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BGG Correspondence for Projective Complete Intersections

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1 Introduction

Let k be a field of characteristic zero and $\mathbb{P}(V)$ a projective space over k with homogeneous coordinate ring $\mathrm{Sym}^\bullet(V^*)$. The classical Bernstein-Gelfand-Gelfand correspondence (cf. [3]) interprets the derived category of coherent sheaves on $\mathbb{P}^n$ in terms of modules over the exterior algebra $\Lambda^\bullet(V)$. This result was later generalized by Kapranov [8], who considered a complete intersection $X \subset \mathbb{P}^n$ of quadrics given by polynomials $W_1, \dots, W_m \in \mathrm{Sym}^2(V^*)$. By a theorem of Serre, coherent sheaves on such X can be described in terms of graded modules over $S_W = \mathrm{Sym}^\bullet(V^*)/\langle W_1, \dots, W_m\rangle$, where $\langle W_1, \dots, W_m\rangle$ is the homogenous ideal generated by $W_1, \dots, W_m$. In this situation, the exterior algebra $\Lambda^\bullet(V)$ is replaced by the graded Clifford algebra $\mathrm{Cl}(W_1, \dots, W_m)$ generated by elements $\widehat{y}_0, \dots, \widehat{y}_n$ of degree 1 and central elements $z_1, \dots, z_m$ of degree 2, subject to relations

$$\widehat{y}_k\widehat{y}_i + \widehat{y}_i\widehat{y}_k = 2\sum_{j=1}^{m} \overline{W}_j\left(\widehat{y}_k, \widehat{y}_i\right)z_j, \tag{1.1}$$

where $\overline{W}_j : V \otimes V \to k$ is the polarization of $W_j : \mathrm{Sym}^2(V) \to k$. We denote this algebra by E_W emphasizing its dependence on the "potential" $W = \sum W_j z_j \in \mathrm{Sym}^2(V^*) \otimes U$, where U is the vector space spanned by $z_1, \dots, z_m$.

In some situations (e.g., those considered in mirror symmetry), one deals with the derived category on a general complete intersection defined by polynomials W_j of

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arbitrary degrees greater than or equal to 2. The goal of this paper is to describe the analogue of the above algebra E_W in this case and establish the corresponding equivalence of categories. In general, E_W is an A_∞-algebra, rather than an associative algebra. This means that E_W is equipped with "higher-order" products, besides the usual multiplication, and this system of products satisfies a sequence of generalized associativity identities, see Appendix A.1. For such objects the notions of modules and derived categories generalize nicely, see [9, 12], and one obtains a description of (the dual to) the derived category of sheaves on X as a quotient of the derived category of the A_∞-algebra E_W. Forgetting all "higher-order" operations on E_W gives an associative algebra isomorphic to the graded Clifford algebra built from the quadratic parts Q_j of W_j. Thus, $Q_j = W_j$ if $\deg W_j = 2$ and $Q_j = 0$ otherwise. In particular, if all W_j have degree greater than or equal to 3, the associative algebra of E_W contains no information about X at all. Thus, A_∞-structures (or something of the sort) are essential in generalizing the BGG correspondence to arbitrary complete intersections.

As in the quadratic case, set $S_W = \mathrm{Sym}^\bullet(V^*)/\langle W_1, \dots, W_m\rangle$. Also, let $(\cdots)^{\mathrm{op}}$ stand for the dual category (with arrows reversed).

Theorem 1.1. The derived category $\mathcal{D}^b(S_W)^{\mathrm{op}}$ is equivalent to the derived category $\mathcal{D}^b(E_W)$ of a minimal A_∞-algebra E_W with the following properties:

(a) as a vector space, E_W is isomorphic to $\Lambda^\bullet(V) \otimes k[z_1, \dots, z_m]$;

(b) the associative algebra (E_W, μ_2) is isomorphic to the graded Clifford algebra $\mathrm{Cl}(Q_1, \dots, Q_m)$ constructed from the above quadratic polynomials Q_j. In particular, if all W_j have degree greater than or equal to 3, the isomorphism of part (a) holds on the level of *algebras*;

(c) for $k \geq 3$, the operations μ_k have the following properties:

 (i) μ_k is multilinear with respect to variables $z_1, \dots, z_m$;

 (ii) $\mu_k(\nu_1, \dots, \nu_k) = 0$ if $\nu_i = 1$ for some i, that is, E_W is *strictly unital*;

 (iii) if $\xi_1, \dots, \xi_k$ are arbitrary vectors in $V \subset E_W$, then

$$\mu_k(\xi_1, \dots, \xi_k) = \sum_{\deg W_j = k} \overline{W}_j(\xi_1, \dots, \xi_k) z_j, \tag{1.2}$$

 where $\overline{W}_j : (V)^{\otimes k} \to \mathbb{C}$ is the polarization of $W_j : \mathrm{Sym}^k(V) \to k$.

