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THE VARIETY OF PAIRS OF COMMUTING NILPOTENT MATRICES IS IRREDUCIBLE

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Abstract. In this paper we prove the dimension and the irreduciblity of the variety parametrizing all pairs of commuting nilpotent matrices. Our proof uses the connection between this variety and the punctual Hilbert scheme of a smooth algebraic surface.

1. Introduction

Let V be a vector space of dimension n over an algebraically closed field k of characteristic zero. Consider the subvariety $\mathcal{N}_2 \subset gl(V) \oplus gl(V)$ of all pairs of commuting nilpotent linear operators. The goal of this note is to give a proof of

Theorem 1. *$\mathcal{N}_2$ is an irreducible variety of dimension $n^2 - 1$.*

Without the nilpotency condition the corresponding statement was first proved by Motzkin and Taussky in [MT] and later generalized by Richardson, cf. [Ri], to a general semisimple Lie algebra. In our case, however, the variety of nilpotent commuting pairs is *not* irreducible for semisimple algebras of types other than A_n and B_2 (this follows from Theorem 2 below). We also note here that an earlier proof of Theorem 1 given in [Gr] contains a serious omission which, to the present author's knowledge, has not been filled.

Our proof of Theorem 1 is similar in spirit to the proof of the Motzkin–Taussky Theorem given in [Gu] (of course, the nilpotency condition introduces some technical difficulties). We reduce Theorem 1 to a similar theorem about a geometric object known as the punctual Hilbert scheme $\mathrm{Hilb}^n_{[s]}$ of a smooth algebraic surface. This variety was studied in detail by Briançon and Iarrobino, cf. [B] and [I].

In fact, our original motivation for the study of $\mathcal{N}_2$ comes from the theory of vector bundles on algebraic surfaces. The moduli space $N(r)$ of (semistable) rank r vector bundles on an algebraic surface, is noncompact and it admits two different compactifications: the *Gieseker compactification* $M^{\mathrm{Gies}}(r)$ and the *Uhlenbeck compactification* $M^{\mathrm{Uhl}}(r)$. These spaces are related via a morphism $\pi : M^{\mathrm{Gies}}(r) \longrightarrow M^{\mathrm{Uhl}}(r)$ which, under certain conditions on the surface and Chern classes of the bundles, is a smooth resolution of $M^{\mathrm{Uhl}}(r)$ that is *semismall* in the sense of Goresky–MacPherson [GM]. The fibers of this morphism are products of varieties known as *punctual Quot schemes* (with reduced scheme structure). Every punctual *Quot* scheme may be obtained as a

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quotient of a Zariski open subset $U \subset \mathcal{N}_2 \times V^{\oplus r}$ by a free action of $GL(V)$. Therefore, Theorem 1 implies irreducibility and dimension of punctual *Quot* schemes, which gives the semismallness property for π, as well as a formula relating the geometry of the two compactifications $M^{\mathrm{Gies}}(r)$ and $M^{\mathrm{Uhl}}(r)$ (see [GS] for the rank 1 case and [Ba2] in general). The statement about irreduciblity and dimension of *Quot* schemes was proved independently by the author in [Ba1], and by Ellingsrud and Lehn in [EL]. The proof of Theorem 1 presented here is a modification of [Ba1].

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2. Commuting nilpotents in a semisimple Lie algebra

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Denote by $\mathcal{N}_2(\mathfrak{g}) \subset \mathfrak{g} \oplus \mathfrak{g}$ the variety of pairs (n_1, n_2) of commuting nilpotent elements. We want to study the dimension and the irreducible components of $\mathcal{N}_2$. First we recall the following:

Definition 1. A nilpotent element $n \in \mathfrak{g}$ is called *distinguished* (cf. [BC]) if its centralizer $\mathfrak{z}_{\mathfrak{g}}(n) \subset \mathfrak{g}$ does not contain any semisimple elements. This means that any $x \in \mathfrak{g}$ which commutes with n is automatically nilpotent. If n is distinguished, then so is any other element in the adjoint orbit of n.

The proof of the following result was kindly communicated to the author by M. Grinberg. The same type of argument was originally used in [P].

Theorem 2. *The dimension of $\mathcal{N}_2$ is equal to* $\dim \mathfrak{g}$. *Moreover, there exists a bijection between the set of irreducible components of $\mathcal{N}_2$ which have the maximal dimension* $\dim \mathfrak{g}$, *and the set of distinguished nilpotent conjugacy classes in* $\mathfrak{g}$.

