathbb{C}\Gamma(x)}(W\otimes_\Gamma \mathbb{C}\Gamma(x))$-equivariant as well. ∎

5 Monads and quiver varieties

The two lines $\mathbb{P}^1_z$ and $\mathbb{P}^1_w$ in $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ intersect at the point P, corresponding to the quotient algebra

$$Q/(zQ + wQ) \cong \big(\mathbb{C}[x] \otimes \mathbb{C}[y]\big)\#\Gamma. \tag{5.1}$$

Moreover, the category $\mathrm{qgr}^2(Q/(zQ + wQ))$ is equivalent to the category of finite dimensional Γ-modules. The point P is given on the line $\mathbb{P}^1_z$ by the equation $w = 0$ and on the line $\mathbb{P}^1_w$ by the equation $z = 0$. Let $i^z_P : \{P\} \hookrightarrow \mathbb{P}^1_w$ and $i^w_P : \{P\} \hookrightarrow \mathbb{P}^1_z$ denote the embeddings. There is a canonical isomorphism of functors

$$\left(i^z_P\right)^* \circ i^*_w \cong \left(i^w_P\right)^* \circ i^*_z : \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right) \longrightarrow \mathrm{coh}(P) = \mathbf{Rep}(\Gamma). \tag{5.2}$$

Let W, V be a pair of Γ-modules as in Definition 2.12.

Definition 5.1. A coherent sheaf E on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ is called W-framed, provided it is equipped with two isomorphisms $i^*_z E \cong W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_z}$ and $i^*_w E \cong W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_w}$, which agree at the point P.

Let $\mathcal{M}_{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(V, W)$ denote the set of isomorphism classes of W-framed torsion-free sheaves E (for the definition of *torsion-free* see [2, Definition 1.1.4]) on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ such that $H^1(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1, -1)) \cong V$.

Theorem 5.2. The set $\mathcal{M}_{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(V, W)$ is in a natural bijection with the quiver variety $\mathfrak{M}^\tau_\Gamma(V, W)$. □

Sketch of the proof. The proof is essentially the same as that of [2, Section 4, Theorem 1.3.10]. So, we will skip most of the details and only mention the points that are different from [2, Section 4].

The first difference is that, in the present situation, the monad representing a framed sheaf has a form slightly different from the one used in [2]. Specifically, for any point (B_1, B_2, I, J) of the quiver variety, our monad is now given by the following complex:

$$0 \longrightarrow V \otimes_\Gamma \mathcal{O}(-1,-1) \xrightarrow{a} \begin{matrix} (V \otimes \epsilon) \otimes_\Gamma \mathcal{O}(0,-1) \\ \bigoplus \\ \left(V \otimes \epsilon^{-1}\right) \otimes_\Gamma \mathcal{O}(-1,0) \\ \bigoplus \\ W \otimes_\Gamma \mathcal{O} \end{matrix} \xrightarrow{b} V \otimes_\Gamma \mathcal{O} \longrightarrow 0, \tag{5.3}$$

$$a = \big(B_1 z - x, B_2 w - y, Jzw\big), \qquad b = \big(-\big(B_2 w - y\big), B_1 z - x, I\big). \tag{5.4}$$

Second, whenever the functor i^* (the restriction to the line at infinity in $\mathbb{P}^2_\tau$) is used in [2], it should be replaced by a pair of functors i_z^*, i_w^*.

Third, [2, Lemma 4.2.12] should be replaced by the following isomorphisms:

$$\begin{aligned}
&H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,0)\big) = H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(0,-1)\big) = H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,-1)\big) = 0,\\
&H^2\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E\big) = H^2\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,0)\big)\\
&\qquad\qquad = H^2\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(0,-1)\big) = H^2\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,-1)\big) = 0,\\
&H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,0)\big) = H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(0,-1)\big) = H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,-1)\big),
\end{aligned} \tag{5.5}$$

and furthermore, there is a canonical exact sequence

$$0 \longrightarrow H^0\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E\big) \longrightarrow W \xrightarrow{f_E} V \longrightarrow H^1\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E\big) \longrightarrow 0. \tag{5.6}$$

Fourth, the Beilinson spectral sequence takes the form (see Appendix B):

$$E_1^{p,q} = \left\{ \mathbf{Ext}^q\big(\mathcal{O}(1,1), E\big) \otimes_\Gamma \mathcal{O}(-1,-1) \longrightarrow \begin{bmatrix} \Big(\mathbf{Ext}^q\big(\mathcal{O}(1,0), E\big) \otimes \epsilon\Big) \otimes_\Gamma \mathcal{O}(-1,0) \\ \bigoplus \\ \Big(\mathbf{Ext}^q\big(\mathcal{O}(0,1), E\big) \otimes \epsilon^{-1}\Big) \otimes_\Gamma \mathcal{O}(0,-1) \end{bmatrix} \longrightarrow \mathbf{Ext}^q(\mathcal{O}, E) \otimes_\Gamma \mathcal{O} \right\}. \tag{5.7}$$

We apply this spectral sequence to obtain a monadic description of an arbitrary framed coherent sheaf E. Using (5.5), we see that, for any W-framed sheaf E such that $H^1(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,-1)) \cong V$, the spectral sequence takes the form

$$\begin{tikzcd}[column sep=small]
V \otimes_\Gamma \mathcal{O}(-1,-1) \arrow[r] \arrow[drr, dashed, "d_2^{-2,1}"'] & (V \otimes \epsilon) \otimes_\Gamma \mathcal{O}(-1,0) \bigoplus (V \otimes \epsilon^{-1}) \otimes_\Gamma \mathcal{O}(0,-1) \arrow[r] & H^1(E) \otimes_\Gamma \mathcal{O} \\
& & H^0(E) \otimes_\Gamma \mathcal{O}.
\end{tikzcd} \tag{5.8}$$

Further, we show that we can replace $H^1(E) = \mathbf{Coker}\, f_E$ and $H^0(E) = \mathbf{Ker}\, f_E$ by V and W, respectively, and lift the differential $d_2^{-2,1} : E_2^{-2,1} \to E_2^{0,0}$ to a morphism $V \otimes_\Gamma \mathcal{O}(-1,-1) \to W \otimes_\Gamma \mathcal{O}$. Finally, replacing the spectral sequence with the total complex, we obtain the desired monadic description (5.3) of the sheaf E. We leave for the reader to check that the maps in (5.3) take the form (5.4) for an appropriately chosen quiver data (B_1, B_2, I, J). ■

Remark 5.3. There is an alternative way to prove Theorem 5.2. using the following tri-graded algebra:

$$S := \mathbb{C}\langle \xi, \eta, \zeta, x, z, y, w\rangle \# \Gamma / I, \tag{5.9}$$

where

$$I = \left\langle \begin{matrix} [\bullet, \zeta] = [\bullet, z] = [\bullet, w] = [\xi, x] = [\eta, y] = 0, \\ [\eta, \xi] = \tau\zeta^2,\ [y, x] = \tau z w,\ [\eta, x] = \tau\zeta z,\ [y, \xi] = \tau\zeta w, \\ \xi z = x\zeta,\ \eta w = y\zeta \end{matrix} \right\rangle. \tag{5.10}$$

Let X be the corresponding noncommutative variety (i.e., such that $\mathrm{coh}(X) = \mathrm{qgr}^3(S)$). Then we have a diagram

$$\mathbb{P}^2_\tau \xleftarrow{\ p\ } X \xrightarrow{\ q\ } \mathbb{P}^1 \times_\tau \mathbb{P}^1, \tag{5.11}$$

where the morphism q is a noncommutative analogue of the blowup of the point P on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$, and the morphism p is a noncommutative analogue of the blowup of a pair of points on the line at infinity. We can show that Fourier-Mukai type functors

$$q_* p^* : \mathrm{coh}\left(\mathbb{P}^2_\tau\right) \longrightarrow \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right), \qquad p_* q^* : \mathrm{coh}\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1\right) \longrightarrow \mathrm{coh}\left(\mathbb{P}^2_\tau\right) \tag{5.12}$$

induce mutually inverse bijections between the corresponding sets of (isomorphism classes of) W-framed torsion-free sheaves. Theorem 5.2 is now immediate from [2, Theorem 1.3.10].

