

ORBIFOLD COHOMOLOGY AS PERIODIC CYCLIC HOMOLOGY

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To the Memory of Andrey Nikolaevich Tyurin

It is known from the work of Feigin–Tsygan, Weibel and Keller that the cohomology groups of a smooth complex variety X can be recovered from (roughly speaking) its derived category of coherent sheaves. In this paper we show that for a finite group G acting on X the same procedure applied to G -equivariant sheaves gives the orbifold cohomology of X/G .

As an application, in some cases we are able to obtain simple proofs of an additive isomorphism between the orbifold cohomology of X/G and the usual cohomology of its crepant resolution (the equality of Euler and Hodge numbers was obtained earlier by various authors). We also state some conjectures on the product structures, as well as the singular case; and a connection with a recent work by Kawamata.

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1. Introduction

Let X be a smooth variety over a field k (for simplicity we assume in this introduction that $k = \mathbb{C}$) and G be a finite group acting on X . If the quotient variety X/G is Gorenstein (i.e. the canonical class is a Cartier divisor) and $\pi : Y \rightarrow X/G$ is a crepant resolution of singularities then Ruan’s cohomological crepant resolution conjecture (which we call cohomological conjecture for short) states, as a particular case, that the cohomology groups $H^*(Y, \mathbb{C})$ should be isomorphic to the orbifold cohomology

$$H_{\text{orb}}^*(X/G, \mathbb{C}) = \left(\bigoplus_{g \in G} H^*(X^g, \mathbb{C}) \right)_G$$

where $(\dots)_G$ denotes the coinvariants, X^g is the fixed point set, and G acts on the above direct sum by conjugating g . This definition of $H_{\text{orb}}^*(X, \mathbb{C})$ is slightly different from the usual one, see [26, Sec. 3], but equivalent to it.

Moreover, Ruan has introduced a product structure on $H^*(Y, \mathbb{C})$ (a deformation of the usual product using the rational curves contracted by π). On the other hand, $H_{\text{orb}}^*(X/G, \mathbb{C})$ has a product structure introduced by Chen and Ruan, see *loc. cit.* as well as [11, 30] for a simplified definition; and the cohomological conjecture states that $H^*(Y, \mathbb{C}) \simeq H_{\text{orb}}^*(X/G, \mathbb{C})$ is actually a ring isomorphism. On the level of Betti (or Hodge) numbers this conjecture was recently proved by Lupercio–Poddar and Yasuda, see [35]. However, with the approach used in the proof (motivic integration) it is not clear how to identify the actual cohomology groups with their product structures. In this paper we will try to outline a different approach to the cohomological crepant resolution conjecture and show that it is in fact a consequence of a categorical resolution conjecture stated (in a form and under a name slightly different from ours) by Kawamata in [19]. We hope that the categorical approach will allow to interpret the product structures. Besides, we believe that it would be very important to establish a link between the categorical and the motivic integration approach, relating the derived category of sheaves to the space of arcs (or possibly the space of formal loops of Kapranov–Vasserot).

For simplicity we only work with global quotients X/G but all statements can be made (and, hopefully, proved) for general smooth Deligne–Mumford stacks and categories of sheaves twisted by a gerbe.

In more detail: we would like to show that the above cohomological conjecture follows from an equivalence of two derived categories: the bounded derived category $D^b(Y)$ of coherent sheaves on Y and the bounded derived category $D_G^b(X)$ of G -equivariant sheaves on X (i.e. sheaves on the quotient stack $[X/G]$). Thus, a possible proof of the cohomological conjecture could consist of three steps:

- (1) Prove an equivalence of derived categories $D^b(Y) \rightarrow D_G^b(X)$ (categorical resolution conjecture — see end of Sec. 5).
- (2) Recover an isomorphism of cohomology groups $H^*(Y, \mathbb{C}) \rightarrow H_{\text{orb}}^*(X/G, \mathbb{C})$ from the above equivalence.
- (3) Identify the two product structures (perhaps with some corrections).

In this paper we mostly deal with the second step. As for the other two, we note that (1) is known in some cases due to the work of Bridgeland–King–Reid and Kawamata (see [4, 19] and Sec. 4 of this paper); while the orbifold product in (3) seems to arise from the convolution product of sheaves (see Sec. 5).

To deal with (2) one needs a construction which recovers the (orbifold) cohomology ring from the (equivariant) derived category. In a sense, we are using some additional structure: the derived category should “remember” that it was obtained as a quotient of two DG-categories forming a *localization pair*, see [20, Sec. 2.4]. However, it follows from a result of Orlov [25] that a fully faithful exact

functor between the (non-equivariant) derived categories automatically preserves this additional structure; and the same holds for equivalences of equivariant derived categories (in the case of a finite group action).

In the non-equivariant case a construction recovering the cohomology groups follows from the work of Feigin–Tsygan, Weibel and Keller. In fact, for any exact category \mathcal{A} which is linear over a commutative ring k Keller constructs in [20] a *mixed complex* $C(\mathcal{A})$ which leads to a family of homology theories $HC_\bullet(\mathcal{A}, W)$ depending on a graded $k[u]$ -module W (usually taken to be of finite projective dimension). We will be interested in the case when \mathcal{A} is the category $\text{Vect}(Y)$ of vector bundles on Y or the category $\text{Vect}_G(X)$ of G -equivariant vector bundles on X ; since k is assumed to be a field, the assumption of “finite projective dimension” is satisfied automatically.

We “recall” the relevant definitions in Sec. 2. Two properties make this construction very attractive in our setup:

- When \mathcal{A} is the category of vector bundles on Y , $k = \mathbb{C}$ and $W = k[u, u^{-1}]$, the above homology group $HC_0(\mathcal{A}, W)$ (respectively $HC_1(\mathcal{A}, W)$) can be identified with $H^{\text{even}}(Y, \mathbb{C})$ (respectively $H^{\text{odd}}(Y, \mathbb{C})$). Note that multiplication by u gives an isomorphism $HC_i(\mathcal{A}, W) \simeq HC_{i+2}(\mathcal{A}, W)$.
- For any W the homology theory $HC_\bullet(\mathcal{A}, W)$ is invariant with respect to equivalences of derived categories coming from functors between localization pairs.

To formulate our results, recall that G acts on $\coprod_{g \in G} X^g$: $h \in G$ sends $x \in X^g$ to $hx \in X^{hgh^{-1}}$. This action is inherited by $\bigoplus_{g \in G} HC_\bullet(\text{Vect}(X^g), W)$.

Theorem 1.1. *Let G be a finite group acting on a smooth quasiprojective variety X over a field k of characteristic not dividing $|G|$. For any graded $k[u]$ -module W there exists an isomorphism functorial with respect to pullbacks under G -equivariant maps:*

$$\psi_X : HC_\bullet(\text{Vect}_G(X), W) \simeq \left(\bigoplus_{g \in G} HC_\bullet(\text{Vect}(X^g), W) \right)_G$$

where $(\dots)_G$ denotes the coinvariants.

In the C^∞ -manifold or C^∞ -etale groupoid setting this result (formulated in terms of modules over smooth functions rather than categories) has a long history. First Feigin–Tsygan, see [12], constructed a spectral sequence computing cyclic homology of a general crossed product algebra, which was later reformulated by Getzler–Jones, see [14]; and also [1] for more general crossed products by Hopf algebras. When the crossed product algebra comes from functions on a smooth manifold the E_2 term of this spectral sequence can be interpreted in terms of fixed point submanifolds: this result was announced in [8] and the first published proof appears in [6]. Later it was generalized in [10]. The case when G is a Lie group was studied by Nistor in [24]; later Block and Getzler have related the corresponding crossed product cyclic homology groups to equivariant differential forms and fixed

points, see [3]. We also mention a closely related computation of G -equivariant topological K -theory by G. Segal, see [18]; and its algebraic K -theory counterparts [28, 31].

In Sec. 3, we adapt the proofs in [3, 14] to fit our case of categories and rings of regular functions. Note that the proof of [31] cannot be applied in our case due to the failure of devissage, see [20, Example 1.11].