Moreover, this induces the equivalence between $\mathcal{D}^b(\mathrm{Coh}(X))^{\mathrm{op}}$ and the quotient $\mathcal{D}^b(E_W)/I$, where $\mathrm{Coh}(X)$ is the category of coherent sheaves on X, and I is the full subcategory consisting of the objects isomorphic to finite complexes of free E_W-modules. □

See Section 4 for the definitions of the categories involved. We only note here that $\mathcal{D}^b(E_W)$ is a slight abuse of notation, in fact, it stands for a subcategory of a larger derived category obtained by imposing conditions on cohomology. For a Noetherian associative algebra A such construction gives a subcategory equivalent to $\mathcal{D}^b(A)$. When X is smooth, $\mathcal{D}^b(\mathrm{Coh}(X))^{\mathrm{op}} \simeq \mathcal{D}^b(\mathrm{Coh}(X))$.

Remark 1.2. The properties stated in Theorem 1.1 do not determine the A_∞-structure on E_W-uniquely. On one hand, for many purposes one only needs to know the A_∞-structure up to homotopy. On the other hand, it turns out that our particular A_∞-structure is obtained by specialization of a family of A_∞-structures on E_W parametrized by V^*. The products in this family have an interesting recursive property, see Proposition 3.2 in Section 4, allowing to determine them uniquely. This phenomenon does not manifest itself in the quadratic case.

Another approach to E_W, not considered here, is to look at the standard cocommutative coproduct on E_W. In fact, the polynomials $W_1, \ldots, W_m$ define an L_∞-structure (cf. [11]) on the super vector space $L = sV \oplus s^2U$ (V in homological degree 1 and U in degree 2). All identities of an L_∞-algebra are satisfied since every term in them vanishes (e.g., the Jacobi identity holds since L is 2-step nilpotent). Applying the standard constructions of differential homological algebra (cf. [5, Chapter 22]) one considers the DG-coalgebra $C = C(L)$ (also used in this paper) and then the free Lie algebra $\mathcal{L}(C(L))$. If all polynomials W_j with $\deg W_j \geq 3$ are set to zero, all higher Lie brackets vanish and there is a quasi-isomorphism of DG-Lie algebras $\mathcal{L}(C(L)) \to L$, see [5, Theorem 22.9]. The universal enveloping algebra functor gives a quasi-isomorphism of DG-Hopf algebras $\Omega(C(L)) \to E_W$, where Ω stands for the cobar construction and E_W is considered with the Clifford algebra structure arising from Q_j. Bringing back all nonquadratic W_j perturbs the differential on $\Omega(C(L))$, leading to a transferred A_∞-structure E_W (cf. Appendix A.3). However, since the perturbed differential agrees again with the Hopf algebra structure on $\Omega(C(L))$, by a result stated in [14, Theorem 2], E_W has a much richer "homotopy bialgebra" structure. At this moment we are not able to identify it explicitly.

This paper is organized as follows. In Section 2, we introduce a DG-algebra A which is quasi-isomorphic to a standard DG-algebra computing $\mathrm{Ext}^\bullet_{S_W}(k,k)$, and related to S_W by a Koszul-type equivalence. In Section 3, we compute an A_∞-structure on $E_W = H^*(A)$. Section 4 contains the proofs of the equivalences stated above. Finally, the appendix states some standard definitions and results used in this paper.

2 A Koszul-equivalent DG-algebra

We fix our notations here. Assume that all tensor products are over k, unless stated otherwise. Fix a vector space V over k with a basis $(y_0, \ldots, y_n)$, another vector space U with a basis $(z_1, \dots, z_m)$ and denote by $(x_0, \dots, x_n)$ and $(w_1, \dots, w_m)$ the dual bases of V^* and U^*, respectively.