We now turn to the proof of Theorem 2.13. Let E be a W-framed torsion-free coherent sheaf on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ such that $H^1(\mathbb{P}^1 \times_\tau \mathbb{P}^1, E(-1,-1)) \cong V$. Theorem 5.2 implies that E can be represented as the cohomology sheaf of monad (5.3). We consider the following maps:

$$\begin{array}{c} (V \otimes \epsilon) \otimes_\Gamma \mathcal{O}(0,-1) \\ \bigoplus \\ \left(W \otimes \epsilon^{-n}\right) \otimes_\Gamma \mathcal{O}(-n,0) \xrightarrow{\Phi} \left(V \otimes \epsilon^{-1}\right) \otimes_\Gamma \mathcal{O}(-1,0) \xrightarrow{\Psi} \left(W \otimes \epsilon^{n}\right) \otimes_\Gamma \mathcal{O}(n,0), \\ \bigoplus \\ W \otimes_\Gamma \mathcal{O} \end{array}$$

$$\Phi = \left(0, -\left(\widehat{B_1 z - x}\right) I, P(x,z)\right), \qquad \Psi = \left(-zwJ\left(\widehat{B_1 z - x}\right), 0, P(x,z)\right), \tag{5.13}$$

where $(\widehat{B_1 z - x})$ stands for the cofactor matrix (i.e., the matrix formed by the $(n-1) \times (n-1)$ minors in the matrix $B_1 z - x$, taken with appropriate sign) and $P(x,z) = \det(B_1 z - x)$. It is easy to see that $\Psi \cdot a = b \cdot \Phi = 0$. Thus, Φ and Ψ induce morphisms

$$\left(W \otimes \epsilon^{-n}\right) \otimes_\Gamma \mathcal{O}(-n,0) \xrightarrow{\phi} E \xrightarrow{\psi} \left(W \otimes \epsilon^{n}\right) \otimes_\Gamma \mathcal{O}(n,0). \tag{5.14}$$

Furthermore, it is easy to show that the composite $\psi \circ \phi : (W \otimes \epsilon^{-n}) \otimes_\Gamma \mathcal{O}(-n,0) \to (W \otimes \epsilon^{n}) \otimes_\Gamma \mathcal{O}(n,0)$ equals multiplication by $P(x,z)^2$. Thus, (ϕ, ψ) is a trivialization of E in a neighborhood of $\mathbb{P}^1_z$. Finally, it is easy to see that the trivialization of $i_w^* E \cong W \otimes_\Gamma \mathcal{O}_{\mathbb{P}^1_w}$ takes the form

$$\left(W \otimes \epsilon^{-n}\right) \otimes_\Gamma \mathcal{O}(-n) \xrightarrow{P(x,z)} W \otimes_\Gamma \mathcal{O} \xrightarrow{P(x,z)} \left(W \otimes \epsilon^{n}\right) \otimes_\Gamma \mathcal{O}(n). \tag{5.15}$$

This trivialization is equivalent to the trivial one, thus the map $i_w^* : \mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W) \to \mathrm{Gr}^{\mathrm{aff}}(W)$ takes (E, ϕ, ψ) to the base point $\mathcal{W}_0$. Thus, we obtain an embedding

$$\beta : \bigsqcup_V \mathcal{M}_{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(V, W) \hookrightarrow \left(i_w^*\right)^{-1}\left(\mathcal{W}_0\right) \subset \mathrm{Gr}^{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(W). \tag{5.16}$$

Theorem 2.13 is an immediate consequence of Theorem 5.2 and the following result.

Theorem 5.4. The map $\beta : \bigsqcup_V \mathcal{M}_{\mathbb{P}^1 \times_\tau \mathbb{P}^1}(V, W) \to (i_w^*)^{-1}(\mathcal{W}_0)$ is a bijection. □

Proof of Theorem 5.4. Let E be a coherent sheaf on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ with a trivialization (ϕ, ψ) in a neighborhood of $\mathbb{P}^1_z$. Then E has a canonical W-framing on $\mathbb{P}^1_z$ (given by restricting the trivialization). If, in addition, $i_w^*(E) = \mathcal{W}_0$, then the sheaf E acquires a canonical W-framing on $\mathbb{P}^1_w$. Moreover, the framings agree at the point P, hence we obtain a map

$$\alpha : \left(i_w^*\right)^{-1}(\mathcal{W}_0) \longrightarrow \bigsqcup_V \mathcal{M}_{\mathbb{P}^1\times_\tau\mathbb{P}^1}(V,W). \tag{5.17}$$

We now show that both $\alpha \circ \beta$ and $\beta \circ \alpha$ are identities. To prove that $\alpha \circ \beta = \mathrm{Id}$, note that the W-framings of the sheaf E on $\mathbb{P}^1_z$ and $\mathbb{P}^1_w$ induced by the trivialization (5.13) coincide with the canonical W-framings.

In order to prove $\beta \circ \alpha = \mathrm{Id}$, we need to show that any trivialization of E which gives the canonical W-framing of E on $\mathbb{P}^1_w$ is equivalent to the trivialization (5.13). Indeed, let E be the cohomology of monad (5.3) and consider an arbitrary trivialization $(W\otimes\epsilon^{-n'})\otimes_\Gamma\mathcal{O}(-n',0) \xrightarrow{\phi'} E \xrightarrow{\psi'} (W\otimes\epsilon^{n'})\otimes_\Gamma\mathcal{O}(n',0)$ of E in a neighborhood of $\mathbb{P}^1_z$. Applying diagram (5.3) to compute $\mathrm{Hom}((W \otimes \epsilon^{-n'}) \otimes_\Gamma \mathcal{O}(-n',0), E)$ and $\mathrm{Hom}(E, (W \otimes \epsilon^{n'}) \otimes_\Gamma \mathcal{O}(n',0))$, we see that the morphisms ϕ' and ψ' can be lifted to morphisms

$$\begin{array}{ccccc} & & (V\otimes\epsilon)\otimes_\Gamma\mathcal{O}(0,-1) & & \\ & & \bigoplus & & \\ \left(W\otimes\epsilon^{-n'}\right)\otimes_\Gamma\mathcal{O}(-n',0) & \xrightarrow{\Phi'} & \left(V\otimes\epsilon^{-1}\right)\otimes_\Gamma\mathcal{O}(-1,0) & \xrightarrow{\Psi'} & \left(W\otimes\epsilon^{n}\right)\otimes_\Gamma\mathcal{O}(n',0) \\ & & \bigoplus & & \\ & & W\otimes_\Gamma\mathcal{O} & & \end{array} \tag{5.18}$$

such that

$$b \cdot \Phi' = \Psi' \cdot a = 0. \tag{5.19}$$

Moreover, the lift Φ' of ϕ is unique while the lift Ψ' of ψ' is unique up to a summand of the form $\lambda \cdot b$, where $\lambda \in \mathrm{Hom}(V \otimes_\Gamma \mathcal{O}, (W \otimes \epsilon^{n'}) \otimes_\Gamma \mathcal{O}(n',0))$. Let

$$\Phi' = \left(\Phi'_1, \Phi'_2, \Phi'_3\right), \qquad \Psi' = \left(\Psi'_1, \Psi'_2, \Psi'_3\right) \tag{5.20}$$

be the components of Φ' and Ψ' with respect to the direct sum decomposition

$$(V\otimes\epsilon)\otimes_\Gamma\mathcal{O}(0,-1)\bigoplus\left(V\otimes\epsilon^{-1}\right)\otimes_\Gamma\mathcal{O}(-1,0)\bigoplus W\otimes_\Gamma\mathcal{O}. \tag{5.21}$$

Since the trivialization (ϕ', ψ') of E restricts to the canonical W-framing of $E_{\mathbb{P}^1_w}$, it follows that

$$\Phi'_3 = \Psi'_3 = P'(x,z), \tag{5.22}$$

where $P'(x,z)$ is a homogeneous polynomial of certain degree n' such that $P'(1,0) = 1$. Furthermore, the vanishing

$$\operatorname{Hom}\left(\left(W \otimes \epsilon^{-n'}\right) \otimes_\Gamma \mathcal{O}(-n',0), \left(V \otimes \epsilon^{-1}\right) \otimes_\Gamma \mathcal{O}(0,-1)\right) = 0 \tag{5.23}$$

implies that $\Phi_1' = 0$. On the other hand, using the freedom in the choice of λ, we can make $\Psi_2' = 0$. Then, (5.19) yield

$$\left(B_1 z - x\right)\Phi_2' + IP'(x,z) = 0, \qquad \Psi_1'\left(B_1 z - x\right) + P'(x,z)zwJ = 0. \tag{5.24}$$

Multiplying the first equation by $(\widehat{B_1 z - x})$ on the left and the second by $(\widehat{B_1 z - x})$ on the right, we obtain

$$P(x,z)\Phi_2' = -P'(x,z)\left(\widehat{B_1 z - x}\right)I, \qquad \Psi_1' P(x,z) = -zwJ\left(\widehat{B_1 z - x}\right)P'(x,z), \tag{5.25}$$

where $P(x,z) = \det(B_1 z - x)$. Thus, we see that the trivialization (ϕ', ψ') is equivalent to trivialization (5.13) via the equivalence given by the polynomials $P(x,z)$ and $P'(x,z)$. ■

6 Projective D-modules

In this section, we prove Theorem 2.4. Throughout, we will assume the parameter τ to be generic. We begin with a description of projective D_{frac}-modules.