In those cases when the derived equivalence $D^b(Y) \rightarrow D_G^b(X)$ is known, we get an isomorphism between $HC_\bullet(\text{Vect}(Y), W)$ and the right hand side expression in Theorem 1.1, obtaining a slightly generalized version of the cohomological conjecture (in general the cyclic homology groups *do not* satisfy the long exact sequence hence even on the level of dimensions the equality cannot be derived using motivic measures and motivic integration). This is the second main result of this paper (see Corollary 4.1).

We expect that, in order to identify the product structures in step (3) above, one should modify ψ_X to make it compatible with pushforwards under G -equivariant closed embeddings, rather than pullbacks (compare with [31, Lemmas 4.2 and 4.3]).

We note here that for a general algebraic group G the equivariant derived category $D_G^b(X)$ is defined *not* by taking complexes of G -equivariant sheaves but by a more delicate localization procedure, see [5]. One can expect that the corresponding cyclic homology groups satisfy nice properties, for example, similar to those proved in [3].

This paper is organized as follows. Section 2 gives some basic information of cyclic homology groups of exact categories. In Sec. 3, we prove Theorem 1.1. In Sec. 4, we show how an equivalence of derived categories implies equality of (orbifold) cohomology groups and also give some examples in which this equivalence is known. Finally, in Sec. 5, we give a conjecture about the singular case and a conjecture on how the orbifold cohomology product can be recovered from the convolution product in the derived category.

2. Generalities on Mixed Complexes

Recall, (cf. [32]) that a *mixed complex* over a commutative ring k is a sequence of k -modules $\{C_m : m \geq 0\}$ with two families of morphisms $b : C_m \rightarrow C_{m-1}$ and $B : C_m \rightarrow C_{m+1}$ satisfying $b^2 = B^2 = Bb + bB = 0$. To any such mixed complex one can apply the following formalism (see [13]): let W be a graded module over the polynomial ring $k[u]$, where $\deg(u) = -2$ (in practice it is always assumed that W has finite homological dimension). Then one can form a complex $C[[u]] \otimes_{k[u]} W$ with a differential $b + uB$ and compute its homology groups, to be denoted by $HC_\bullet(C, W)$. The following are important examples:

- $W = k[u]/uk[u]$ gives the Hochschild homology $HH_\bullet(C)$
- $W = k[u, u^{-1}]/uk[u]$ gives cyclic homology $HC_\bullet(C)$
- $W = k[u, u^{-1}]$ gives periodic cyclic homology $HP_\bullet(C)$
- $W = k[u]$ gives negative cyclic homology $HN_\bullet(C)$ (sometimes also denoted by $HC_\bullet^-(C)$).

The following lemma shows that for some purposes it suffices to consider only the first case.

Lemma 2.1. *Let $f : (C, b, B) \rightarrow (C', b', B')$ be a map of mixed complexes such that f induces an isomorphism $H(C, b) \rightarrow H(C', b')$. Then for any coefficients W of finite projective dimension over $k[u]$,*

$$f : H_{\bullet}(C[[u]] \otimes_{k[u]} W, b + uB) \rightarrow H_{\bullet}(C'[[u]] \otimes_{k[u]} W, b' + uB')$$

is an isomorphism.

Proof. See [13, Proposition 2.4].

This lemma justifies the following point of view on mixed complexes, see [20, Sec. 1.2]. Let Λ be the DG-algebra generated by an indeterminate ε of chain degree 1 with $\varepsilon^2 = 0$ and $d\varepsilon = 0$. Then a mixed complex may be identified with a left Λ -module whose underlying DG k -module is (C, b) and where ε acts by B . Moreover, if we are interested only in the resulting homology groups (as is the case in this paper), we can view a mixed complex as an object in the derived category of the DG algebra Λ .

In what follows we will need a definition of the *mapping cone* over a map $f : C \rightarrow C'$ of mixed complexes. It is given by the mixed complex

$$\left(C' \oplus C[1], \begin{bmatrix} b_{C'} & f \\ 0 & -b_C \end{bmatrix}, \begin{bmatrix} B_{C'} & 0 \\ 0 & -B_C \end{bmatrix} \right).$$

Now we briefly recall Keller’s construction of the mixed complex $C(\mathcal{A})$ of an exact category \mathcal{A} over a field k (actually in [20], the complex is defined for any commutative ring k but the general definition is somewhat more involved). Starting from \mathcal{A} one can construct its category $\mathcal{C}^b\mathcal{A}$ of all bounded complexes over \mathcal{A} and the category $\mathcal{A}c^b\mathcal{A}$ of bounded acyclic complexes over \mathcal{A} . Both $\mathcal{C}^b\mathcal{A}$ and $\mathcal{A}c^b\mathcal{A}$ are *DG categories*, i.e. for any pair of objects X, Y the group $\text{Hom}(X, Y)$ is \mathbb{Z} -graded (by degree of a map) with a differential, which satisfies some natural axioms (see [22]).

For any small DG category \mathcal{B} over a field k Keller constructs a mixed complex as follows. Denote for notational convenience $\text{Hom}_{\mathcal{B}}(X, Y)$ by $(X \rightarrow Y)$ and consider for each $n \in \mathbb{N}$ a vector space

$$C_n(\mathcal{B}) = \bigoplus (B_0 \rightarrow B_1) \otimes (B_1 \rightarrow B_2) \otimes \cdots \otimes (B_{n-1} \rightarrow B_n) \otimes (B_n \rightarrow B_0)$$

where the sum runs over all sequences B_0, \dots, B_n of objects of \mathcal{B} . The face maps

$$d_i(f_0, \dots, f_i, f_{i+1}, \dots, f_n) = \begin{cases} (f_0, \dots, f_i f_{i+1}, \dots, f_n) & \text{if } 0 \leq i \leq n-1 \\ (-1)^{n+\sigma} (f_n f_0, \dots, f_{n-1}) & \text{if } i = n \end{cases}$$

(where $\sigma = \text{deg } f_n \cdot (\text{deg } f_0 + \cdots + \text{deg } f_{n-1})$); together with the degeneracy maps

$$s_i(f_0, \dots, f_i, f_{i+1}, \dots, f_n) = (f_0, \dots, f_i, \text{id}_{B_i}, f_{i+1}, \dots, f_n) \quad i = 0, \dots, n$$

and the cyclic operator

$$t_n(f_0, \dots, f_n) = (-1)^{n+\sigma} (f_n, f_0, f_1, \dots, f_{n-1}),$$

define a mixed complex $(C(\mathcal{B}), b, (1 - t)sN)$ as in [14, Sec. 2]. Note that, unlike in [20], we write fg for a composition of $f : A \rightarrow B$ and $g : B \rightarrow C$ instead of the usual gf . This non-traditional notation will allow us later to match our formulas with those of Getzler–Jones in [14]. □

Next, for any DG subcategory $\mathcal{C} \subset \mathcal{B}$ one has a mixed complex

$$C(\mathcal{C}, \mathcal{B}) = \text{Cone}(C(\mathcal{C}) \rightarrow C(\mathcal{B})).$$

We will always use this definition when $(\mathcal{C}, \mathcal{B})$ is a *localization pair* in the sense of [20, Sec. 2.4]. Every such localization pair leads to a triangulated category $\mathcal{T} = \mathcal{B}/\mathcal{C}$ associated to it. For example, the pair $(\mathcal{A}c^b\mathcal{A}, \mathcal{C}^b\mathcal{A})$ gives rise to the derived category \mathcal{T} of \mathcal{A} . One of the main properties of the mixed complex $C(\mathcal{C}, \mathcal{B})$ is expressed in the following proposition (see [20, Theorem 2.4(a)]):

Proposition 2.1. *Let $(\mathcal{C}, \mathcal{B})$ and $(\mathcal{C}', \mathcal{B}')$ be two localization pairs and $\mathcal{T}, \mathcal{T}'$ their derived categories. If $F : \mathcal{B} \rightarrow \mathcal{B}'$ is an exact functor which takes \mathcal{C} to \mathcal{C}' and induces an equivalence up to factors $\mathcal{T} \rightarrow \mathcal{T}'$ (cf. [20, 1.5]), then F induces an isomorphism $C(\mathcal{C}, \mathcal{B}) \rightarrow C(\mathcal{C}', \mathcal{B}')$ in the derived category of Λ .*

If \mathcal{A} is an exact category then its mixed complex is defined as $C(\mathcal{A}c^b\mathcal{A}, \mathcal{C}^b\mathcal{A})$. We will write $\mathcal{A}c^b(X)$ (respectively $\mathcal{C}^b(X)$) instead of $\mathcal{A}c^b \text{Vect}(X)$ (respectively $\mathcal{C}^b \text{Vect}(X)$) and similarly for $\mathcal{A}c_G^b(X)$ and $\mathcal{C}_G^b(X)$. Define $C(X) = C(\mathcal{A}c^b(X), \mathcal{C}^b(X))$; $C_G(X) = C(\mathcal{A}c_G^b(X), \mathcal{C}_G^b(X))$.