While V and U are placed in homological degree zero, later we consider vector spaces with nontrivial homological grading (denoted by upper indices). We also use the suspension operation $V \mapsto sV$, where $(sV)^p = V^{p-1}$. Odd copies of variables $(y_0, \dots, y_n)$ will be denoted by $(\widehat{y}_0, \dots, \widehat{y}_n)$ and similarly with other bases. We write $S(X)$ for the *graded* symmetric (co)algebra of the graded vector space X. For instance, $S(sV)$ may be identified with the exterior (co)algebra $\Lambda^\bullet(V)$ if we forget about the gradings.

Eventually we will use internal grading denoted by lower indices. The graded dual of $M = \oplus M_p$ is defined as $M^* = \oplus M^*_p$, where $M^*_p = \mathrm{Hom}_k(M_{-p}, k)$. Similarly, the bigraded dual of $M = \oplus M^p_q$ is defined as $M^* = \oplus (M^*)^p_q$, where $(M^*)^p_q = \mathrm{Hom}_k(M^{-p}_{-q}, k)$.

Consider a *regular* sequence of homogeneous polynomials $(W_1, \dots, W_m) \in S(V^*)$ of degrees $d_j \geq 2$, $j = 1, \dots, m$, and define the "total potential" $W = \sum_{j=1}^m W_j z_j \in S(V^*) \otimes U$. Most formulas in this paper will be written in terms of W rather than individual W_j's.

One of the goals of this paper is to reinterpret the category of graded modules over the graded ring S_W defined in Section 1. In this section, construct a "small" DG-algebra A which is derived equivalent to S_W (when we consider derived categories with appropriate finiteness conditons). The definition of this algebra was originally obtained by studying endomorphisms of the Koszul complex of an S_W-module k. As an algebra,

$$A = S\big(V^* \oplus s^2 U\big) \otimes \mathrm{Cl}\left(s^{-1}V^*, sV\right) \simeq k\big[x_0, \dots, x_n, z_1, \dots, z_m\big] \otimes \mathrm{Cl}\left(s^{-1}V^*, sV\right), \tag{2.1}$$

where the Clifford algebra $\mathrm{Cl}(s^{-1}V^*, sV^*)$ is isomorphic to $\Lambda^\bullet(\widehat{x}_0, \dots, \widehat{x}_n, \widehat{y}_0, \dots, \widehat{y}_n)$ as a vector space and has commutation relations $\widehat{y}_p \widehat{x}_p + \widehat{x}_p \widehat{y}_p = 1$, $p = 0, \dots, n$ (other pairs of variables anticommute).

Before we define the differential of A, denote by $\widehat{\partial}_i$ the "corrected partial derivative" on $S(V^*)$, which satisfies $\widehat{\partial}_i(1) = 0$, takes a homogeneous degree n polynomial $f(x)$ to $(1/n)\partial_i f(x)$ and extends to nonhomogeneous polynomials by linearity. Later, for any multi-index P, the operator $\widehat{\partial}_P$ will denote the obvious composition of corrected partial derivatives. We extend these operators to $k[x_0, \dots, x_n, z_1, \dots, z_m]$ by linearity with respect to z-variables. The differential δ_A on A is a derivation with only nonzero values on generators given by

$$\delta_A\left(\widehat{x}_p\right) = x_p, \qquad \delta_A\left(\widehat{y}_p\right) = -\widehat{\partial}_p(W). \tag{2.2}$$

To relate S_W and A, first consider the Koszul resolution B of S_W, that is, the supercommutative algebra $S(V^* \oplus s^{-1}U^*)$ with its Koszul differential δ_B, that is, a derivation satisfying $\delta_B(\widehat{w}_j) = W_j$ and equal to zero on the generators x_i. One can view δ_B as given by the "half-suspension" potential $\widehat{W} = \sum W_j \widehat{z}_j$, where $\widehat{z}_j \in sU$ acts on $\Lambda^\bullet(U^*) = S(s^{-1}U^*)$ by contraction. Since $(W_1, \dots, W_m)$ is a regular sequence, the natural algebra map $B \to S_W$ sending $\widehat{w}_j$ to zero is a quasi-isomorphism.