Proof. Note that $\mathcal{N}_2(\mathfrak{g})$ is a closed subvariety of

$$\widetilde{\mathcal{N}_2}(\mathfrak{g}) := \big\{(n, x) \in \mathfrak{g} \oplus \mathfrak{g} \mid [n, x] = 0 \text{ and } n \text{ is nilpotent}\big\}$$

(i.e., we do not require that x is nilpotent as well). By identifying the second copy of $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form, we can view $\widetilde{\mathcal{N}_2}(\mathfrak{g})$ as a subvarity of $\mathfrak{g} \oplus \mathfrak{g}^*$. Let ρ be map from $\widetilde{\mathcal{N}_2}(\mathfrak{g})$ onto the cone $\mathcal{N}(\mathfrak{g})$ of nilpotent elements in $\mathfrak{g}$ given by the first projection $(n, x) \mapsto n$. If $\mathcal{O} \subset \mathcal{N}(\mathfrak{g})$ is an adjoint orbit, then its preimage $\rho^{-1}(\mathcal{O})$ can be naturally identified (as a subvariety of $\mathfrak{g} \oplus \mathfrak{g}^*$) with the conormal bundle $T^*_{\mathcal{O}}\mathfrak{g}$ of $\mathcal{O}$ in $\mathfrak{g}$. Therefore, $\dim \widetilde{\mathcal{N}_2}(\mathfrak{g}) = \dim \mathfrak{g}$ and hence $\dim \mathcal{N}_2(\mathfrak{g}) \leq \dim \mathfrak{g}$.

This also shows that the irreducible components of $\mathcal{N}_2$ of maximal dimension are closures of those $T^*_{\mathcal{O}}\mathfrak{g}$ which belong to $\mathcal{N}_2 \subset \widetilde{\mathcal{N}_2}$. If n is a point of such an orbit $\mathcal{O}$, then the condition $[x, n] = 0$, $x \in \mathfrak{g}$ implies that x is also nilpotent, which means that $\mathcal{O}$ is distinguished. $\square$

Conjecture. *All irreducible components of the variety $\mathcal{N}_2(\mathfrak{g})$ of pairs of commuting nilpotent elements in $\mathfrak{g}$ have maximal dimension* $\dim \mathfrak{g}$.

The Lie algebra $\mathfrak{g} = sl_n$ has a unique distinguished nilpotent class (the class of the Jordan $n \times n$ cell). Thus, Theorem 1 proves the above conjecture for sl_n. In general,

$\mathfrak{g}$ has several distinguished nilpotent classes, cf. [BC], and therefore $\mathcal{N}_2(\mathfrak{g})$ has several irreducible components.

3. Reduction to pairs admitting a cyclic vector

From now on, we consider only the case $\mathfrak{g} = sl_n$. In this section we show that the irreducibility of $\mathcal{N}_2$ is equivalent to the irreducibility of an open subvariety $U \subset \mathcal{N}_2 \times V$ formed by all triples (B_1, B_2, w) such that w is a *cyclic* vector for $(B_1, B_2) \in \mathcal{N}_2$, i.e., such that any subspace $W \subset V$ that contains w and is invariant with respect to B_1 and B_2, should necessarily coincide with the whole V.

Lemma 3. (cf. [Ba1]) *Let B_1, B_2 be two commuting nilpotent operators on a vector space V. There exists a third nilpotent operator B_2' and a vector $w \in V$ such that*
(i) *B_2' commutes with B_1;*
(ii) *any linear combination $\alpha B_2 + \beta B_2'$ is nilpotent;*
(iii) *w is a cyclic vector for the pair of operators (B_1, B_2').*

Proof. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k$ be the sizes of Jordan blocks for B_1. We will construct a basis $e_{j,i}$ of V, where $1 \leq i \leq k$ and $1 \leq j \leq \mu_i$ such that

(a) $B_1(e_{j,i}) = e_{j+1,i}$ for $j < \mu_i$ and $B_1(e_{\mu_i,i}) = 0$ (i.e., B_1 has Jordan canonical form in this basis);

(b) $B_2(e_{j,i})$ is a linear combination of $e_{p,q}$ where either $p = j$ and $q > i$ or $p > j$ and q is arbitrary.

In other words, the elements of the basis correspond to the cells of a Young digram having μ_i cells in the i-th column (e.g., in the diagrams below we have $\mu_1 = 4$, $\mu_2 = \mu_3 = \mu_4 = 3$, $\mu_5 = 2$). Condition (a) means that B_1 acts by shifting cells one step down (see Diagram 1) and (b) means the vector corresponding to the cell marked by *, is mapped by B_2 into a subspace generated by the vectors in the shaded area (see Diagram 2). Note that due to (a) it suffices to check (b) on the vectors of the first row $e_{1,i}$.