3. Mixed Complex for a Finite Group Action

In this section we will prove Theorem 1.1. In view of Lemma 2.1 and the subsequent remarks, it suffices to prove the following proposition:

Proposition 3.1. *Let G be a finite group acting on a smooth quasiprojective variety X over a field k of characteristic not dividing $|G|$. There exists a quasiisomorphism in the derived category of Λ :*

$$\psi_X : C_G(X) \rightarrow \left(\bigoplus_{g \in G} C(X^g) \right)_G$$

which is functorial with respect to pullbacks under G -equivariant maps of smooth varieties $f : Y \rightarrow X$.

The proof of the above proposition will be carried out in several steps: first we replace $\text{Vect}_G(X)$ by a categorical analogue of a crossed product ring $A \rtimes G$ and construct a functorial map of objects in the derived category (Steps 1 and 2). Once this map is constructed, we use Mayer–Vietoris, Luna’s fundamental lemma and etale descent of Weibel–Geller to reduce to the case when X is a vector space with a linear action of G ; and conclude the argument using exactness of a Koszul complex (Steps 3–6). Our proof is an adaptation of the argument in [3] from the differentiable to the algebraic situation.

Step 1: Change of Categories

Note that the action of G on X is inherited by the categories $\text{Vect}_G(X)$, $\mathcal{C}^b(X)$ and $\mathcal{A}c^b(X)$: for any $g \in G$ and any bundle (or a complex of bundles) \mathcal{F} we can consider $g\mathcal{F} := (g^{-1})^*\mathcal{F}$ and for any $\psi : \mathcal{F} \rightarrow \mathcal{G}$ we have $g(\psi) : g\mathcal{F} \rightarrow g\mathcal{G}$.

For any \mathcal{F} consider the object $\tilde{\mathcal{F}} = \bigoplus_{g \in G} g\mathcal{F}$ with its natural G -equivariant structure. Then each object \mathcal{H} in $\mathcal{C}^b \text{Vect}_G(X)$ is isomorphic to a direct factor of some $\tilde{\mathcal{F}}$. In fact, take $\mathcal{F} = \mathcal{H}$ viewed as an object in $\mathcal{C}^b \text{Vect}(X)$. Then the G -equivariant structure on \mathcal{H} defines isomorphisms $i_g : \mathcal{H} \rightarrow g\mathcal{H}$ for all $g \in G$. Now consider the G -equivariant maps

$$\mathcal{H} \xrightarrow{a} \tilde{\mathcal{H}} = \bigoplus_{g \in G} g\mathcal{H} \xrightarrow{b} \mathcal{H}$$

where a is given by a direct sum of i_g and $b = \frac{1}{|G|} \sum i_g$. Since $b \circ a = \text{id}_{\mathcal{H}}$, \mathcal{H} splits off as a direct factor of $\tilde{\mathcal{H}}$.

Denote by $\mathcal{C}^b(X) \rtimes G$, respectively $\mathcal{A}c^b(X) \rtimes G$ the full subcategory of $\mathcal{C}^b_G(X)$, respectively $\mathcal{A}c^b_G(X)$, formed by all $\tilde{\mathcal{F}}$. Since $\mathcal{C}^b(X) \rtimes G$ and $\mathcal{A}c^b(X) \rtimes G$ are closed under degree-wise split extensions and shifts, both are exact DG categories (by [20, Example 2.2(a)]). The above argument shows that the natural embedding $\mathcal{C}^b(X) \rtimes G \rightarrow \mathcal{C}^b_G(X)$ induces an equivalence *up to factors* of associated derived categories. Now [20, Theorem 2.4(a)] gives

Proposition 3.2. *There exists an isomorphism in the mixed derived category*

$$C(\mathcal{A}c^b(X) \rtimes G, \mathcal{C}^b(X) \rtimes G) \rightarrow C_G(X)$$

which is functorial with respect to G -equivariant pullbacks of vector bundles.

Step 2: Construction of the Quasiisomorphism

To proceed further, we take a closer look at $\mathcal{C}^b(X) \rtimes G$. The objects in this category can be identified with objects in $\mathcal{C}^b(\text{Vect}(X))$ while the morphisms are given by

$$(\mathcal{F} \rightarrow \mathcal{G})_{\mathcal{C}^b(X) \rtimes G} = \bigoplus_{g \in G} (\mathcal{F} \rightarrow g\mathcal{G})_{\mathcal{C}^b(X)}$$

(the expression on the right obviously coincides with all G -equivariant morphisms from $\tilde{\mathcal{F}}$ to $\tilde{\mathcal{G}}$). If we denote by $\psi \cdot g$ and $\varphi \cdot h$ the components of $(\mathcal{F} \rightarrow \mathcal{G})_{\mathcal{C}^b(X) \rtimes G}$ and $(\mathcal{G} \rightarrow \mathcal{H})_{\mathcal{C}^b(X) \rtimes G}$ living in $(\mathcal{F} \rightarrow g\mathcal{G})_{\mathcal{C}^b(X)}$ and $(\mathcal{G} \rightarrow h\mathcal{H})_{\mathcal{C}^b(X)}$, respectively, then the composition of $\psi \cdot g$ and $\varphi \cdot h$ is given by

$$\psi g(\varphi) \cdot gh : \mathcal{F} \rightarrow g\mathcal{G} \rightarrow gh\mathcal{H}.$$

Now define the map

$$C(\mathcal{C}^b(X) \rtimes G) \rightarrow \left(\bigoplus_{g \in G} C(\mathcal{C}^b(X^g)) \right)_G$$

by sending $(\varphi_0 \cdot g_0, \varphi_1 \cdot g_1, \dots, \varphi_n \cdot g_n)$ to $(\varphi_0|_{X^g}, g_0(\varphi_1)|_{X^g}, g_0g_1(\varphi_2)|_{X^g}, \dots, (g_0 \cdots g_{n-1})(\varphi_n)|_{X^g})$ where $g = g_0 \cdots g_n$. Note that only the map to the coinvariants is indeed a map of mixed complexes since the individual maps $C(\mathcal{C}^b(X) \rtimes G) \rightarrow C(\mathcal{C}^b(X^g))$ do not respect the last face map d_n and the cyclic operator t_n . Since the above morphism obviously sends $C(\mathcal{A}c^b(X) \rtimes G)$ to $(\bigoplus_{g \in G} C(\mathcal{A}c^b(X^g)))_G$, we obtain a morphism of mixed complexes

$$C(\mathcal{A}c^b(X) \rtimes G, \mathcal{C}^b(X) \rtimes G) \rightarrow \left(\bigoplus_{g \in G} C(\mathcal{A}c^b(X^g), \mathcal{C}^b(X^g)) \right)_G.$$

Composing this map with the inverse of the quasiisomorphism in Proposition 3.2, we obtain a morphism of objects in the derived category of Λ

$$\psi_X : C_G(X) \rightarrow \left(\bigoplus_{g \in G} C(X^g) \right)_G$$

which is functorial with respect to pullbacks under G -equivariant maps $f : Y \rightarrow X$.

Step 3: Mayer–Vietoris and Luna’s Fundamental Lemma

Now we use a Mayer–Vietoris sequence to reduce to the case when X is affine.