Next, consider the graded dual *coalgebra* $C = B^* = S(V \oplus sU) \simeq k[y_0, \dots, y_n] \otimes \Lambda(\widehat{z}_1, \dots, \widehat{z}_m)$. Note that $S(V)$ acts on $S(V^*)$ by differential operators with constant coefficients and we can write the pairing $S(V) \otimes S(V^*) \to k$ as $(y^P, g(x)) = \partial_P g(0)$. Similarly, $S(V^*)$ acts on $S(V)$ by differential operators with constant coefficients and the same pairing may be written as $(f(y), x^Q) = \partial_Q f(0)$ (both P and Q are multi-indices). The same applies to $S(s^{-1}U^*)$ and $S(sU)$, which act on each other by contractions. It is immediate that the differential δ_C of C is given again by $\widehat{W} = \sum W_j \widehat{z}_j$, if now we interpret W_j as differential operators on $S(V)$ and $\widehat{z}_j$ as scalars in $S(sU)$.

Now consider a linear map $\tau = \tau_1 + \tau_2 : C \to A$ of degree $+1$, where τ_1 is the suspension operator identifying $V \oplus sU \subset C$ with $sV \oplus s^2U \subset A$, extended by zero to the rest of C, and

$$\tau_2 : S(V)_{\geq 1} \longrightarrow A, \qquad y^P \longmapsto |P|!\widehat{d}\big(\widehat{\partial}_P W\big), \quad |P| \geq 1 \tag{2.3}$$

(again, τ_2 is extended by zero from $S(V)_{\geq 1}$ to C). Here, $\widehat{d}$ is the "corrected exterior derivative," $\sum_{i=0}^{n} \widehat{\partial}_i(\cdots) \cdot \widehat{x}_i$, which satisfies $\delta_A \widehat{d}(f(x)g(z)) = f(x)g(z) - f(0)g(z)$. Below, $\Omega(\cdots)$ stands for the reduced cobar construction (cf. [**5**, Chapter 19]).

Lemma 2.1. The linear map $\tau : C \to A$ satisfies the twisted cochain condition of Appendix A.4. Moreover, the natural multiplicative extension $\Omega(C) \to A$ is a quasi-isomorphism of DG-algebras. □

Proof. Since $z_1, \dots, z_m$ are central in A, the nontrivial case is when the twisted cochain condition is applied to y^P, $|P| \geq 2$. Then everything follows from the identities:

$$\begin{aligned}
&\tau_2\big(y^P\big)\tau_2\big(y^Q\big) + \tau_2\big(y^Q\big)\tau_2\big(y^P\big) = 0;\\
&\delta_A \tau_2\big(y^P\big) = |P|!\widehat{\partial}_P W - \tau_1 \delta_C\big(y^P\big);\\
&\widehat{y}_p \widehat{x}_p + \widehat{x}_p \widehat{y}_p = 1.
\end{aligned} \tag{2.4}$$

To prove the quasi-isomorphism $\Omega(C) \to A$ by [**12**, Proposition 2.2.1.4], it suffices to establish that the natural map $F_0 : k \to A \otimes C$ is a quasi-isomorphism (the differential on $A \otimes C = \mathcal{F}_\tau(A)$ is as in Appendix A.4). To that end, first set formally all W_j to zero.

Then $A \otimes C$ becomes a tensor product of classical Koszul complexes with a standard contracting homotopy H_0 and projection $G_0 : A \otimes C \to k$, satisfying the side conditions $H_0^2 = 0$, $H_0F_0 = 0$, $G_0H_0 = 0$. Returning to the original W_j amounts to perturbing the differential on $A \otimes C$, and the quasi-isomorphism follows from the basic pertubation lemma. ■

Corollary 2.2. If C_W is the graded dual coalgebra of S_W and $\tau_W : C_W \to C \to A$ is the composition of the adjoint to $B \to S_W$ and τ, then its canonical multiplicative extension $\Omega(C_W) \to A$ is a quasi-isomorphism of DG-algebras. □