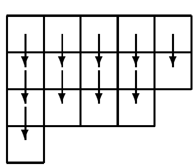

DIAGRAM 1

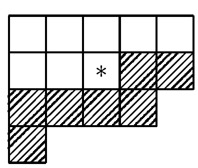

DIAGRAM 2

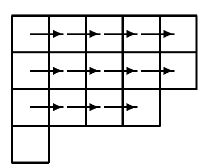

DIAGRAM 3

Once such a basis is constructed we define B_2' by $B_2'(e_{j,i}) = e_{j,i+1}$ if $j \leq \mu_{i+1}$ and 0 otherwise (i.e., the operator B_2' shifts cells to the right, see Diagram 3). Then $[B_1, B_2'] = 0$ and $w = e_{1,1}$ is a cyclic vector for (B_1, B_2') since $e_{j,i} = B_1^{j-1}(B_2')^{i-1}(w)$. Moreover, any linear combination $\alpha B_2 + \beta B_2'$ is nilpotent since it is given by a strictly lower-triangular matrix in the basis $\langle e_{1,1}, e_{1,2}, \ldots, e_{1,k}, e_{2,1}, e_{2,2} \ldots, e_{3,1} \ldots \rangle$ (i.e., we form a basis from the diagram by going from left to right, starting from the top row).

To construct $\{e_{i,j}\}$, recall one way to find a Jordan basis for B_1. Let $V_i = \mathrm{Ker}(B_1^{n-i})$. The subspaces V_i form a decreasing filtration $V = V_0 \supset V_1 \supset V_2 \ldots$. Moreover, $B_1 \cdot V_i \subset V_{i+1}$.

First, choose a basis $(w_1, \dots, w_{a_1})$ of $W_1 := V_0/V_1$ (if $W_1 = 0$ go to the next step). Lift this basis to some vectors $e_{1,1}, e_{1,2}, \dots, e_{1,a_1}$ in V_0. Secondly, choose a basis $(w_{a_1+1}, \dots, w_{a_2})$ of $W_2 := V_1/(B_1 \cdot V_0 + V_2)$ (again, if $W_2 = 0$ go to the next step). Lift this basis to some vectors $e_{1,a_1+1}, \dots, e_{1,a_2}$ in V_1. Continue in this manner by choosing bases of the spaces $W_{i+1} = V_i/(B_1 \cdot V_{i-1} + V_{i+1})$ and lifting them to V_i. This procedure gives us vectors $e_{1,1}, e_{1,2}, \dots, e_{1,k}$, and the sizes of Jordan blocks are given by $\mu_{a_i+1} = \dots = \mu_{a_{(i+1)}} = n - i$ (in particular, on the first few steps the spaces W_i are likely to be zero). The formula (a) tells us how to define $e_{i,j}$ for $i \geq 2$. It is easy to check that the system of vectors $\{e_{i,j}\}$ is in fact a basis of V.

To guarantee (b) as well we should go back and choose w_i more carefully. Note that all the subspaces V_i and $B_1 \cdot V_i$ are B_2-invariant. Therefore we have an induced action of B_2 on each of the W_i. We can choose our basis $(w_{a_{i-1}+1}, \dots, w_{a_i})$ of W_i in such a way that, for all $l \in \{a_{i-1}+1, \dots, a_i\}$, $B_2(w_l)$ is a linear combination of w_j with $j > l$. This ensures that (b) holds as well. $\square$

Theorem 4. *The subset U defined above is dense in $\mathcal{N}_2 \times V$.*

Proof. Let (B_1, B_2, v) be any point in $\mathcal{N}_2 \times V$. Consider the triple $(B_1, B_2', w) \in U$ provided by the above lemma. Connect the two triples (B_1, B_2, v) and (B_1, B_2', w) by an affine line L inside the vector space $gl(V) \oplus gl(V) \oplus V$. Due to the choice of B_2', the whole line L belongs to the closed subspace $\mathcal{N}_2 \times V \subset gl(V) \oplus gl(V) \oplus V$. Since $U \cap L$ is open in L and nonempty, (B_1, B_2, v) belongs to the closure of U. $\square$

Thus, to prove Theorem 1 it suffices to show that U is irreducible of dimension $n^2 + n - 1$.

4. The punctual Hilbert scheme

Note that $GL(V)$ acts on $\mathcal{N}_2 \times V$ by conjugating the operators and acting on the vector. The subset U is $GL(V)$-invariant by its definition.