Proposition 6.1. Assume that τ is generic. Then

(i) any projective finitely generated D_{frac}-module has the form $\mathcal{M} \cong W \otimes_\Gamma D_{\text{frac}}$, for a finite dimensional Γ-module W;

(ii) two D_{frac}-modules $W \otimes_\Gamma D_{\text{frac}}$ and $W' \otimes_\Gamma D_{\text{frac}}$ are isomorphic if and only if $\dim W = \dim W'$. □

Proof. Let $e = (1/|\Gamma|)\sum_{\gamma\in\Gamma}\gamma \in \mathbb{C}\Gamma$ denote the averaging idempotent. Consider the subalgebra $eD_{\text{frac}}e \subset D_{\text{frac}}$. It is clear that this algebra is isomorphic to the algebra of differential operators on $\mathbb{C}/\Gamma$ with rational coefficients. Set $\xi = ex^m$, and $\eta = ex^{1-m}y$. We have an isomorphism

$$eD_{\text{frac}}e \cong \mathbb{C}(\xi)\langle\eta\rangle/\left\langle[\eta,\xi] = |\tau|\right\rangle \tag{6.1}$$

(ξ can be considered as a coordinate on $\mathbb{C}/\Gamma$ and η as a vector field on $\mathbb{C}/\Gamma$ generating the algebra of differential operators). We see that $eD_{\text{frac}}e$ is a skew polynomial ring over the field $\mathbb{C}(\xi)$, hence it is Euclidean. Therefore, $eD_{\text{frac}}e$ is a principal ideal domain, hence any projective $eD_{\text{frac}}e$-module is free.

We claim next that the algebras D_{frac} and $eD_{\mathrm{frac}}e$ are Morita equivalent. To prove this, observe first that, since Γ-action on $\mathbb{C}\setminus\{0\}$ is free, the field $\mathbb{C}(x)$ is a Galois extension of $\mathbb{C}(x)^{\Gamma}$, with Γ being the Galois group. It follows that the algebra $\mathbb{C}\Gamma(x) = \mathbb{C}(x)\#\Gamma$ is a simple $\mathbb{C}(x)^{\Gamma}$-algebra. Hence, $\mathbb{C}\Gamma(x)\cdot e\cdot\mathbb{C}\Gamma(x)$, a two-sided ideal in $\mathbb{C}\Gamma(x)$, must be equal to $\mathbb{C}\Gamma(x)$. We see that there exist elements $a_j, b_j \in \mathbb{C}\Gamma(x)$, $j = 1,\ldots,l$ such that $\sum a_j\cdot e\cdot b_j = 1$. Therefore, since $a_j, b_j \in \mathbb{C}\Gamma(x) \subset D_{\mathrm{frac}}$, we deduce $D_{\mathrm{frac}}\cdot e\cdot D_{\mathrm{frac}} = D_{\mathrm{frac}}$. This implies, by a standard argument, that the functor $N \mapsto N \otimes_{eD_{\mathrm{frac}}e} eD_{\mathrm{frac}}$ provides a Morita equivalence between the algebras $eD_{\mathrm{frac}}e$ and D_{frac}. Our claim follows.

Using the Morita equivalence, we deduce that any projective D_{frac}-module is isomorphic to

$$\left(e\cdot D_{\mathrm{frac}}\cdot e\right)^{\oplus r} \otimes_{e\cdot D_{\mathrm{frac}}\cdot e} \left(e\cdot D_{\mathrm{frac}}\right) \cong \left(e\cdot D_{\mathrm{frac}}\right)^{\oplus r} \cong \left(\mathrm{triv}^{\oplus r}\right)\otimes_{\Gamma} D_{\mathrm{frac}}, \tag{6.2}$$

where $\mathrm{triv} = \epsilon^0$ is the trivial 1-dimensional Γ-module. This proves the first part of the proposition.

To prove the second part, let $W \cong \oplus W_i \otimes \epsilon^i$ be a decomposition of W with respect to the irreducible Γ-modules ϵ^i. Then, $W \otimes_\Gamma D_{\mathrm{frac}}$ goes under Morita equivalence to

$$\begin{aligned}\left(W\otimes_\Gamma D_{\mathrm{frac}}\right)\otimes_{D_{\mathrm{frac}}} D_{\mathrm{frac}}e \cong W\otimes_\Gamma D_{\mathrm{frac}}e &= \left(\bigoplus W_i\otimes\epsilon^i\right)\otimes_\Gamma D_{\mathrm{frac}}e\\ &\cong \bigoplus W_i\otimes e_i D_{\mathrm{frac}}e,\end{aligned} \tag{6.3}$$

where $e_i \in \mathbb{C}\Gamma$ is the projector onto ϵ^i. Now it is easy to see that $e_iD_{\mathrm{frac}}e$ is a free rank 1 $eD_{\mathrm{frac}}e$-module (with $e\cdot x^i$ being a generator). Hence

$$\bigoplus_i W_i \otimes e_i D_{\mathrm{frac}}e \cong \left(eD_{\mathrm{frac}}e\right)^{\oplus \dim W}. \tag{6.4}$$

In particular, it follows that $W\otimes_\Gamma D_{\mathrm{frac}}$ and $W'\otimes_\Gamma D_{\mathrm{frac}}$ are isomorphic D_{frac}-modules if and only if $\dim W = \dim W'$. ■

Recall that we have a natural isomorphism $K(\mathbb{C}\Gamma) \xrightarrow{\sim} K(D)$, $W \mapsto W\otimes_\Gamma D$. Let $[N] \in K(\mathbb{C}\Gamma)$ denote the class of a D-module N under the inverse isomorphism, and write $\dim : K(\mathbb{C}\Gamma) \to \mathbb{Z}$ for the dimension homomorphism. Proposition 6.1 can be reformulated in the following way.

Corollary 6.2. There is a natural isomorphism $K(D_{\mathrm{frac}}) = \mathbb{Z}$. Moreover, the morphism $K(D) \to K(D_{\mathrm{frac}})$ induced by the localization functor $N \mapsto N\otimes_D D_{\mathrm{frac}}$ gets identified with the dimension homomorphism $\dim : K(\mathbb{C}\Gamma) \to \mathbb{Z}$. □

Lemma 6.3. If N is a projective finitely generated D-module such that $[N] = W$, then $N\otimes_D D_{\mathrm{frac}} \cong W\otimes_\Gamma D_{\mathrm{frac}}$. □

Proof. It is clear that $N \otimes_D D_{\mathrm{frac}}$ is a projective finitely generated D_{frac}-module, hence Proposition 6.1(i) yields $N \otimes_D D_{\mathrm{frac}} \cong W' \otimes_\Gamma D_{\mathrm{frac}}$ for some W'. Moreover, Corollary 6.2 implies that $\dim W' = \dim W$. Finally, Proposition 6.1(ii) shows that $N \otimes_D D_{\mathrm{frac}} \cong W \otimes_\Gamma D_{\mathrm{frac}}$. ■

Lemma 6.4. Let W be a Γ-module. Any projective finitely generated D-module N with $\dim[N] = \dim W$ can be embedded into $W \otimes_\Gamma D_{\mathrm{frac}}$ as a fat D-submodule. Furthermore, the embedding is unique up to the action of the group G_W. □

Proof. To prove the existence of embedding, we consider the natural map $N \to N \otimes_D D_{\mathrm{frac}} = N \otimes_{\mathbb{C}\Gamma[x]} \mathbb{C}\Gamma(x)$. Since N is projective, it follows that N is torsion free (as a $\mathbb{C}\Gamma[x]$-module), hence the above map is an embedding. Further, by Lemma 6.3, it follows that $N \otimes_D D_{\mathrm{frac}} \cong W \otimes_\Gamma D_{\mathrm{frac}}$. Finally, taking an arbitrary set of generators (over D) of $N \subset W \otimes_\Gamma D_{\mathrm{frac}}$ and denoting by $p_1(x)$ some Γ-invariant multiple of all their denominators, we see that

$$N \subset \frac{1}{p_1} \cdot (W \otimes_\Gamma D) \subset W \otimes_\Gamma D_{\mathrm{frac}}. \tag{6.5}$$

Similarly, considering the dual D-module, we can check that there exists a Γ-invariant polynomial $p_2(x)$ such that

$$p_2 \cdot (W \otimes_\Gamma D) \subset N \subset W \otimes_\Gamma D_{\mathrm{frac}}. \tag{6.6}$$

Finally, taking $p(x) = p_1(x)p_2(x)$, we see that N is a fat D-submodule in $W \otimes_\Gamma D_{\mathrm{frac}}$.