3 A transferred A_∞-structure

In this section, we provide an explicit contraction identifying the cohomology of A with the graded vector space $E_W = k[z_1, \dots, z_m] \otimes \Lambda(\widehat{y}_0, \dots, \widehat{y}_n)$. Using the obvious supercommutative product $\widehat{y}_i \wedge \widehat{y}_j$ in E_W, define $G : E_W \to A$ by

$$G\big((\widehat{y}_{i_1} \wedge \dots \wedge \widehat{y}_{i_s})z^P\big) = \frac{1}{s!} \sum_{\sigma \in \Sigma_s} (-1)^\sigma \big(\widehat{y}_{i_{\sigma(1)}} + \widehat{d}\widehat{\partial}_{i_{\sigma(1)}} W\big) \cdots \big(\widehat{y}_{i_{\sigma(s)}} + \widehat{d}\widehat{\partial}_{i_{\sigma(s)}} W\big) z^P, \tag{3.1}$$

where Σ_s is the symmetric group and P is a multi-index. Then, G is a map of complexes (if E_W has zero differential) since individual factors on the right-hand side are annihilated by δ_A.

Let $F : A \to E_W$ be the quotient map by the right ideal generated by $x_i, \widehat{x}_i$. To establish $FG = 1_{E_W}$ for any subset $I = \{i_1, \dots, i_s\}$, denote $\widehat{y}_{i_1} \wedge \dots \wedge \widehat{y}_{i_s}$ by $\widehat{y}^I$, and for $I_1 \subset I$ let $[\widehat{y}^{I_1} \setminus \widehat{y}^I] = \pm \widehat{y}^{I \setminus I_1}$ with the sign determined by the formula $\widehat{y}^{I_1} \wedge [\widehat{y}^{I_1} \setminus \widehat{y}^I] = \widehat{y}^I$. Then

$$G(\widehat{y}^I) = \sum_{I_1 = \{i_1, \dots, i_p\} \subset I} \widehat{d}\widehat{\partial}_{i_1}(W) \cdots \widehat{d}\widehat{\partial}_{i_p}(W) \big[\widehat{y}^{I_1} \setminus \widehat{y}^I\big] \tag{3.2}$$

which implies $FG = 1_{E_W}$. To define a homotopy, split A into a tensor product of complexes

$$(A, \delta_A) \simeq \big(S(V^* \oplus sV^*), \delta_x\big) \otimes \big(G(E_W), 0\big), \tag{3.3}$$

where δ_x is the restriction of δ_A to the subalgebra $S(V^* \oplus sV^*)$ generated by the x- and $\widehat{x}$-variables. Define $H : A \to A$ as $\widehat{d} \otimes 1_{G(E_W)}$, where the corrected exterior derivative $\widehat{d}$ is defined before **Lemma 2.1**. It follows immediately that $\delta_A H + H\delta_A = 1_A - GF$ and $F(H(a)b) = 0$, for all $a, b \in A$.

Following the procedure of **Appendix A.3**, one uses H to compute the "kernels" $\mathbf{p}_n : A^{\otimes n} \to A$ and then defines an A_∞-structure on E_W by $\mu_k = F \circ \mathbf{p}_k \circ G^{\otimes k}$. The next proposition computes $\mu_2(v,u) = F(G(v)G(u))$ (which is associative since E_W has zero differential) and states some properties of the higher products. To unload notation, from now on we set

$$W^{(i_1,\ldots,i_k)} := \widehat{\partial}_{i_1} \cdots \widehat{\partial}_{i_k}(W). \tag{3.4}$$

Proposition 3.1. The associative algebra (E_W, μ_2) is isomorphic to $\mathrm{Cl}(Q_1,\ldots,Q_m)$, the graded Clifford algebra built from the quadratic parts Q_j of the homogeneous polynomials W_j. Moreover, the higher products μ_k, $k \geq 3$ have the following properties:

(a) $\mu_k(v_1,\ldots,v_k) = 0$ if $v_i = 1$, for some i. Thus, E_W is a *strictly unital* A_∞-algebra;

(b) $\mu_k(v_1,\ldots,v_k)$ are multilinear with respect to the z-variables. □

Proof. Since the maps F, G are linear with respect to the z-variables which also belong to the center of A, it suffices to compute $\mu_2(\widehat{y}^I, \widehat{y}^J)$. Since F annihilates elements of the form $\widehat{x}_i b$,

$$\mu_2(\widehat{y}^I, \widehat{y}^J) = F\left(\widehat{y}^I \sum_{J_1=\{j_1,\ldots,j_s\}\subset J} \widehat{d}W^{(j_1)} \cdots \widehat{d}W^{(j_s)} \left[\widehat{y}^{J_1} \setminus \widehat{y}^J\right]\right). \tag{3.5}$$