Proposition 3.3. *Let X be a quasiprojective scheme, $V, W \subset X$ two G -invariant open subschemes and $U = V \cap W$. There is a distinguished triangle in the mixed derived category:*

$$C_G(X) \rightarrow C_G(V) \oplus C_G(W) \rightarrow C_G(U) \rightarrow C_G(X)[1].$$

Outline of the Proof. Most of the argument is identical to the non-equivariant case proved in [21, Proposition 5.8]. First one shows that for a quasiprojective scheme X , $C_G(X)$ is quasiisomorphic to the complex obtained from the category of G -equivariant perfect complexes (see [21, Sec. 5.1]). Moreover, let $\mathcal{T}_G(X)$ be the derived category of G -equivariant perfect complexes and for any closed $Z \subset X$ by $\mathcal{T}_G(X \text{ on } Z)$ the subcategory of complexes which are exact on the complement to Z . If $Z = X \setminus W$ and $j : V \rightarrow X$ is the open embedding, one shows that the lines of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{T}_G(X \text{ on } Z) & \longrightarrow & \mathcal{T}_G X & \longrightarrow & \mathcal{T}_G W & \longrightarrow & 0 \\ & & j^* \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_G(V \text{ on } Z) & \longrightarrow & \mathcal{T}_G V & \longrightarrow & \mathcal{T}_G(V \cap W) & \longrightarrow & 0 \end{array}$$

are exact up to factors and the functor j^* is an equivalence up to factors (the proof in [29, Secs. 5.4 and 5.5] may be repeated almost word-by-word; note also that working “up to factors” allows one to ignore all problems with non-surjectivity of $K_G^0(X) \rightarrow K_G^0(W)$ and so on, since for any complex E , the sheaf $E \oplus E[1]$ has zero class in K -theory). Finally, one applies [20, Theorem 2.7].

Alternatively, one can work with $\mathcal{C}^b(X) \rtimes G$ and deduce the above fact from the non-equivariant version [20, Proposition 5.8]. □

The above Mayer–Vietoris sequence shows that if ψ_U, ψ_V and ψ_W are quasiisomorphisms then the same holds for ψ_X (of course, this is only possible by functoriality of ψ_X applied to open embeddings). By induction on the number of elements in a G -invariant affine covering of X we can assume that $X = \text{Spec } A$ is affine. In this case we can reinterpret the map ψ_X as follows. The group G acts on the k -algebra A and we can define $A \rtimes G$ to be the crossed product algebra $A \otimes k[G]$ with the product defined by

$$(a_1 \cdot g_1)(a_2 \cdot g_2) = (a_1 g_1(a_2)) \cdot g_1 g_2.$$

Then the category $\text{Vect}_G(X)$ is equivalent to the category of $A \rtimes G$ -modules which are projective as A -modules. Since G is finite, this is equivalent to being projective as $A \rtimes G$ -modules.

Now consider the category $dgfree A \rtimes G$ of complexes of free $A \rtimes G$ -modules. By [20, Sec. 2.4], the natural functor $dgfree A \rtimes G \rightarrow \mathcal{C}^b \text{Vect}_G(X)$ induces a quasiisomorphism of mixed complexes

$$C(0, dgfree A \rtimes G) \rightarrow C_G(X).$$

Moreover, if one considers $A \rtimes G$ as a subcategory of $dgfree A \rtimes G$ with one object (a free rank one module viewed as a complex in degree 0), then by [20, Theorem 2.4(a)], one gets a quasiisomorphism

$$C(A \rtimes G) \rightarrow C(0, dgfree A \rtimes G)$$

where the mixed complex $C(B)$ of any k -algebra B is defined as in Sec. 2 (if one views B as a category with one object). Thus, we obtain a chain of quasiisomorphisms

$$C(A \rtimes G) \rightarrow C(0, dgfree A \rtimes G) \rightarrow C(\mathcal{A}c^b(X) \rtimes G, \mathcal{C}^b(X) \rtimes G) \rightarrow C_G(X)$$

induced by embeddings of subcategories.

Similarly, if $g \in G$ and $J_g \subset A$ denotes the ideal of the fixed point set $X^g \subset X$ then we have a chain of quasiisomorphisms

$$C(A/J_g) \rightarrow C(0, dgfree A/J_g) \rightarrow C(X^g).$$

Restricting the map ψ_X constructed in the previous step, it suffices to prove that the following map is a quasiisomorphism in the derived category of Λ :

$$\psi_A : C(A \rtimes G) \rightarrow \left(\bigoplus_{g \in G} C(A/J_g) \right)_G,$$

where $\psi_A(a_0 \cdot g_0, \dots, a_n \cdot g_n)$ is the image of $(a_0, g_0(a_1), \dots, (g_0 \cdots g_{n-1})(a_n)) \in A^{\otimes n+1}$ in $(A/J_g)^{\otimes n+1} = C_n(A/J_g)$ and $g = g_0 \cdots g_n$.

Eventually we will assume that X is not only affine but has some additional properties. The following proposition is probably well-known but the author was unable to find a convenient reference.

Proposition 3.4. *Let G be a finite group acting on a smooth quasiprojective variety X over a field k . Assume that $|G|$ is invertible in k . There exists a covering of X by G -invariant affine open subsets U_1, \dots, U_n such that for any $i = 1, \dots, n$ there is a point $x_i \in U_i$ satisfying the following properties:*

- (a) *the fixed point scheme U_i^g is empty unless $g \in G_{x_i}$ (the stabilizer of x_i);*
- (b) *if T_{x_i} is the tangent space to U_i at x_i with its natural G_{x_i} action, then there exists a G_{x_i} -equivariant etale morphism $\varphi_i : U_i \rightarrow T_{x_i}$ such that for any subgroup $H \subset G_{x_i}$ the induced morphism $U_i/H \rightarrow T_{x_i}/H$ is etale and the diagram*

$$\begin{array}{ccc}
 U_i & \longrightarrow & T_{x_i} \\
 \downarrow & & \downarrow \\
 U_i/H & \longrightarrow & T_{x_i}/H
 \end{array}$$

is Cartesian;

- (c) *for any $H \subset G_{x_i}$ the fixed point scheme U_i^H is a scheme-theoretic preimage of $T_{x_i}^H$.*

Proof. For any point $x \in X$ we can choose an open affine G -invariant neighborhood U such that (a) holds. By [9, Lemma 8.3], there exists a G_x -equivariant map $\varphi : U \rightarrow T_x$ such that $\varphi(x) = 0$ and $d\varphi(x)$ is equal to identity. Now by [9, Theorem 6.2] (Luna’s fundamental lemma in finite characteristic) applied to the finite set of subgroups $H \in G_{x_i}$ we can shrink U so that (b) holds as well.

For (c) let $U_i = \text{Spec } B$ and $T_{x_i} = \text{Spec } A$. Denote by I_A the A^H -submodule of A generated by elements $a - h(a)$, $a \in A, h \in H$. Let also J_A be the ideal in A generated by I_A ; then J_A is the ideal of the fixed point scheme $T_{x_i}^H$. We use the similar notation I_B, J_B for the objects corresponding to B . Since B is flat over A , $J_A \otimes_A B$ is naturally an ideal in $B = A \otimes_A B$ and it suffices to prove that it coincides with J_B (*a priori* we just have an inclusion $J_A \otimes_A B \subset J_B$). Note that since $|H|$ is invertible $A = I_A \oplus A^H$ as A^H -modules. Applying $\otimes_{A^H} B^H$ and using the fact that $B = A \otimes_{A^H} B^H$ by Part (b) we conclude that $I_B = I_A \otimes_{A^H} B^H$ which implies $J_B = J_A \otimes_A B$ by definition of J_A, J_B . □

Thus, later we may assume that X is affine and there exists $x \in X$ such that (a), (b) and (c) above are satisfied.

Step 4: Eilenberg–Zilber Theorem

The mixed complex $C(A \rtimes G)$ was studied in a more general situation by Jones and Getzler [14]. In the next proposition we present their results (with some simplifications possible due to the fact that $|G|$ is invertible in k ; also our isomorphism is more explicit than in [14]).