Now assume that we have two embeddings $\psi_1, \psi_2 : N \hookrightarrow W \otimes_\Gamma D_{\mathrm{frac}}$. Tensoring with D_{frac}, we obtain two isomorphisms $\psi_1, \psi_2 : N \otimes_D D_{\mathrm{frac}} \xrightarrow{\sim} W \otimes_\Gamma D_{\mathrm{frac}}$. Then $g = \psi_2 \circ \psi_1^{-1} \in GL_{D_{\mathrm{frac}}}(W \otimes_\Gamma D_{\mathrm{frac}}) = G_W$ and it is clear that $\psi_2 = g \circ \psi_1$. ■

Proof of Theorem 2.4. It follows from Lemma 6.4 that any projective D-module N such that $\dim[N] = r$ can be embedded into $W \otimes_\Gamma D_{\mathrm{frac}}$ as a fat D-submodule. On the other hand, for generic τ, the homological dimension of the algebra D equals 1 (see [7]), hence any fat D-submodule $N \subset W \otimes_\Gamma D_{\mathrm{frac}}$ is projective. Moreover, it is clear that we have $N \otimes_D D_{\mathrm{frac}} = W \otimes_\Gamma D_{\mathrm{frac}}$, hence by Proposition 6.1 and Corollary 6.2, we deduce $\dim[N] = \dim W = r$. Finally, by Lemma 6.4, two fat D-submodules in $W \otimes_\Gamma D_{\mathrm{frac}}$ are isomorphic as D-modules if and only if they are conjugate by the action of the group G_W. It follows that the set of isomorphism classes of projective D-modules N with $\dim[N] = r$ is in a natural bijection with the coset space $G_W \backslash \mathrm{Gr}^D(W)$. It remains to apply the isomorphisms of diagram (2.10). ■

Appendices

A Formalism of polygraded algebras

Let $A = \oplus_{p\geq 0} A_p$ be a graded algebra over a field $\mathbb{K}$. Let $\mathrm{gr}(A)$ denote the category of graded finitely generated right A-modules. For any $n \in \mathbb{Z}$ and any $M \in \mathrm{gr}(A)$, let $M_{\geq n} = \oplus_{p\geq n} M_p$ be the tail of M. An element $x \in M$ is called torsion if $x \cdot A_{\geq n} = 0$ for some n. A module M is called torsion if every element of M is torsion. Let $\mathrm{tor}(A)$ denote the full subcategory of $\mathrm{gr}(A)$ formed by all torsion A-modules. Then $\mathrm{tor}(A)$ is a Serre subcategory, hence we can consider the quotient category $\mathrm{qgr}(A) = \mathrm{gr}(A)/\mathrm{tor}(A)$. If A is commutative and generated over A_0 by A_1, then by the Serre theorem, the category $\mathrm{qgr}(A)$ is equivalent to the category $\mathrm{coh}(X)$ of coherent sheaves on $X = \mathrm{Proj}(A)$, the projective spectrum of the algebra A.

In the series of papers [1, 15] (cf. the references therein), a formalism has been developed, that allows to consider the category $\mathrm{qgr}(A)$ as a category of coherent sheaves in the case when A is a noncommutative graded algebra. This means that the category $\mathrm{qgr}(A)$ shares many of the general properties of categories of coherent sheaves, provided that the algebra A satisfies some "reasonable" properties. In this case, we say that $\mathrm{qgr}(A)$ is the category of coherent sheaves on a noncommutative algebraic variety X and denote it by $\mathrm{coh}(X)$.

We extend the formalism of [1] to the polygraded case as follows. Let $A = \oplus_{\mathbf{p}\in\mathbb{N}^r} A_{\mathbf{p}}$ be an $\mathbb{N}^r$-graded algebra (we will denote vector indices by bold letters). Let $\mathrm{gr}^r(A)$ denote the category of finitely generated $\mathbb{Z}^r$-graded A-modules. For any $\mathbf{n} \in \mathbb{Z}^r$ and any $M \in \mathrm{gr}^r(A)$, let $M_{\geq \mathbf{n}} = \oplus_{\mathbf{p}\geq\mathbf{n}} M_{\mathbf{p}}$ be the tail of M, where $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_r) \geq \mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_r)$ if and only if $\mathbf{p}_i \geq \mathbf{n}_i$ for all $1 \leq i \leq r$. An element $x \in M$ is called torsion if $x \cdot A_{\geq \mathbf{n}} = 0$ for some $\mathbf{n}$. A module M is called torsion if every element of M is torsion. Let $\mathrm{tor}^r(A)$ denote the full subcategory of $\mathrm{gr}^r(A)$ formed by all torsion A-modules. Thus, $\mathrm{tor}^r(A)$ is a Serre subcategory. We set $\mathrm{qgr}^r(A) = \mathrm{gr}^r(A)/\mathrm{tor}^r(A)$.

In the polygraded situation, we have to make the following modifications in the definitions used in [1]. First, a $\mathbb{Z}^r$-graded $\mathbb{K}$-module $V = \oplus_{\mathbf{p}\in\mathbb{Z}^r} V_{\mathbf{p}}$ should be called left bounded if $V = V_{\geq \mathbf{n}}$ for some $\mathbf{n} \in \mathbb{Z}^r$ (such $\mathbf{n}$ is called a left bound for V). Similarly, V should be called right bounded if $V_{\geq \mathbf{n}} = 0$ for some $\mathbf{n} \in \mathbb{Z}^r$ (such $\mathbf{n}$ is called a right bound for V). Note, that a finitely generated module M over a finitely generated algebra A is torsion if and only if it is both left and right bounded. Thus, $\mathrm{tor}^r(A)$ is the category of bounded $\mathbb{Z}^r$-graded A-modules.

Most essential changes involve the definition of property $\chi_i(M)$ (cf. [1, Definition 3.2]). First, introduce the following notation: For each $i = 1, \ldots, r$, write $e_i \in \mathbb{Z}^r$ for the ith basis vector, and let $I \subset \{1, \ldots, r\}$ denote a nonempty subset of indices. For any

$M \in \mathrm{gr}^r(A)$, put

$$M_n^I = \left(\bigoplus_{p \geq n,\ p_i = n_i \text{ for } i \in I} M_p \right) = M_{\geq n} \Big/ \sum_{i \in I} M_{\geq n + e_i}. \tag{A.1}$$

Definition A.1. We say that property $\chi_i(M)$ holds for a $\mathbb{Z}^r$-graded A-module M provided $\mathrm{Ext}^j(A_0^{\{k\}}, M)$ a bounded $\mathbb{K}$-module for any $j \leq i$ and any $1 \leq k \leq r$.

We say that property χ_i holds for the graded algebra A provided property $\chi_i(M)$ holds for every finitely generated $\mathbb{Z}^r$-graded A-module M.

We say that property χ holds for A provided property χ_i holds for every i.

In [1], a graded algebra A was said to be *regular algebra of dimension* d if the following holds:

(0) A is connected (i.e., $A_0 = \mathbb{K}$);

(1) A has finite global dimension d;

(2) A has polynomial growth;

(3) A is Gorenstein, that is,

$$\mathrm{Ext}^i_{\mathrm{mod}(A)}(\mathbb{K}, A) = \begin{cases} \mathbb{K}[l], & \text{if } i = d, \\ 0, & \text{otherwise.} \end{cases} \tag{A.2}$$

It was demonstrated in [1] that, for regular algebras A, the category $\mathrm{qgr}(A)$ has good properties, in particular, we can compute cohomology of the sheaves $\mathcal{O}(i) = \pi(A(i))$, where $\pi : \mathrm{gr}(A) \to \mathrm{qgr}(A)$ is the projection functor and (i) stands for the degree shift by $i \in \mathbb{Z}$. Further, in [2], we explained that the conditions (0) and (3) above can be replaced, respectively, by the following conditions:

(0′) A_0 is a finite-dimensional semisimple $\mathbb{K}$-algebra;

(3′) A is generalized Gorenstein, that is,

$$\mathrm{Ext}^i_{\mathrm{mod}(A)}(\mathbb{K}, A) = \begin{cases} R[l], & \text{if } i = d, \\ 0, & \text{otherwise,} \end{cases} \tag{A.3}$$

where R is a finite dimensional A_0-bimodule isomorphic to A_0 as right A_0-module.

In this paper, we will need a further generalization of the notion of regular algebra to the setup of polygraded algebras. To this end, we have to replace condition (3′) above by the following condition:

(3″) A is strongly Gorenstein with parameters $\mathbf{d} = (d_1, \dots, d_r)$, $\mathbf{l} = (l_1, \dots, l_r)$ such that $d = \sum_{i=1}^{r} d_i$. This means that for any subset $I \subset \{1, \dots, r\}$, we have

$$\mathrm{Ext}^{i}_{\mathrm{mod}(A)}\left(A_0^I, A\right) = \begin{cases} \left(R_I \otimes_{A_0} A_0^I\right)(\mathbf{l}_I), & \text{if } i = d_I, \\ 0, & \text{otherwise,} \end{cases} \tag{A.4}$$

where

$$d_I = \sum_{k \in I} d_k, \qquad \mathbf{l}_I = \sum_{k \in I} l_k e_k, \qquad R_I = \bigotimes_{k \in I} \epsilon^k \quad (\text{tensor product over } A_0), \tag{A.5}$$

where ϵ^k are A_0-bimodules isomorphic to A_0 as right A_0-modules, and such that $\epsilon^k \otimes_{A_0} \epsilon^l \cong \epsilon^l \otimes_{A_0} \epsilon^k$ as A_0-bimodules.