Taking into account $\widehat{y}_i \widehat{d} = -\widehat{d}\widehat{y}_i + \partial_i$, one gets

$$\begin{aligned} &G(\widehat{y}^I) \cdot G(\widehat{y}^J) \\ &\quad = \sum_{k\geq 0} \sum_{\substack{I_1=\{i_1,\ldots,i_k\}\subset I \\ J_1=\{j_1,\ldots,j_k\}\subset J}} (-1)^{(|I|-k)k} \det\left(W^{(i_p,j_q)}\right)_{p,q=1,\ldots,k} G\left(\left[\widehat{y}^{I_1} \setminus \widehat{y}^I\right] \wedge \left[\widehat{y}^{J_1} \setminus \widehat{y}^J\right]\right). \end{aligned} \tag{3.6}$$

Applying F to the expression on the right-hand side amounts to evaluating the determinant at $(x_0,\ldots,x_n) = (0,\ldots,0)$ and removing G. Thus, only the quadratic defining equations W_j will give a nonzero contribution to μ_2. For quadratic polynomials, one has $\widehat{\partial}_i \widehat{\partial}_j(Q) = (1/2)(\partial^2 Q/\partial x_i \partial x_j)$. In particular, this gives a formula

$$\mu_2(\widehat{y}_p, \widehat{y}_q) = \widehat{y}_p \widehat{y}_q + \frac{1}{2} \sum_{j=1}^m \frac{\partial^2 Q_j}{\partial x_p \partial x_q} z_j. \tag{3.7}$$

In particular, $\mu_2(\widehat{y}_p, \widehat{y}_q) + \mu_2(\widehat{y}_q, \widehat{y}_p) = \sum_{j=1}^m (\partial^2 Q_j/\partial x_p \partial x_q) z_j$. Therefore, the homomorphism

$$\rho : T(V^*) \otimes k[z_1,\ldots,z_m] \longrightarrow (E_W, \mu_2), \quad \widehat{y}_p \longmapsto \widehat{y}_p, \quad z_j \longmapsto z_j, \tag{3.8}$$

descends to an algebra map $\mathrm{Cl}(Q_1,\dots,Q_m) \to (E_W,\mu_2)$. By a standard argument involving filtration by monomials of degree less than or equal to k in $\widehat{y}_p$, the map $\rho : \mathrm{Cl}(Q_1,\dots,Q_m) \to (E_W,\mu_2)$ is an isomorphism.

Part (a) follows by an easy induction from the definition of $\mathbf{p}_n$ and the side conditions $H^2 = 0$, $FH = 0$, $HG = 0$, which hold in our case. Part (b) is a consequence of linearity of F, G, and H with respect to the central z-variables. ■

The easiest way to describe the A_∞-structure on E_W completely is to include it in a family of A_∞-structures which we now proceed to describe. Add extra central variables $(\boldsymbol{x}_0,\dots,\boldsymbol{x}_n)$ to A and E_W to obtain a $k[\boldsymbol{x}_0,\dots,\boldsymbol{x}_n]$-algebra $A[\boldsymbol{x}]$ and a free $k[\boldsymbol{x}_0,\dots,\boldsymbol{x}_n]$-module $E_W[\boldsymbol{x}]$, respectively. Consider a new potential $W(x,\boldsymbol{x}) \in A[\boldsymbol{x}]$ obtained from W by replacing every x_i by $(x_i + \boldsymbol{x}_i)$. Define the differential $\delta_{A[\boldsymbol{x}]}$ and the contraction $F_{\boldsymbol{x}}, G_{\boldsymbol{x}}, H_{\boldsymbol{x}}$ from $A[\boldsymbol{x}]$ to $E_W[\boldsymbol{x}]$ by the same formulas as before, but using $W(x,\boldsymbol{x})$ instead of $W(x)$. In particular, all operators just introduced are linear with respect to the $\boldsymbol{x}$-variables, and the corrected partial and exterior derivatives $\widehat{\partial}_i$, $\widehat{d}$ act only on x_i, not $\boldsymbol{x}_i$. The corrected partial derivatives which *do* act on $\boldsymbol{x}_i$ will be denoted by $\widehat{d}/d\boldsymbol{x}_i$ to avoid confusion.