To state the lemma we need to fix some notation. For any $g \in G$ consider the sequence of vector spaces $(A_g^h)_n = A^{\otimes n+1}$ together with the face, degeneracy and

cyclic operators defined by the formulas similar to those in Sec. 2:

$$d_i(a_0, \dots, a_i, a_{i+1}, \dots, a_n) = \begin{cases} (a_0, \dots, a_i a_{i+1}, \dots, a_n) & \text{if } 0 \leq i \leq n-1 \\ (g^{-1}(a_n) a_0, \dots, a_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(a_0, \dots, a_n) = (a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n) \quad i = 0, \dots, n,$$

$$t_n(a_0, \dots, a_n) = (g^{-1}(a_n), a_0, \dots, a_{n-1}).$$

Note that the operator t_n does not satisfy the cyclic identity $t^{n+1} = 1$. However, we can still construct a mixed complex from the spaces A_g^{\natural} by considering a G -action on the direct sum $\bigoplus_{g \in G} A_g^{\natural}$ such that $h \in G$ sends $(a_0, \dots, a_n) \in A_g^{\natural}$ to $(h(a_0), \dots, h(a_n)) \in A_{hgh^{-1}}^{\natural}$. Then this G -action commutes with d_i, s_i, t hence we obtain the face, degeneracy and cyclic operators on the quotient space $(\bigoplus_g A_g^{\natural})_G$ of G -coinvariants. In this quotient space the operator t_n does satisfy $t^{n+1} = 1$ and we denote the resulting mixed complex by $C(\bigoplus_{g \in G} A_g^{\natural})_G$.

Proposition 3.5. *The map of mixed complexes*

$$\varphi_A : C(A \rtimes G) \rightarrow C\left(\bigoplus_{g \in G} A_g^{\natural}\right)_G$$

defined by $\varphi(a_0 \cdot g_0, \dots, a_n \cdot g_n) = (a_0, g_0(a_1), g_0 g_1(a_2), \dots, g_0 \cdots g_{n-1}(a_n)) \in (A_g^{\natural})_n$ with $g = g_0 \cdots g_n$; is a quasiisomorphism in the derived category of Λ .

Proof. In [14], Getzler and Jones define a bi-graded object $A\sharp G(p, q) = k[G^{p+1}] \otimes A^{\otimes q+1}$ with two families of face maps $d^h : A\sharp G(p, q) \rightarrow A\sharp G(p-1, q)$ (horizontal maps) and $d^v : A\sharp G(p, q) \rightarrow A\sharp G(p, q-1)$ (vertical maps), and similarly for degeneracies and cyclic operators (we will not use the precise definitions of these operators). These two families of operators give $A\sharp G(p, q)$ the structure of a cylindrical module, see the part before [14, Proposition 1.1].

This cylindrical module has a total complex $\text{Tot}_n(A, G) = \bigoplus_{p+q=n} A\sharp G(p, q)$ and a diagonal complex $\Delta_n(A, G) = A\sharp G(n, n)$, both being mixed complexes (see [14] for more details). The cylindrical module structure on $A\sharp G(p, q)$ is defined in such a way that $C(A \rtimes G)$ is isomorphic to the diagonal complex $\Delta(A, G)$ via the map

$$(a_0 \cdot g_0, \dots, a_n \cdot g_n) \mapsto (g_0, \dots, g_n | h_0^{-1} a_0, \dots, h_n^{-1} a_n) \in A\sharp G(n, n) = \Delta_n(A, G)$$

where $h_i = g_i \cdots g_n$. Applying the Eilenberg–Zilber theorem for paracyclic modules, see [14, Theorem 3.1], one gets an explicit quasiisomorphism

$$AW : \Delta(A, G) \rightarrow \text{Tot}(A, G)$$

given by the Alexander–Whitney map defined in [32, Sec. 8.5.2]. We will only need one component of this map (with values in $A\sharp G(0, n)$) which is given simply by $d_1^h \cdots d_n^h$, where $d_i^h : A\sharp G(i, q) \rightarrow A\sharp G(i-1, q)$ sends $(g_0, \dots, g_i | a_0, \dots, a_q)$ to $(g_i g_0, \dots, g_{i-1} | g_i(a_0), \dots, g_i(a_q))$.

Now the transformation

$$(g_0, \dots, g_p | a_0, \dots, a_q) \mapsto (g_1, \dots, g_p | g_0 g_1 \cdots g_p | a_0, \dots, a_q)$$

identifies $A\sharp G(p, q)$ with a cylindrical module the vertical maps of which are given by the above operators on $\bigoplus_{g \in G} A_g^\sharp$ (and the index g corresponds to $(\cdots | g | \cdots)$ in the above notation), while the rows can be identified with the standard homological complex of G acting on $\bigoplus_{g \in G} A_g^\sharp$. See [14, Sec. 4] for details. Since $|G|$ is invertible in k , the projection of $\text{Tot}(A, G)$ onto its first column $A\sharp(0, \bullet)$ together with the projection to the coinvariants, gives a quasiisomorphism

$$\text{Tot}(A, G) \rightarrow C\left(\bigoplus_{g \in G} A_g^\sharp\right)_G.$$

The fact that it is indeed a quasiisomorphism can be proved using the homotopy

$$h : A\sharp G(p, q) \rightarrow A\sharp G(p + 1, q),$$

$$(g_1, \dots, g_p | g | a_0 \cdots a_q) \mapsto \frac{1}{|G|} \sum_{g' \in G} (g', g_1, \dots, g_p | g | a_0, \dots, a_q).$$

The composition of (quasi)isomorphisms:

$$C(A \rtimes G) \rightarrow \Delta(A, G) \rightarrow \text{Tot}(A, G) \rightarrow C\left(\bigoplus_g A_g^\sharp\right)_G$$

is given by

$$(a_0 \cdot g_0, \dots, a_n \cdot g_n) \rightarrow (g_0^{-1}(a_0), a_1, g_1(a_2), \dots, (g_1 \cdots g_{n-1})(a_n)) \in (A_g^\sharp)_n,$$

$$g = g_1 g_2 \cdots g_n g_0.$$

Note that this differs from the map in the statement of our proposition exactly by the action of $g_0 \in G$. Since on the space of coinvariants g_0 acts by identity, this finishes the proof. □

Thus, we have reduced Proposition 3.1 to the claim that the map

$$C\left(\bigoplus_{g \in G} A_g^\sharp\right)_G \rightarrow \left(\bigoplus_{g \in G} C(A/J_g)\right)_G$$

defined by the natural surjections $(A_g^\sharp)_n \rightarrow (A/J_g)^{\otimes n+1}$, is a quasiisomorphism. Note also that we have not used the smoothness assumption yet.

Step 5: Shapiro’s Lemma and Etale Descent

Let $\mathcal{O} \subset G$ be a conjugacy class. It is easy to see that $\bigoplus_{g \in \mathcal{O}} A_g^\sharp$ and $\bigoplus_{g \in \mathcal{O}} C(A/J_g)$ are G -invariant subspaces, hence it suffices to prove the quasiisomorphism

$$C\left(\bigoplus_{g \in \mathcal{O}} A_g^\sharp\right)_G \rightarrow \left(\bigoplus_{g \in \mathcal{O}} C(A/J_g)\right)_G$$

for all conjugacy classes \mathcal{O} . To that end, choose $g \in \mathcal{O}$ and denote by $C_g = \{h \in G \mid gh = hg\}$ the centralizer of g . Then C_g acts on A_g^{\natural} ; and on the space of coinvariants $(A/J_g)_{C_g}$ the cyclic operator t_n satisfies $t_n^{n+1} = 1$ therefore we obtain a mixed complex $C(A_g^{\natural})_{C_g}$. Moreover, we have natural isomorphisms of G -modules

$$\bigoplus_{g' \in \mathcal{O}} A_{g'}^{\natural} \simeq \text{Ind}_{C_g}^G C(A/J_g); \quad \bigoplus_{g' \in \mathcal{O}} C(A/I_{g'}) \simeq \text{Ind}_{C_g}^G C(A/J_g)$$

where $\text{Ind}_{C_g}^G$ denotes the induction map from C_g -modules to G -modules. By Shapiro’s lemma we are reduced to proving the quasiisomorphism

$$C(A_g^{\natural})_{C_g} \simeq (C(A/J_g))_{C_g},$$

which would follow once we prove that the natural surjection $A \rightarrow A/J_g$ induces a quasiisomorphism

$$C(A_g^{\natural})_{\langle g \rangle} \simeq C(A/J_g)$$

where $\langle g \rangle \subset C_g$ is the cyclic subgroup generated by g . Note that $\langle g \rangle$ trivially on A/J_g (in fact, J_g is generated by elements $a - g(a)$ with $a \in A$), so we do not have to take coinvariants on the right hand side.