Now, with all these modifications made, we can verify that most of the results of [1] can be extended to $\mathbb{N}^r$-graded algebras by the same arguments as in [1]. In particular, we have an analogue of [1, Theorem 8.1].

Theorem A.2. Let A be an $\mathbb{N}^r$-graded Noetherian regular algebra of dimension d over a semisimple algebra A_0. Let $\mathcal{O}(\mathbf{p}) = \pi(A(\mathbf{p})) \in \mathrm{qgr}^r(A) = \mathrm{coh}(X)$. Then

(1) property χ holds for A;

(2) $H^0(X, \mathcal{O}(\mathbf{p})) = A_{\mathbf{p}}$ and $H^{>0}(X, \mathcal{O}(\mathbf{p})) = 0$ for all $\mathbf{p} \geq 0$;

(3) the cohomological dimension of the category $\mathrm{coh}(X) = \mathrm{qgr}^r(A)$ equals $d - r$. □

Remark A.3. As opposed to the single-graded case studied in [1], in the polygraded case, it is impossible to determine the cohomology of the sheaves $\mathcal{O}(\mathbf{p})$ for nonpositive $\mathbf{p}$ without some extra information about the structure of the algebra A (it is necessary to know the A_0^I-module structure on A_n^I for all $\mathbf{n} \geq 0$ and $I \subset \{1, \dots, r\}$).

Definition A.4. We say that an $\mathbb{N}^r$-graded algebra A is strongly generated by its first component if for any $1 \leq i \leq r$, both maps below are surjective for any $\mathbf{p} \geq \mathbf{0}$

$$A_{e_i} \otimes A_{\mathbf{p}} \longrightarrow A_{\mathbf{p}+e_i}, \qquad A_{\mathbf{p}} \otimes A_{e_i} \longrightarrow A_{\mathbf{p}+e_i}. \tag{A.6}$$

Remark A.5. It is easy to see that any $\mathbb{N}$-graded algebra, which is generated by its first component, is strongly generated. Thus, in the case $r = 1$, we obtain nothing new.

An element $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$ is called strictly positive if $p_i > 0$ for all $1 \leq r$.

Proposition A.6. If A is an $\mathbb{N}^r$-graded Noetherian algebra strongly generated by its first component and satisfying the χ-condition, then for any strictly positive $\mathbf{p}$, the shift functor $s(M) = M(\mathbf{p})$ in the category $\mathrm{qgr}^r(A)$ is ample in the sense of [1, condition (4.2.1)]. □

Proof. It follows, from an $\mathbb{N}^r$-graded analogue of [1, Theorem 4.5], that the collection of shift functors $s_i(M) = M(e_i)$, $i = 1, \ldots, r$ is ample. Now let $\mathcal{E}$ be an object of $\mathrm{qgr}^r(A)$. Then it follows, from the ampleness of the collection (s_i), that there exists a surjection $\oplus_{i=1}^{p} \mathcal{O}(-\mathbf{l}_i) \to \mathcal{E}$ for some $\mathbf{l}_i \geq 0$. Now for each $\mathbf{l}_i$, we can choose $k_i \in \mathbb{N}$ such that $k_i \cdot \mathbf{p} \geq \mathbf{l}_i$. Then the strong regularity of the algebra A implies that the canonical map

$$A_{k_i \cdot \mathbf{p} - \mathbf{l}_i} \otimes_{A_0} \mathcal{O}\big(-k_i \cdot \mathbf{p}\big) \longrightarrow \mathcal{O}\big(-\mathbf{l}_i\big) \tag{A.7}$$

is surjective. Further, since $A_{k_i \cdot \mathbf{p} - \mathbf{l}_i}$ is a finitely generated A_0-module, it follows that we have a surjection $\oplus_{i=1}^{p} \mathcal{O}(-k_i \cdot \mathbf{p})^{\oplus m_i} \to \mathcal{E}$ and part (a) of the ampleness property for the functor s follows. Part (b) of the ampleness for the functor s follows trivially from the ampleness of the collection s_i. ■

Remark A.7. For any strictly positive $\mathbf{p}$, we put $\Delta_{\mathbf{p}}(A) := \bigoplus_{k=0}^{\infty} A_{k \cdot \mathbf{p}}$. Thus, $\Delta_{\mathbf{p}}(A)$ is a *single-graded* subalgebra of A. The following is immediate from Proposition A.6 and [1, Theorem 4.5].

Corollary A.8. If A is an $\mathbb{N}^r$-graded Noetherian algebra which is strongly generated by its first component and such that condition χ holds, then for any strictly positive $\mathbf{p}$, the algebra $\Delta_{\mathbf{p}}(A)$ is Noetherian, satisfies condition χ. Furthermore, there is an equivalence of categories

$$\mathrm{qgr}^r(A) \cong \mathrm{qgr}\left(\Delta_{\mathbf{p}}(A)\right). \tag{A.8}$$

□

Remark A.9. We would like to emphasize that, inspite of Corollary A.8, the above developed formalism of quotient categories for polygraded algebras *does not* reduce to that for single-graded algebras. The point is that though algebras A and $\Delta_{\mathbf{p}}(A)$ give rise to equivalent quotient categories, the algebra $\Delta_{\mathbf{p}}(A)$ may not be regular or Koszul, for instance, even when A is.

B The geometry of $\mathbb{P}^1 \times_\tau \mathbb{P}^1$

The goal of this appendix is to study homological properties of the algebra Q, see (2.7) and to establish Serre duality and Beilinson spectral sequence for $\mathbb{P}^1 \times_\tau \mathbb{P}^1$.

Proposition B.1. The bigraded algebra Q is Noetherian and it is strongly generated by its first component. Furthermore, Q is a regular algebra of dimension 4. $\square$

To prove this proposition, we introduce some notation. Given a semisimple algebra A_0 and an A_0-bimodule M, we write $T_{A_0}(M)$ for the tensor algebra of M over A_0.

Definition B.2. Let A be the $\mathbb{N}^r$-graded algebra generated by a bimodule $\oplus_{i=1}^r A_{e_i}$ over a semisimple algebra A_0. We say that A is quadratic if $A = T_{A_0}(\oplus_{i=1}^r A_{e_i})/\langle R\rangle$, where $\langle R\rangle$ denotes the two-sided ideal generated by a graded vector subspace $R = \oplus_{1\le i,j\le r} R_{e_i+e_j}$, called the space of "quadratic relations."

Assume that A is a quadratic $\mathbb{N}^r$-graded algebra. Let $A^!$ denote its quadratic dual algebra (with respect to the total grading). Then $A^!$ is also a quadratic $\mathbb{N}^r$-graded algebra. Recall that the algebra A is called Koszul if the following Koszul complex $\mathcal{K}^\bullet(A)$ is exact:

$$\begin{aligned}\cdots &\longrightarrow \oplus_{1\le i,\, j\le r}\big(A^!_{e_i+e_j}\big)^* \otimes_{A_0} A\big(-e_i-e_j\big)\\ &\longrightarrow \oplus_{1\le i\le r}\big(A^!_{e_i}\big)^* \otimes_{A_0} A\big(-e_i\big) \longrightarrow A \longrightarrow A_0 \longrightarrow 0.\end{aligned} \tag{B.1}$$

Definition B.3. We call the algebra A strongly Koszul if for any subset $I \subset \{1,\dots,r\}$ the following partial Koszul complex $\mathcal{K}_I^\bullet(A)$ is exact

$$\begin{aligned}\cdots &\longrightarrow \oplus_{i,j\in I}\big(A^!_{e_i+e_j}\big)^* \otimes_{A_0} A\big(-e_i-e_j\big)\\ &\longrightarrow \oplus_{i\in I}\big(A^!_{e_i}\big)^* \otimes_{A_0} A\big(-e_i\big) \longrightarrow A \longrightarrow A_0^I \longrightarrow 0.\end{aligned} \tag{B.2}$$

It is clear from the definition of the quadratic dual algebra that $(A^!)_0^I$ is dual to $A_0^{\bar{I}}$, where $\bar{I} = \{1,\dots,r\}\setminus I$. Thus if A is strongly Koszul, then for any $I \subset \{1,\dots,r\}$, the algebra A_0^I is Koszul as well. Fix $\mathbf{d} = (d_1,\dots,d_r)$, and for any subset I, write $\mathbf{d}_I = \sum_{i\in I} d_i e_i$. Set $A^!_p := (A^!)_p$.

Definition B.4. We say that $A^!$ is strongly Frobenius of index $\mathbf{d}$ if the following holds

(i) $A^!_{\mathbf{p}} = 0$ unless $0 \le \mathbf{p} \le \mathbf{d}$;

(ii) the component $A^!_{\mathbf{d}_I}$ of $A^!$ is isomorphic to A_0 as right A_0-module, for any subset $I \subset \{1,\dots,r\}$;

(iii) the multiplication map $A^!_{\mathbf{p}} \otimes_{A_0} A^!_{\mathbf{d}_I-\mathbf{p}} \to A^!_{\mathbf{d}_I}$ gives a nondegenerate pairing, for any $0 \le \mathbf{p} \le \mathbf{d}_I$.