By **Appendix A.3**, $E_W[\boldsymbol{x}]$ acquires a transferred A_∞-structure $\{\eta_k\}_{k\geq 2}$ from $A[\boldsymbol{x}]$. For example, repeating the arguments leading to (3.6), we obtain the following expression for the product η_2 in $E_W[\boldsymbol{x}]$:

$$\eta_2(\widehat{y}^I,\widehat{y}^J) = \sum_{k\geq 0} \sum_{\substack{I_1=\{i_1,\dots,i_k\}\subset I \\ J_1=\{j_1,\dots,j_k\}\subset J}} (-1)^{(|I|-k)k} \det\left(W^{(i_p,j_q)}[\boldsymbol{x}]\right)_{p,q=1,\dots,k} \left[\widehat{y}^{I_1} \setminus \widehat{y}^I\right] \wedge \left[\widehat{y}^{J_1} \setminus \widehat{y}^J\right], \tag{3.9}$$

where $W[\boldsymbol{x}]$ is obtained from W by substitution $x_i \mapsto \boldsymbol{x}_i$ and

$$W^{(i_p,j_q)}[\boldsymbol{x}] = \frac{\widehat{d}}{d\boldsymbol{x}_{i_p}} \frac{\widehat{d}}{d\boldsymbol{x}_{i_q}} W[\boldsymbol{x}]. \tag{3.10}$$

In other words, $(E_W[\boldsymbol{x}],\eta_2)$ is a Clifford algebra of a symmetric quadratic form $V \otimes V \to \mathrm{Sym}^\bullet(V^*) \otimes U$, a partial polarization of W.

There is a natural surjective map $\pi_E : E_W[\boldsymbol{x}] \to E_W$ obtained by sending $\boldsymbol{x}_i$ to zero.

Proposition 3.2. The A_∞-structure $\{\eta_k\}$ on $E_W[\boldsymbol{x}]$ has the following properties:

(a) $\mu_k(\pi_E(v_1),\dots,\pi_E(v_k)) = \pi_E \eta_k(v_1,\dots,v_k)$ for $k \geq 2$;

(b) the generators $z_1,\dots,z_m$ and $\boldsymbol{x}_0,\dots,\boldsymbol{x}_n$ are central for the associative product η_2, while for $k \geq 3$, the higher products η_k are linear with respect to these generators;

(c) the following recursive formula determines uniquely the A_∞-structure $\{\eta_k\}$ (and hence by (a) the A_∞-structure $\{\mu_k\}$):

$$\eta_k(\widehat{y}_i, \widehat{y}^{I_2}, \dots, \widehat{y}^{I_k}) = \frac{\widehat{d}}{d\boldsymbol{x}_i}\eta_{k-1}(\widehat{y}^{I_2}, \dots, \widehat{y}^{I_k}). \tag{3.11}$$

□

Proof. Let $\pi_A : A[\boldsymbol{x}] \to A$ be the quotient map with respect to the DG-ideal generated by $\boldsymbol{x}_0, \dots, \boldsymbol{x}_n$. Then, π_A is multiplicative and $\pi_A H_{\boldsymbol{x}} = H\pi_A$, $\pi_E F_{\boldsymbol{x}} = F\pi_A$, $\pi_A G_{\boldsymbol{x}} = G\pi_E$. The definitions of the corresponding "kernels" $\mathbf{p}_n[\boldsymbol{x}]$ on $A[\boldsymbol{x}]$ (cf. Appendix A.3) give (a) immediately.

Part (b) follows from the fact that $F_{\boldsymbol{x}}, G_{\boldsymbol{x}}$, and $H_{\boldsymbol{x}}$ commute with multiplication by $\boldsymbol{x}_i$.