Now we finally use Proposition 3.4 (and the smoothness assumption which stands behind it). Since the properties (a), (b) and (c) are preserved by finite intersections of G -invariant affine open subsets, by Mayer–Vietoris argument we can assume that there exists a point $x \in X$ and a G_x -equivariant map $\varphi : X = \text{Spec } B \rightarrow T_x = \text{Spec } A$ which is etale and satisfies (a), (b) and (c) of Proposition 3.4. Let J_B and J_A denote the ideal of g -fixed points in B and A , respectively. We have two cases:

When $g \notin G_x$, g has no fixed points by Proposition 3.4(a), i.e. $J_B = B$. Then by [23, Theorem 6] $C(B \rtimes \langle g \rangle)$ is quasiisomorphic to $(C(B))_{\langle g \rangle}$ (the piece obtained from the conjugacy class of identity), so the argument of Step 4 applied to $G = \langle g \rangle$ shows that $C(B_g^{\natural})$ is quasiisomorphic to zero (if $g \neq 1$).

When $g \in G_x$ we use the etale map $\varphi : X \rightarrow T_x$ to reduce to the case of the flat space T_x . Note that until now all maps were defined in the derived category of Λ . However, it suffices to check that the map $C(B_g^{\natural})_{\langle g \rangle} \rightarrow C(B/J_B)$ defined above is a composition of quasiisomorphisms of complexes over k . Note that the action $b(b_0, \dots, b_n) = (bb_0, b_1, \dots, b_n)$ actually turns both $C(B_g^{\natural})$ and $C(B/J_B)$ into complexes of B -modules. We will show that they are isomorphic in the derived category of B (thus, taking coinvariants in $C(B_g^{\natural})_{\langle g \rangle}$ is only necessary to define an *a priori* mixed complex structure).

Proposition 3.6. *Let $X = \text{Spec } B$, $x \in X$, $g \in G_x$ and $T_x = \text{Spec } A$ be as above. If the map $A \rightarrow A/J_A$ induces a quasiisomorphism $C(A_g^{\natural}) \rightarrow C(A/J_A)$ in the derived category of A then the map $B \rightarrow B/J_B$ induces a quasiisomorphism $C(B_g^{\natural}) \rightarrow C(B/J_B)$ in the derived category of B .*

Proof. This proof is a minor modification of the etale descent result of [16]. In fact, if $C(A_g^{\natural}) \rightarrow C(A/J_A)$ is a quasiisomorphism of complexes of A -modules then consider the following commutative diagram:

$$\begin{CD} B \otimes_A C(A_g^{\natural}) @>>> B \otimes_A C(A/J_A) \simeq B/J_B \otimes_{A/J_A} C(A/J_A) \\ @VVV @VVV \\ C(B_g^{\natural}) @>>> C(B/J_B) \end{CD}$$

where the two vertical arrows are given by $b \otimes (a_0, \dots, a_n) = (ba_0, a_1, \dots, a_n)$.

The top arrow is a quasiisomorphism since B is flat over A . The top left corner isomorphism holds since $B \simeq B/J_B \otimes_{A/J_A} A$ by Part (b) of Proposition 3.4. To show that the bottom arrow is a quasiisomorphism we will show that this property holds for the two vertical arrows. Moreover, it suffices to prove it for the left arrow only since then one can set $g = 1$, replace the pair (B, A) by $(B/J_B, A/J_A)$, respectively, and get the proof for the right arrow.

To prove the assertion about $B \otimes_A C(A_g^{\natural}) \rightarrow C(B_g^{\natural})$ we borrow some formulas from [3, pp. 513 and 514]. Let A_{Δ} and A_g denote bimodules over $A \otimes A$ isomorphic to A as abelian groups, with the module structure given by $(a_0, a_1) \cdot a = a_0 a a_1$ for $a \in A_{\Delta}$ and $(a_0, a_1) \cdot a = a_0 a g^{-1}(a_1)$ for $a \in A_g$. Consider the free resolution P_{\bullet}^A of A_{Δ} as $A \otimes A$ -module:

$$\dots \rightarrow A \otimes A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \xrightarrow{\Delta} A \rightarrow 0$$

where the $A \otimes A$ -module structure on $A^{\otimes(k+2)}$ is given by

$$(\bar{a}_0, \bar{a}_1) \cdot (a_0, \dots, a_{k+1}) = (\bar{a}_0 a_0, a_1, \dots, a_k, a_{k+1} \bar{a}_1),$$

the map $\Delta : A \otimes A \rightarrow A$ sends (a_0, a_1) to $a_0 a_1$ and

$$\partial(a_0, \dots, a_{k+1}) = \sum_{i=0}^k (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_{k+1}).$$

Exactness of P_{\bullet}^A is proved using the homotopy $s : (a_0, \dots, a_{k+1}) \mapsto (1, a_0, \dots, a_{k+1})$. Let B_g and P_{\bullet}^B be the similar objects over B .

Now we have a chain of isomorphisms:

$$B \otimes_A C(A_g^{\natural}) \simeq B \otimes_A A_g \otimes_{A \otimes A} P_{\bullet}^A \simeq B_g \otimes_{A \otimes A} P_{\bullet}^A \simeq B_g \otimes_{B \otimes B} (B \otimes B) \otimes_{A \otimes A} P_{\bullet}^A.$$

The first isomorphism follows from the definitions of P_{\bullet}^A , $C(A_g^{\natural})$ and A_g . The second uses $\langle g \rangle$ -equivariance of $\text{Spec } B \rightarrow \text{Spec } A$ and the $A \otimes A$ -module structure on B_g which comes from $A \otimes A \rightarrow B \otimes B$. Taking into account that $C(B_g^{\natural}) \simeq B_g \otimes_{B \otimes B} P_{\bullet}^B$, we need to prove that the natural injective map of complexes $\rho : (B \otimes B) \otimes_{A \otimes A} P_{\bullet}^A \rightarrow P_{\bullet}^B$ becomes a quasiisomorphism after applying $B_g \otimes_{B \otimes B} (\dots)$. Explicitly, we have

$$\begin{CD} \dots \rightarrow B \otimes A \otimes B @>>> B \otimes B @>>> B \otimes_A B @>>> 0 \\ @. @VV \rho V @VV \rho V @VV a V \\ \dots \rightarrow B \otimes B \otimes B @>>> B \otimes B @>>> B @>>> 0 \end{CD}$$

where the upper row is exact since $B \otimes B$ is flat over $A \otimes A$ and the right vertical arrow a is the natural surjective map $B \otimes_A B \rightarrow B$.

Since B is étale over A , the kernel C of a is a $B \otimes B$ -module supported away from the diagonal $X_\Delta \subset X \times X = \text{Spec}(B \otimes B)$. Note that the support of B_g coincides with the graph of the map $g^{-1} : X \rightarrow X$. We now claim that the supports of B_g and C are disjoint. In fact, if $(x_1, x_2) \in \text{Supp}(C) \subset X \times X$ then $x_1 \neq x_2$ but $\varphi(x_1) = \varphi(x_2) \in T_x$. If $(x_0, g^{-1}(x_0)) \in \text{Supp}(C)$ then g^{-1} stabilizes $\varphi(x_0) \in T_x$ but does not stabilize $x_0 \in X$, which contradicts property (c) in Proposition 3.4. Hence the supports of B_g and C are disjoint and after tensoring $B_g \otimes_{B \otimes B}$ the two rows in the above diagram become quasiisomorphic since $B_g \otimes_{B \otimes B} \text{Coker } \rho$ computes $\text{Tor}_{B \otimes B}^\bullet(B_g, C) = 0$ (a bit more rigorously, one could first show that the localization at each maximal prime vanishes — see the last lines on [16, p. 374]). This finishes the proof of the proposition. \square

Thus, it suffices to prove the quasiisomorphism $C(A_g^\natural) \rightarrow C(A/J_A)$ when $A = k[T_x]$ is a polynomial ring with an action of the cyclic group $\langle g \rangle$ induced from its linear action on T_x .