Proposition B.5. The algebra Q is strongly Koszul, and $Q^!$ is a strongly Frobenius algebra of index $(2, 2)$. Moreover, the bigraded components of $Q^!$ are

$$Q^!_{i,j} = \begin{cases} \mathbb{C}\Gamma, & (i,j) = (0,0), \\ \mathbb{C}\langle \xi, \zeta \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (1,0), \\ \mathbb{C}\langle \xi \wedge \zeta \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (2,0), \\ \mathbb{C}\langle \eta, \omega \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (0,1), \\ \mathbb{C}\langle \xi \wedge \eta, \xi \wedge \omega, \zeta \wedge \eta, \zeta \wedge \omega \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (1,1), \\ \mathbb{C}\langle \xi \wedge \zeta \wedge \eta, \xi \wedge \zeta \wedge \omega \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (2,1), \\ \mathbb{C}\langle \eta \wedge \omega \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (0,2), \\ \mathbb{C}\langle \xi \wedge \eta \wedge \omega, \zeta \wedge \eta \wedge \omega \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (1,2), \\ \mathbb{C}\langle \xi \wedge \zeta \wedge \eta \wedge \omega \rangle \otimes \mathbb{C}\Gamma, & (i,j) = (2,2), \\ 0, & \text{otherwise}, \end{cases} \tag{B.3}$$

where ξ is a generator of the Γ-bimodule ϵ^{-1}, η is a generator of the Γ-bimodule ϵ, and ζ and ω stand for generators of the trivial Γ-bimodule. □

Proof. Consider τ as a variable, and view the algebra Q as an algebra, depending on a parameter τ. We will indicate the value of τ by a superscript. For example, Q^0 stands for the algebra Q with $\tau = 0$.

First, it is easy to show that for any τ, the components of the dual algebra are given by the above formulas. Further, note that for $\tau = 0$, we have an isomorphism $Q^0 \cong \mathbb{C}[x, z, y, w]\#\Gamma$. In this case, it is quite easy to show that Q^0 is strongly Koszul. Finally, we note that we may view the family of partial Koszul complexes $\mathcal{K}^\bullet_I(Q^\tau)$ of the algebras Q^τ as a family of varying (with τ) differentials on the partial Koszul complex $\mathcal{K}^\bullet_I(Q^0)$. Since the complex is exact for $\tau = 0$, the same holds for all values of τ close enough to zero. However, the algebras Q^τ and $Q^{\alpha\cdot\tau}$ are isomorphic for any $\alpha \in \mathbb{C}^*$. Thus, Q^τ is strongly Koszul for any τ.

Similarly, to show that $(Q^\tau)^!$ is strongly Frobenius for any τ, we note that this holds for $\tau = 0$. Further, we consider the family of pairings $(Q^\tau)^!_p \otimes_\Gamma (Q^\tau)^!_{d_I - p} \to (Q^\tau)^!_{d_I}$ as a family of varying (with τ) pairings $(Q^0)^!_p \otimes_\Gamma (Q^0)^!_{d_I - p} \to (Q^0)^!_{d_I}$. Since the pairings are nondegenerate for $\tau = 0$, the same is true for all values of τ close enough to zero. However, the algebras $(Q^\tau)^!$ and $(Q^{\alpha\cdot\tau})^!$ are isomorphic for any $\alpha \in \mathbb{C}^*$. Thus, $(Q^\tau)^!$ is strongly Frobenius for any τ. ■

Proposition B.6. If an $\mathbb{N}^r$-graded algebra A is strongly Koszul and the dual algebra $A^!$ is strongly Frobenius of index $(d_1, \dots, d_r)$, then A is strongly Gorenstein with parameters $\mathbf{d} = (d_1, \dots, d_r)$ and $\mathbf{l} = (d_1, \dots, d_r)$. □

Proof. If A is strongly Koszul, then the partial Koszul complex $\mathcal{K}_I^\bullet(A)$ can be considered as a projective resolution of A_0^I. It follows that $\mathrm{Ext}^\bullet_{\mathrm{mod}(A)}(A_0^I, A)$ coincides with the cohomology of complex

$$0 \longrightarrow A \longrightarrow \oplus_{i \in I} A^!_{e_i} \otimes_{A_0} A\big(-e_i\big) \longrightarrow \cdots \longrightarrow \oplus_{i\in I} A^!_{\mathbf{d}_I - e_i} \otimes_{A_0} A\big(\mathbf{d}_I - e_i\big) \\ \longrightarrow A^!_{\mathbf{d}_I} \otimes_{A_0} A\big(\mathbf{d}_I\big) \longrightarrow 0. \tag{B.4}$$

On the other hand, the strong Frobenius property of the algebra $A^!$ shows that $A^!_{\mathbf{p}} \cong A^!_{\mathbf{d}_I} \otimes_{A_0} (A^!_{\mathbf{d}_I - \mathbf{p}})^*$ as A_0-bimodule. Hence, the above complex is isomorphic to the complex $A^!_{\mathbf{d}_I} \otimes_{A_0} \mathcal{K}_I^\bullet(A)(\mathbf{d}_I)$ truncated at the rightmost term. Therefore, it has a single nonzero cohomology group in degree d_I, which is isomorphic to $A^!_{\mathbf{d}_I} \otimes_{A_0} A_0^I(\mathbf{d}_I)$. It follows that A satisfies the strong Gorenstein property with parameters $(\mathbf{d}, \mathbf{d})$ and with $R_I = A^!_{\mathbf{d}_I}$. ■

Proof of Proposition B.1. It is clear that Q is strongly generated by its first component. So, it remains to prove regularity and the Noetherian property.

First, note that $Q^\tau_{(0,0)} = \mathbb{C}\Gamma$ is a semisimple algebra. Thus $(0')$ holds. Second, we have to show that Q^τ is Noetherian. This follows from the fact that Q^τ can be represented as a consecutive Ore extension, starting with the base field $\mathbb{C}$. Further, it is easy to show that $\dim_{\mathbb{C}} Q^\tau_{i,j} = (i+1)(j+1)|\Gamma|$. In particular, Q^τ has polynomial growth. Thus (2) holds.

The strong Gorenstein property $(3'')$ for the algebra Q^τ follows immediately from Propositions B.5 and B.6. The Gorenstein parameters are given by $\mathbf{d} = (2,2)$ and $\mathbf{l} = (2,2)$.

Finally, it follows from [10] that the global dimension of Q^τ equals the length of the minimal free resolution of $Q^\tau_{(0,0)}$. But the Koszul complex $\mathcal{K}^\bullet(Q^\tau)$ provides such a resolution of length 4, hence the global dimension of Q^τ is bounded by 4 from above. On the other hand, since Q^τ is Gorenstein with parameters $\mathbf{d} = (2,2)$, $\mathbf{l} = (2,2)$, it follows that $\mathrm{Ext}^4(Q^\tau_{0,0}, Q^\tau) \neq 0$, hence the global dimension equals 4. ■

Thus, the cohomological dimension of the category $\mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1) = \mathrm{qgr}^2(Q^\tau)$ equals 2, and it is clear that we have

$$H^p\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, \mathcal{O}(i,j)\big) = \begin{cases} Q_{i,j}, & p = 0, \\ 0, & p > 0, \end{cases} \tag{B.5}$$

for all $i, j \geq 0$. More generally, we prove the following result.

Lemma B.7. The cohomology groups of the sheaves $\mathcal{O}(i,j)$ are given by

$$H^p\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1, \mathcal{O}(i,j)\right) = \begin{cases} Q_{i,j}, & \text{if } p = 0,\ i,j \geq 0, \\ \epsilon^{-1} \otimes Q^*_{-2-i,0} \otimes_\Gamma Q_{0,j}, & \text{if } p = 1,\ i \leq -2,\ j \geq 0, \\ \epsilon \otimes Q^*_{0,-2-j} \otimes_\Gamma Q_{i,0}, & \text{if } p = 1,\ i \geq 0,\ j \leq -2, \\ Q^*_{-2-i,-2-j}, & \text{if } p = 2,\ i,j \leq -2, \\ 0, & \text{otherwise.} \end{cases} \tag{B.6}$$

□

Sketch of proof. In order to compute the global cohomology of $\mathcal{O}(i,j)$ for not necessarily positive values of (i,j), we use partial Koszul complexes. In more detail, the projections of the partial Koszul complexes to the category $\mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1)$ yield exact sequences

$$\begin{gathered} 0 \longrightarrow \epsilon \otimes \mathcal{O}(-2,0) \longrightarrow Q_{1,0} \otimes_\Gamma \mathcal{O}(-1,0) \longrightarrow \mathcal{O} \longrightarrow 0, \\ 0 \longrightarrow \epsilon^{-1} \otimes \mathcal{O}(0,-2) \longrightarrow Q_{0,1} \otimes_\Gamma \mathcal{O}(0,-1) \longrightarrow \mathcal{O} \longrightarrow 0, \end{gathered} \tag{B.7}$$

(we used here the fact that $Q_0^I \in \mathrm{tor}(Q)$ for any nonempty $I \subset \{1,2\}$, and that $Q^!_{0,1} \cong Q^*_{0,1}$, $Q^!_{1,0} \cong Q^*_{1,0}$, $Q^!_{0,2} \cong \epsilon^{-1}$, $Q^!_{2,0} \cong \epsilon$).