Lemma 5. *The $GL(V)$ action on U is free.*

Proof. Suppose that $(B_1, B_2, w) \in U$ and $g \in GL(V)$ are such that $gB_ig^{-1} = B_i$, $i = 1, 2$ and $g(w) = w$. Then $W = \mathrm{Ker}(1 - g)$ is a subspace which contains w and is invariant with respect to B_1, B_2. By definition of U we have $W = V$, hence $g = 1$. $\square$

To interpret the space of orbits of $GL(V)$ on U, we consider the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{P}^2)$ of all quotient sheaves $\mathcal{O}_{\mathbb{P}^2} \to A$ of finite length n (cf. [G], [S]) on the projective plane $\mathbb{P}^2$. Let $s \in \mathbb{P}^2$ be a point and let (x, y) be local coordinates at s. Any triple $(B_1, B_2, w) \in U$ represents V as a quotient module $\phi : \mathbb{C}[x, y] \to V$ if we set $\phi\big[P(x, y)\big] = P(B_1, B_2)(w) \in V$ for any $P(x, y) \in \mathbb{C}[x, y]$. The map ϕ is surjective since w is a cyclic vector. The fact that B_1 and B_2 are nilpotent means that the corresponding quotient module $\mathcal{O}_{\mathbb{P}^2} \to V$ on $\mathbb{P}^2$ is supported at s. Therefore we obtain a map ψ from U to the fiber $\mathrm{Hilb}^n_{[s]} \subset \mathrm{Hilb}^n(\mathbb{P}^2)$ of the Hilbert–Chow morphism, cf. [F1], $\mathrm{Hilb}^n(\mathbb{P}^2) \to Sym^n(\mathbb{P}^2)$ over the point $n \cdot s \in Sym^n(\mathbb{P}^2)$ (we give $\mathrm{Hilb}^n_{[s]}$ the reduced scheme structure). Of course, we could have taken any smooth point on any surface instead of $\mathbb{P}^2$. One can easily check (cf. [N]) the following:

Lemma 6. *The fibers $\psi : U \to \mathrm{Hilb}^n_{[s]}$ are exactly the $GL(V)$-orbits on U.* $\square$

Hence the irreducibility of U is equivalent to the irreducibility of $\mathrm{Hilb}^n_{[s]}$. The latter was first proved in [B] in the case of characteristic zero. Later this result was extended in [I] to the case when $char\ k = p$ is large enough. The proof given below is a version of the proof in [Gra] (the use of Theorem 2 here replaces a dimension calculation of $\mathrm{Hilb}^n_{[s]}$ from [Gra]).

Theorem 7. $\mathrm{Hilb}^n_{[s]}$ *is irreducible of dimension* $(n-1)$.

Proof. Consider the universal subscheme $Z \subset \mathrm{Hilb}^n(\mathbb{P}^2) \times \mathbb{P}^2$. On the set-theoretic level Z is given by all pairs (A, x) such that $x \in Supp\ A$. By definition of the Hilbert scheme Z is finite and flat over $\mathrm{Hilb}^n(\mathbb{P}^2)$. Denote by Z_{n-1} the subscheme of all points in Z where n sheets of the map $f : Z \to \mathrm{Hilb}^n(\mathbb{P}^2)$ come together (i.e., Z_{n-1} is the $(n-1)$st ramification locus of f, cf. [GL] for a rigorous definition). Then $(Z_{n-1})_{red}$ is a locally trivial bundle over $\mathbb{P}^2$ with fibers isomorphic to $\mathrm{Hilb}^n_{[s]}$. Since Z is normal and $\mathrm{Hilb}^n(\mathbb{P}^2)$ is smooth (cf. [F2]), we can apply the following theorem due to Lazarsfeld (cf. [GL] for the statement of this result, the proof of which is contained in Lazarsfeld's PhD thesis):

Let $f : Z \to H$ be a finite surjective morphism of irreducible varieties, with Z normal and H nonsingular. If Z_{n-1} is not empty, then every irreducible component of Z_{n-1} has codimension $\leq (n-1)$ in Z.

Since in our case $\dim Z = 2n$, it follows that any irreducible component of $\mathrm{Hilb}^n_{[s]}$ should be at least $(n-1)$-dimensional. However, every irreducible component of $\mathrm{Hilb}^n_{[s]}$ of dimension k gives an irreducible component of $\mathcal{N}_2$ of dimension $k + \dim GL(V) - \dim V = k + n^2 - n \geq n^2 - 1$. Applying Theorem 2 in the case of $\mathfrak{g} = sl_n$ (when there is only one distinguished nilpotent class), we conclude that $\mathrm{Hilb}^n_{[s]}$ has a unique component of dimension $k = n - 1$. $\square$

Remark 1. In the case of a not necessarily algebraically closed field of characterstic $p > n$ our argument shows that that number of irreducible components of $\mathcal{N}_2$ is equal to the number of irreducible components of $\mathrm{Hilb}^n_{[s]}$. In particular, when k is algebraically closed and $char\ k > n$, the variety $\mathcal{N}_2$ is irreducible. For $char\ k \leq n$, as far as the author knows, the Hilbert–Chow morphism has not been greatly studied. Moreover, the Killing form used in the proof of Theorem 2 may become degenerate.

Remark 2. It is known that in the case of the real numbers $\mathrm{Hilb}^{2n}_{[s]}$ has at least $[n/2]$ irreducible components, cf. [I].

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