Step 6: The Linear Case

As a last step we consider the case of a flat space $V = T_x = \text{Spec } A$ with a linear action of the cyclic group $H = \langle g \rangle$. Let $V = V_0 \oplus V_1$ where $V_0 = V^H$ and V_1 is the H -invariant complement. Note that $\bar{A} = A/J_A$ is the algebra of regular functions on V_0 .

Recall that for any variety Y , a vector bundle E and its section s one has a Koszul complex

$$\dots \rightarrow \Lambda^3 E^* \xrightarrow{\partial} \Lambda^2 E^* \xrightarrow{\partial} E^* \xrightarrow{\partial} \mathcal{O}_Y \rightarrow 0$$

where the differential is given by contraction with s . We denote this Koszul complex by $\text{Kos}(Y, E, s)$. It is well-known that for a regular section s and affine Y , $\text{Kos}(Y, E, s)$ is a projective resolution of \mathcal{O}_Z where Z is the zero scheme of s .

Recall from the previous step that $C(A_g^\natural)$ is obtained by taking a particular projective resolution P_\bullet^A of the diagonal copy $V_\Delta \subset V \times V$ and tensoring it with the $A \otimes A$ -module corresponding to the graph of $g^{-1} : V \rightarrow V$. Similarly, $C(\bar{A})$ is obtained by taking a particular resolution $P_\bullet^{\bar{A}}$ of $(V_0)_\Delta \subset V_0 \times V_0$ and tensoring it with the module of functions on $(V_0)_\Delta$. As in [3], we prove the quasiisomorphism $C(A_g^\natural) \rightarrow C(\bar{A})$ by looking at the Koszul resolutions instead of P_\bullet .

Indeed, $V_\Delta \subset V \times V$ is a zero scheme of a section s of the trivial vector bundle with fiber V , given by $s(v_1, v_2) = v_1 - v_2$; and similarly $(V_0)_\Delta$ is a zero scheme of the section $s_0 = s|_{V_0 \times V_0}$ which takes values in the trivial vector bundle with the fiber V_0 . Then we have a commutative diagram

$$\begin{array}{ccc} P_\bullet^A & \longrightarrow & P_\bullet^{\bar{A}} \\ \downarrow \alpha_A & & \downarrow \alpha_{\bar{A}} \\ \text{Kos}(V \times V, V, s) & \longrightarrow & \text{Kos}(V_0 \times V_0, V_0, s_0) \end{array}$$

where α_A is an extension of the identity map $A_\Delta \rightarrow A_\Delta$ to the projective resolutions, and $\alpha_{\bar{A}}$ is its reduction modulo J_A (for example, we define α by the formula on [3, p. 515]). Since α_A (respectively $\alpha_{\bar{A}}$) is a quasiisomorphism by definition of a projective resolution, and remains one after applying $A_g \otimes_{A \otimes A}$ (respectively $\bar{A}_\Delta \otimes_{\bar{A} \otimes \bar{A}}$) we only have to check that the induced map

$$A_g \otimes_{A \otimes A} \text{Kos}(V \times V, V, s) \rightarrow \bar{A}_\Delta \otimes_{\bar{A} \otimes \bar{A}} \text{Kos}(V_0 \times V_0, V_0, s_0)$$

is a quasiisomorphism. But by direct computation (see [27, Expose VII, Proposition 2.5], for example) one can see that the left hand side is isomorphic to the left hand side tensored by $\text{Kos}(V_1, V_1, s')$ where s' is the section of a trivial vector bundle over V_1 with fiber V_1 , given by $s'(v) = v - g^{-1}(v)$. Since $V_1^H = 0$, $\text{Kos}(V_1, v_1, s')$ is quasiisomorphic to k . This finishes the proof of Step 6, and the proof of Proposition 3.1.

4. Equivalences of Derived Categories and Cohomology

First we state a result which says that the cohomology of a complex quasiprojective variety can be recovered from an enhanced version of its derived category of vector bundles. Parts (a) and (b) of the next theorem are not stated explicitly in the papers [20, 21] but easily follow from their results.

Theorem 4.1. (a) *Let X be a quasiprojective variety over the field of complex numbers. In notation of Sec. 1, the cyclic homology group $HC_i(\text{Vect}(X), \mathbb{C}[u, u^{-1}])$ is isomorphic to $H^{\text{even}}(X, \mathbb{C})$ for $i = 2k$ and $H^{\text{odd}}(X, \mathbb{C})$ for $i = 2k + 1$.*

(b) *If $F : D^b(X) \rightarrow D^b(Y)$ is an equivalence of bounded derived categories of sheaves on two smooth projective varieties X, Y over a field k then F induces an isomorphism $HC_\bullet(\text{Vect}(X), W) \rightarrow HC_\bullet(\text{Vect}(Y), W)$ for any graded $k[u]$ -module W . In particular, if $k = \mathbb{C}$ then F induces an isomorphism of complex cohomology groups.*

(c) *Let X, Y be as in (b) and assume that a finite group G acts on X . If $\text{char } k$ does not divide $|G|$ and there exists an equivalence of derived categories $F : D^b(Y) \rightarrow D_G^b(X)$ then F induces an isomorphism*

$$HC_\bullet(\text{Vect}(Y), W) \simeq \left(\bigoplus_{g \in G} HC_\bullet(\text{Vect}(X^g), W) \right)_G .$$

Proof. By [21, Corollary 5.2], the mixed complex $C(X)$ is quasiisomorphic to the mixed complex obtained by sheafifying the standard mixed complex of an algebra (see [32, Sec. 9]). This result holds for any field k . If $\text{char } k = 0$, then by [12] (in the affine case) and [34] (general quasiprojective schemes) the homology groups $HC_i(\text{Vect}(X), k[u, u^{-1}])$ are given by the crystalline cohomology of X (see [34, Theorem 3.4]). In particular, when $k = \mathbb{C}$ we get the Betti cohomology of $X(\mathbb{C})$. This proves (a).

To prove (b) note that by smoothness the triangulated category obtained from $(\mathcal{A}^b(X), \mathcal{C}^b(X))$ is equivalent to the bounded derived category $D^b(X)$. By a fundamental theorem of Orlov (see [25, Theorem 2.2]) any equivalence F as above is induced by a Fourier–Mukai transform with respect to some sheaf on $X \times Y$. Thus, any such equivalence F automatically comes from a morphism of localization pairs hence by the invariance property (Proposition 2.1), F induced the isomorphism of all cyclic homology groups. If $k = \mathbb{C}$ and we only care about the isomorphism of cohomology as $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces we can conclude by Part (a). If we want to obtain an isomorphism of \mathbb{Z} -graded vector spaces we set $W = k[u]/uk[u]$ (which gives the Hochschild homology groups HH_\bullet) and then use [34, Corollary 1.4] which implies that for *smooth and projective* varieties the Hochschild homology coincides with the De Rham homology.

Part (c) follows from Proposition 2.1; Theorem 1.1 and the fact that any equivalence F still comes from a functor between localization pairs is due to [22, 8.1 and 8.2]. □

Example 4.1. Let $X \dashrightarrow Y$ be an elementary flop of Bondal–Orlov, see [7, Theorem 3.6]. By *loc. cit.* any such flop induces an equivalence of derived categories $F : D^b(X) \rightarrow D^b(Y)$ hence an isomorphism of cohomology groups by Part (b) of the above theorem. Note that the motivic integration approach only gives an equality of Betti numbers or classes in the K -theory of Hodge structures. This equivalence was extended to more general flops in dimensions 3 and 4 by Bridgeland and Namikawa.

Next we describe some situations when a derived category of sheaves on one variety is equivalent to the equivariant derived category of sheaves (with respect to a finite group action) on another variety. From now on we assume that $k = \mathbb{C}$. Let G be a finite group with a unimodular action on a smooth irreducible quasiprojective variety X (i.e. we require that for each $x \in X$ the image of the stabilizer G_x in $GL(T_x)$ actually belongs to the subgroup $SL(T_x)$). Denote by $G \text{Hilb}(X) \subset \text{Hilb}^{|G|}(X)$ the scheme of all G -invariant 0-dimensional subschemes $Z \subset X$ on multiplicity $|G|$ such that $H^0(\mathcal{O}_Z)$ is isomorphic to a regular representation of $|G|$. Assume further that the generic point of $x \in X$ has trivial stabilizer and let $Y(G, X) \subset G \text{Hilb}(X)$ be the irreducible component containing all free G -orbits.