To complete the proof of the lemma, we apply descending induction in (i,j) using the above sequences twisted by $(i+2,j)$ and $(i,j+2)$, respectively, and the fact that the multiplication map $Q_{i,0} \otimes_\Gamma Q_{0,j} \to Q_{i,j}$ is an isomorphism of Γ-bimodules. ■

Serre duality for $\mathbb{P}^1 \times_\tau \mathbb{P}^1$. A natural approach to Serre duality theorems for noncommutative schemes corresponding to regular noncommutative algebras would be via the concept of balanced dualizing complex (see [14, 15]). Generalizing this concept to the case of $\mathbb{N}^r$-graded algebras does not seem to be straightforward, however. The reason is that, while the notion of dualizing complex easily extends to the polygraded case, it is rather difficult to find the relevant meaning of "balanced" property in this case. The problem is similar to that of computing the cohomology of sheaves $\mathcal{O}(\mathbf{p})$ for nonpositive values of $\mathbf{p}$, see Remark A.3.

In the special case of $\mathbb{P}^1 \times_\tau \mathbb{P}^1$, these problems can be circumvented as follows: We consider the algebra $A = \Delta_{(1,1)}(Q)$. It follows from Proposition B.1 and Corollary A.8 that this algebra is Noetherian and satisfies condition χ. Moreover, by Corollary A.8, Theorem A.2, and Proposition B.1, the cohomological degree of the category $\mathrm{qgr}^2(A) = \mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1)$ equals 2. Hence, we can use [15, Theorem 2.3] which implies that the category $\mathrm{coh}(\mathbb{P}^1 \times_\tau \mathbb{P}^1)$ enjoys the Serre duality with the dualizing sheaf

$$\omega^0 = \pi\Big(\oplus_{k=0}^\infty H^2\big(\mathbb{P}^1 \times_\tau \mathbb{P}^1, \mathcal{O}(-k,-k)\big)^* \Big). \tag{B.8}$$

But Lemma B.7 yields

$$\pi\left(\bigoplus_{k=0}^{\infty} H^2\left(\mathbb{P}^1 \times_\tau \mathbb{P}^1, \mathcal{O}(-k,-k)\right)^*\right) \cong \pi\left(\bigoplus_{k=0}^{\infty} Q_{k-2,k-2}\right) \cong \mathcal{O}(-2,-2). \tag{B.9}$$

Thus, the dualizing sheaf on $\mathbb{P}^1 \times_\tau \mathbb{P}^1$ is isomorphic to $\mathcal{O}(-2,-2)$.

Beilinson spectral sequence. A noncommutative analogue of Beilinson spectral sequence has been introduced in [9] for a certain class of graded Koszul algebras, using a double Koszul bicomplex. Below, we explain how to adapt the approach of [9] to the case of $\mathbb{N}^r$-graded Koszul algebras. We will freely use the notation and definitions of [9], in particular, the notion of *Yang-Baxter operator*.

An exact functor from a tensor category T to the tensor category of vector spaces will be called a *noncommutative fiber functor* if this functor is compatible with the tensor product structures and associativity constraint, but is not necessarily compatible with the commutativity constraint. Given a Yang-Baxter operator on a finite-dimensional vector space, we can construct as has been explained in [11] (see also [9, Section 8]), a tensor category T equipped with a *noncommutative fiber functor*. Then the category of (either graded, or $\mathbb{N}^r$-graded, or ...) commutative algebras in the category T gives a class of (graded, $\mathbb{N}^r$-graded, ...) noncommutative algebras in the category of vector spaces. The class of noncommutative algebras, thus obtained, shares a lot of properties of the category of commutative algebras. For example, for any two algebras in the class, their tensor product admits a canonical algebra structure.

Remark B.8. Instead of Yang-Baxter operator in a vector space we may start with an A_0-invariant Yang-Baxter operator in a finitely generated A_0-bimodule, for any semisimple finite dimensional algebra A_0. Then we obtain an A_0-linear tensor category T with a functor to the category of A_0-bimodules. The category of commutative A_0-algebras in T gives a class of noncommutative A_0-algebras.

Example B.9. Consider a free-right Γ-module V of rank 4 with generators x, y, z, and w and endow it with a Γ-bimodule structure as in (2.8). Then the Γ-linear operator $V \otimes_\Gamma V \to V \otimes_\Gamma V$ defined on generators as

$$\begin{aligned} x \otimes y &\longmapsto y \otimes x - \frac{\tau}{2} z \otimes w - \frac{\tau}{2} w \otimes z, \\ y \otimes x &\longmapsto x \otimes y + \frac{\tau}{2} z \otimes w + \frac{\tau}{2} w \otimes z, \\ u \otimes v &\longmapsto v \otimes u \quad \text{otherwise} \end{aligned} \tag{B.10}$$

is a Yang-Baxter operator. It is easy to see that the algebra Q comes from a bigraded commutative algebra in the tensor category corresponding to this Yang-Baxter operator.

Let A be an $\mathbb{N}^r$-graded algebra obtained in such a way. Then $A \otimes_{A_0} A$ is an $\mathbb{N}^r \oplus \mathbb{N}^r$-graded algebra and the maps $p_1^*(a) = a \otimes 1$ and $p_2^*(a) = 1 \otimes a$ are homomorphisms of algebras $A \to A \otimes_{A_0} A$. Let X denote the noncommutative variety, corresponding to the algebra A and let $X \times X$ denote the noncommutative variety, corresponding to the algebra $A \otimes_{A_0} A$. Thus $\mathrm{qgr}^r(A) = \mathrm{coh}(X)$ and $\mathrm{qgr}^{2r}(A \otimes_{A_0} A) = \mathrm{coh}(X \times X)$.

Now, if M is a right $\mathbb{N}^r$-graded A-module, we define $p_1^* M = M \otimes_A (A \otimes_{A_0} A)$. Then $p_1^* M$ is a right $(\mathbb{N}^r \oplus \mathbb{N}^r)$-graded $A \otimes_{A_0} A$-module. It is clear that $p_1^* M(\mathbf{p}, \mathbf{q}) = M_{\mathbf{p}} \otimes_{A_0} A_{\mathbf{q}}$, hence for any $M \in \mathrm{tor}^r(A)$, we have $p_1^* M \in \mathrm{tor}^{2r}(A \otimes_{A_0} A)$. Thus p_1^* can be considered as a functor $\mathrm{qgr}^r(A) \to \mathrm{qgr}^{2r}(A \otimes_{A_0} A)$, that is, as a functor $\mathrm{coh}(X) \to \mathrm{coh}(X \times X)$.

Similarly, if $M = \oplus_{\mathbf{p},\mathbf{q}} M_{\mathbf{p},\mathbf{q}}$ is a right $(\mathbb{N}^r \oplus \mathbb{N}^r)$-graded $(A \otimes_{A_0} A)$-module, then we define $((p_2)_* M)_{\mathbf{q}} := \Gamma(X, \pi(\oplus_{\mathbf{p}} M_{\mathbf{p},\mathbf{q}}))$ where

$$\Gamma\big(X, \pi(\bullet)\big) = \mathrm{Hom}_{\mathrm{qgr}^r(A)}\big(\pi(A), \pi(\bullet)\big), \tag{B.11}$$

and the A-module structure of $\oplus_{\mathbf{p}} M_{\mathbf{p},\mathbf{q}}$ is obtained via the homomorphism p_1^*. It is clear that $(p_2)_* M = \oplus_{\mathbf{q}} ((p_2)_* M)_{\mathbf{q}}$ is an $\mathbb{N}^r$-graded A-module. Furthermore, if the module $M \in \mathrm{tor}^{2r}(A \otimes_{A_0} A)$, then it is clear that $(p_2)_* M \in \mathrm{tor}^r(A)$. Thus, $(p_2)_*$ can be considered as a functor $\mathrm{qgr}^{2r}(A \otimes_{A_0} A) \to \mathrm{qgr}^r(A)$, that is, a functor $\mathrm{coh}(X \times X) \to \mathrm{coh}(X)$.