Theorem 4.2. (1) *Assume that X, G and $Y = Y(G, X)$ are as above and one of the following conditions is satisfied*

- (a) $\dim X \leq 3$;
- (b) X is a complex symplectic variety, G preserves the symplectic structure and Y is a crepant resolution of X ;
- (c) X is the n th cartesian power of a smooth quasiprojective surface S ; $G = \Xi_n$ (the symmetric group) with the natural permutation action on X .

Then there exists an equivalence $F : D^b(Y) \rightarrow D_G^b(X)$ coming from a morphism of localization pairs.

(2) Assume that $\dim X \leq 3$; X is projective and X' is another smooth projective variety with an action of a finite group G' . Suppose that X/G and X'/G' are Gorenstein and there exists a common resolution of singularities $\pi : Z \rightarrow X/G$, $\pi' : X'/G' \rightarrow Z$ with $\pi^*K_X \simeq (\pi')^*K_{X'}$. Then there exists an equivalence of categories $D_G^b(X) \simeq D_{G'}^b(X')$. In particular, if $Z = X'$ and $G' = \{1\}$ (i.e. Z is a crepant resolution) there exists an equivalence $D_G^b(X) \simeq D^b(Z)$.

Proof. Parts (1a), (1b) correspond to [4, Theorem 1.2 and Corollary 1.3], respectively. To prove (1c), note that for $S = \mathbb{C}^2$ by a result of Haiman, cf. [17, Theorem 5.1], the variety Y is isomorphic to the Hilbert scheme $\text{Hilb}^n(S)$. Then the morphism $Y \rightarrow X/\Xi_n$ is semismall, see [15], and the assertion follows from a general criterion of [4, Theorem 1.1]. In general the equality $\text{Hilb}^n(S) = Y$ is derived from the \mathbb{C}^2 case by considering completions of local rings at points $s \in S$ which are the same as for \mathbb{C}^2 since S is smooth.

Part (2) is a simplified version of [19, Theorem 1.7] due to Kawamata. □

Corollary 4.1. *In the first (respectively the second) case of the above theorem one has an isomorphism of the cyclic homology groups*

$$\begin{aligned}
 & HC_\bullet(\text{Vect}_G(X), W) \simeq HC_\bullet(\text{Vect}(Y), W), \\
 & (\text{respectively } HC_\bullet(\text{Vect}_G(X), W) \simeq HC_\bullet(\text{Vect}_{G'}(X'), W))
 \end{aligned}$$

for all graded $k[u]$ -modules W . For $W = \mathbb{C}[u, u^{-1}]$ this reduces to isomorphisms

$$H_{\text{orb}}^*(X, \mathbb{C}) \simeq H^*(Y, \mathbb{C}); \quad H_{\text{orb}}^*(X, \mathbb{C}) \simeq H_{\text{orb}}^*(X', \mathbb{C})$$

of (orbifold) cohomology groups.

Proof. By inspecting the proofs in [4, 19], one can see that all the derived equivalences above are given by functors between localization pairs. Hence the first two assertions follow from the invariance property (Proposition 2.1). To obtain the last pair of equalities one applies Theorem 1.1 and Theorem 4.1(a). □

Finally we note that by a recent conjecture of Kawamata, the derived equivalences Theorem 4.2 should be a part of more general statement. Here we formulate a part of [19, Conjecture 1.2] in a slightly generalized form:

Conjecture 4.1 (Categorical Resolution Conjecture). *Let \mathcal{X}, \mathcal{Y} be smooth Deligne–Mumford stacks with Gorenstein moduli spaces X, Y . Suppose that there exist birational maps $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $f^*(K_X) = g^*(K_Y)$. Then*

the derived categories $D^b(\mathcal{X})$ and $D^b(\mathcal{Y})$ of coherent sheaves on the stacks \mathcal{X} and \mathcal{Y} , are equivalent.

5. Concluding Remarks

The isomorphisms of Theorem 1.1 are additive counterparts of a K -theoretic statement in [31]. However, in *loc. cit.* one also has a statement concerning K' -theory of singular varieties. This motivates the following conjecture:

Conjecture 5.1. *Let X be a quasiprojective scheme over a field k , G a finite group acting on X and assume that $\text{char } k$ does not divide $|G|$. Let $\text{Coh}(X)$ be the exact category of coherent sheaves on X and $\text{Coh}_G(X)$ the exact category of G -equivariant sheaves. Then for any graded module W over $k[u]$ there exists an isomorphism*

$$\phi_X : \left(\bigoplus_{g \in G} HC_\bullet(\text{Coh}(X^g), W) \right)_G \rightarrow HC_\bullet(\text{Coh}_G(X), W)$$

which is functorial with respect to the (derived) pushforwards under G -equivariant proper maps.

We expect that for smooth X and $W = k[u, u^{-1}]$ (periodic cyclic homology case) this isomorphism satisfies the multiplicative part of the cohomological crepant resolution conjecture, see [26].

We have seen in the proof of Theorem 1.1 that the isomorphism ψ_X is defined using pullbacks to the fixed point sets. We expect that, as in [31], the isomorphism ϕ_X can be defined using direct images with respect to the closed embeddings $X^g \rightarrow X$. Note that ϕ_X and ψ_X are not expected to be mutually inverse, but their composition should be invertible similarly to [31, Lemmas 4.2 and 4.3]. For singular X the cyclic homology of $\text{Coh}(X)$ may be different from the sheafified cyclic homology of the algebras $\mathcal{O}(U)$, $U \subset X$; therefore the general strategy of proof has to be completely different.

Finally we state another conjecture aimed at a better understanding of the orbifold product on orbifold cohomology, see [26, Sec. 3] and also [11, 30].

Let \mathcal{X} be a smooth Deligne–Mumford stack and $\pi : U \rightarrow \mathcal{X}$ be an étale cover by a scheme. Consider $Z = U \times_\pi U \subset U \times U$. If $\pi_i : U \times U \times U \rightarrow U \times U$ is the projection omitting the i th factor, $i = 1, 2, 3$, then $\pi_2(\pi_3^{-1}(Z) \cap \pi_1^{-1}(Z)) = Z$, i.e. Z is an idempotent correspondence from U to itself.

In particular, we have a (finite) morphism $m : Z \times_U Z \rightarrow Z$ which allows to define for any two objects $\mathcal{F}, \mathcal{G} \in D^b(Z)$ a third object

$$\mathcal{F} * \mathcal{G} := m_*(p_1^* \mathcal{F} \otimes p_2^* \mathcal{G})$$

where p_1, p_2 are the projections of $Z \times_U Z$ to Z , and all operations are understood in the derived sense.

The $*$ -product in general is not commutative, but it is associative (more precisely, there is a canonical isomorphism between $\mathcal{F} * (\mathcal{G} * \mathcal{H})$ and $(\mathcal{F} * \mathcal{G}) * \mathcal{H}$). When

\mathcal{X} is the quotient stack $[X/G]$ we can take $U = X$ and then $Z = \coprod_{g \in G} X^g$ while $m : Z \times_U Z \rightarrow Z$ is given by pointwise group product within stabilizers.

Conjecture 5.2. *When $\mathcal{X} = [X/G]$ and $U = X$ the above product $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} * \mathcal{G}$ induces the product of the periodic cyclic homology of $Z = \coprod_{g \in G} X^g$ which coincides with the product on $A(X, G) = H^*(Z, \mathbb{C})$ described in [11, 30]. Thus, taking the (co)invariants with respect to G -action gives the orbifold product structure on $H_{\text{orb}}^*(X/G, \mathbb{C})$.*

Note that the only place where smoothness of \mathcal{X} is important is the relation between cyclic homology of coherent sheaves and the orbifold cohomology groups. However, we could *define* orbifold cohomology for singular Deligne–Mumford stacks via cyclic homology of coherent sheaves; such a definition, perhaps, would give invariants which behave better than those defined via the usual cohomology.

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After the original version of this paper has appeared as an e-print, the author has learned that the idea of using derived categories to extract information about orbifold cohomology was recently used by A. Caldararu (work in progress); he emphasizes the role of Hochschild homology, rather than periodic cyclic homology. The author apologizes for any overlaps with this work.

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