Now, if N is an $(\mathbb{N}^r \oplus \mathbb{N}^r)$-graded $(A \otimes_{A_0} A)$-bimodule, then the assignment $M \mapsto (p_2)_*(p_1^* M \otimes_{A \otimes_{A_0} A} N)$ gives a functor $\Phi_N : \mathrm{qgr}^r(A) \to \mathrm{qgr}^r(A)$, that is, a functor $\mathrm{coh}(X) \to \mathrm{coh}(X)$.

Lemma B.10. (i) Let $(\Delta_A)_{\mathbf{p},\mathbf{q}} = A_{\mathbf{p}+\mathbf{q}}$ and $\Delta_A = \oplus_{\mathbf{p},\mathbf{q} \geq 0} (\Delta_A)_{\mathbf{p},\mathbf{q}}$. Then there is a natural isomorphism of functors $\Phi_{\Delta_A} \cong \mathbf{Id}$.

(ii) If N_1 and N_2 are A-bimodules and $N_1 \otimes_{A_0} N_2$ has a canonical structure of an $(A \otimes_{A_0} A)$-bimodule, then

$$\Phi_{N_1 \otimes_{A_0} N_2}(M) = \Gamma\big(X, \pi\big(M \otimes_A N_1\big)\big) \otimes_{A_0} N_2. \tag{B.12}$$

□

Proof. (i) Let $M' = \Phi_{\Delta_A}(M)$. Note that Δ_A, considered as an A-module, is isomorphic to $\oplus_{\mathbf{q} \in \mathbb{N}^r} A(\mathbf{q})_{\geq 0}$. Hence $M'_{\mathbf{q}} = \Gamma(X, \pi(M \otimes_A A(\mathbf{q})_{\geq 0}))$. On the other hand, it is clear that

$$\pi\big(M \otimes_A A(\mathbf{q})_{\geq 0}\big) \cong \pi\big(M \otimes_A A(\mathbf{q})\big) \cong \pi\big(M(\mathbf{q})\big) \cong \pi(M)(\mathbf{q}). \tag{B.13}$$

This means that $M' = \oplus_{\mathbf{q}} \Gamma(X, \pi(M)(\mathbf{q}))$, hence $\pi(M') \cong \pi(M)$. Furthermore, it is clear that the isomorphism constructed above gives an isomorphism of functors $\Phi_{\Delta_A} \to \mathbf{Id}$.

(ii) Let $M' = \Phi_{N_1 \otimes_{A_0} N_2}(M)$. Then

$$M'_q = \Gamma\Big(X, \pi\Big(M \otimes_A N_1 \otimes_{A_0} \left(N_2\right)_q\Big)\Big) = \Gamma\big(X, \pi\big(M \otimes_A N_1\big)\big) \otimes_{A_0} \left(N_2\right)_q, \tag{B.14}$$

hence $M' = \Gamma(X, \pi(M \otimes_A N_1)) \otimes_{A_0} N_2$. ■

Remark B.11. It is clear that Δ_A can be endowed with an algebra structure. Furthermore, it is easy to show that $\mathrm{qgr}^{2r}(\Delta_A) \cong \mathrm{qgr}^r(A)$. Finally, the multiplication in A gives an epimorphism $A \otimes_{A_0} A \to \Delta_A$. This way, we may view Δ_A as a diagonal embedding $\Delta_X : X \hookrightarrow X \times X$.

Once the diagonal $X \hookrightarrow X \times X$ has been defined, we could apply standard techniques provided that we find a resolution of diagonal. If A is Koszul, we may obtain a resolution of diagonal as follows. Consider the double Koszul bicomplex of A,

$$\begin{tikzcd}
\cdots \arrow[r, "d_R"] & \oplus_{i,j} A \otimes \left(A^!_{e_i+e_j}\right)^* \otimes A(-e_i-e_j) \arrow[r, "d_R"] \arrow[d, "d_L"] & \oplus_i A \otimes \left(A^!_{e_i}\right)^* \otimes A(-e_i) \arrow[r, "d_R"] \arrow[d, "d_L"] & A \otimes A \\
\cdots \arrow[r, "d_R"] & \oplus_{i,j} A(e_i) \otimes \left(A^!_{e_j}\right)^* \otimes A(-e_i-e_j) \arrow[r, "d_R"] \arrow[d, "d_L"] & \oplus_i A(e_i) \otimes A(-e_i) & \\
\cdots \arrow[r, "d_R"] & \oplus_{i,j} A(e_i+e_j) \otimes A(-e_i-e_j), & &
\end{tikzcd} \tag{B.15}$$

where both d_R and d_L are induced by the differential in the Koszul complex of A. Write

$$\mathcal{K}^p(A) = \mathbf{Ker}\left(A(-p) \otimes \left(A^!_p\right)^* \longrightarrow \bigoplus_{\{i \mid e_i \leq p\}} A(e_i - p) \otimes \left(A^!_{p-e_i}\right)^*\right) \tag{B.16}$$

for the cohomology of the truncated Koszul complex. Using the Koszul property of the algebra A and mimicking the proof of [9, Proposition 4.7], we deduce the following proposition.

Proposition B.12. The following complex is exact

$$\begin{aligned} \cdots \longrightarrow \oplus_{i,j} \mathcal{K}^{e_i+e_j}(e_i+e_j) \otimes A(-e_i-e_j) \longrightarrow \oplus_i \mathcal{K}^{e_i}(e_i) \otimes A(-e_i) \\ \longrightarrow A \otimes A \longrightarrow \Delta_A \longrightarrow 0, \end{aligned} \tag{B.17}$$

where the map $A \otimes A \to \Delta_A$ is given by the multiplication in A. □

Let $\mathcal{Q}^p = \pi(\mathcal{K}^p(A))^*$. Combining Proposition B.12 with Lemma B.10, we obtain the Beilinson spectral sequence.

Corollary B.13. Assume that A is Koszul and $A^!$ is Frobenius. Then for any $F \in X$, there exists a spectral sequence with the first term

$$E_1^{p,q} = \bigoplus_{\{\mathbf{p} \mid |\mathbf{p}|=p\}} \operatorname{Ext}^q\left(\mathcal{Q}^{\mathbf{p}}, F\right) \otimes_{A_0} \mathcal{O}(-\mathbf{p}) \Longrightarrow E_\infty^i = \begin{cases} F, & i = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{B.18}$$

□

In the special case of the algebra Q, the only nonvanishing components of $\mathcal{Q}^{\mathbf{p}}$ are

$$\mathcal{Q}^0 = \mathcal{O}, \qquad \mathcal{Q}^{e_1} = \epsilon^{-1} \otimes \mathcal{O}(1,0), \qquad \mathcal{Q}^{e_2} = \epsilon^{1} \otimes \mathcal{O}(0,1), \qquad \mathcal{Q}^{e_1+e_2} = \mathcal{O}(1,1). \tag{B.19}$$

Thus, Beilinson spectral sequence takes the form of (5.7).

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References

[1] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), no. 2, 228–287.

[2] V. Baranovsky, V. Ginzburg, and A. Kuznetsov, *Quiver varieties and a noncommutative* $\mathbb{P}^2$, Compositio Math. **134** (2002), 283–318.

[3] Yu. Berest and G. Wilson, *Automorphisms and ideals of the Weyl algebra*, Math. Ann. **318** (2000), no. 1, 127–147.

[4] ———, *Ideal classes of the Weyl algebra and noncommutative projective geometry*, Internat. Math. Res. Notices **2002** (2002), no. 26, 1347–1396.

[5] R. C. Cannings and M. P. Holland, *Right ideals of rings of differential operators*, J. Algebra **167** (1994), no. 1, 116–141.

[6] ———, *Limits of compactified Jacobians and D-modules on smooth projective curves*, Adv. Math. **135** (1998), no. 2, 287–302.

[7] W. Crawley-Boevey and M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635.

[8] P. Etingof and V. Ginzburg, *Symplectic reflection algebras*, *Calogero-Moser space*, *and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348.

[9] A. Kapustin, A. Kuznetsov, and D. Orlov, *Noncommutative instantons and twistor transform*, Comm. Math. Phys. **221** (2001), no. 2, 385–432.

[10] H. Li, *Global dimension of graded local rings*, Comm. Algebra **24** (1996), no. 7, 2399–2405.

[11] V. V. Lyubashenko, *Hopf algebras and vector-symmetries*, Uspekhi Mat. Nauk **41** (1986), no. 5, 185–186.

[12] A. Pressley and G. Segal, *Loop Groups*, Oxford Mathematical Monographs, Oxford University Press, New York, 1986.

[13] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), no. 1, 1–41.

[14] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Algebra **153** (1992), no. 1, 41–84.

[15] A. Yekutieli and J. J. Zhang, *Serre duality for noncommutative projective schemes*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 697–707.

Vladimir Baranovsky: Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
E-mail address: baranovs@caltech.edu

Victor Ginzburg: Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
E-mail address: ginzburg@math.uchicago.edu

Alexander Kuznetsov: Institute for Information Transmission Problems, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow 101447, Russia
E-mail address: sasha@kuznetsov.mccme.ru