To prove (c), one first shows $F_{\boldsymbol{x}}(H_{\boldsymbol{x}}(a)b) = 0$ for all $a, b \in A[\boldsymbol{x}]$, hence only the first term in the inductive formula for $\mathbf{p}_k[\boldsymbol{x}]$ (cf. Appendix A.3) gives a nonzero contribution to $F_{\boldsymbol{x}} \circ \mathbf{p}_k[\boldsymbol{x}] \circ G_{\boldsymbol{x}}^{\otimes k}$. Therefore, setting $v_i' = G_{\boldsymbol{x}}(v_i)$, we get

$$\begin{aligned}
\eta_k(\widehat{y}_i, v_2, \dots, v_k) &= F_{\boldsymbol{x}}\big(G_{\boldsymbol{x}}(\widehat{y}_i)H_{\boldsymbol{x}}\mathbf{p}_{k-1}[\boldsymbol{x}](v_2', \dots, v_k')\big)\\
&= F_{\boldsymbol{x}}\big(\widehat{y}_i H_{\boldsymbol{x}}\mathbf{p}_{k-1}[\boldsymbol{x}](v_2', \dots, v_k')\big)\\
&= F_{\boldsymbol{x}}\big(-H_{\boldsymbol{x}}\widehat{y}_i\mathbf{p}_{k-1}[\boldsymbol{x}](v_2', \dots, v_k') + \widehat{\partial}_i\mathbf{p}_{k-1}[\boldsymbol{x}](v_2', \dots, v_k')\big)\\
&= F_{\boldsymbol{x}}\big(\widehat{\partial}_i\mathbf{p}_{k-1}[\boldsymbol{x}](v_2', \dots, v_k')\big).
\end{aligned} \tag{3.12}$$

Inspecting the definition of $\mathbf{p}_k[\boldsymbol{x}]$, we see that $\mathbf{p}_{k-1}[\boldsymbol{x}](G_{\boldsymbol{x}}(\widehat{y}^{I_2}), \dots, G_{\boldsymbol{x}}(\widehat{y}^{I_k}))$ is a sum of products involving $\widehat{d}W^P(x, \boldsymbol{x})$, $W^P(x, \boldsymbol{x})$, and $\widehat{y}_i$. We can assume that all factors $\widehat{y}_i$ stand to the right of $\widehat{d}W^P(x, \boldsymbol{x})$ and then disregard those terms which contain $\widehat{d}W^P(x, \boldsymbol{x})$ since they are annihilated by $F_{\boldsymbol{x}}$. All other terms can be reduced to the form $R(x_0 + \boldsymbol{x}_0, \dots, x_n + \boldsymbol{x}_n)$, where R is a polynomial with coefficients in $k[z_1, \dots, z_m] \otimes \Lambda(y_0, \dots, y_n)$. Applying $F_{\boldsymbol{x}}$ just amounts to setting $x_i = 0$, for $i = 0, \dots, n$. Now (c) follows from the formula

$$\big(\widehat{\partial}_i R(x_0 + \boldsymbol{x}_0, \dots, x_n + \boldsymbol{x}_n)\big)\big|_{x=0} = \frac{\widehat{d}}{d\boldsymbol{x}_i}\Big(R(x_0 + \boldsymbol{x}_0, \dots, x_n + \boldsymbol{x}_n)\big|_{x=0}\Big). \tag{3.13}$$

Finally, to show that the above formula allows to recover the general values of η_k, we note that $E_W[\boldsymbol{x}]$ is generated by $\widehat{y}_i$ as a $k[z_1, \dots, z_m, \boldsymbol{x}_0, \dots, \boldsymbol{x}_n]$-algebra and that by A_∞-identities, one has

$$\begin{aligned}
\eta_k(\widehat{y}_i * \widehat{y}^I, v_2, \dots, v_k) = &\pm\eta_k(\widehat{y}_i, \widehat{y}^I * v_2, \dots, v_k) \pm \eta_k(\widehat{y}_i, \widehat{y}^I, v_2 * v_3, \dots, v_k)\\
&\pm \eta_k(\widehat{y}_i, \widehat{y}^I, v_2, \dots, v_{k-1} * v_k) + (\text{smth}),
\end{aligned} \tag{3.14}$$

where $*$ denotes the product η_2 and (smth) is an expression which depends on $\eta_{k'}$ with $k' < k$. Hence, using induction on k and cardinality of I, as well as explicit formula for $\widehat{y}_i * \widehat{y}^I$, we see that the associative product $\eta_2 = *$ and property (c) determine the A_∞-structure uniquely. ■

Corollary 3.3.

$$\mu_k(\widehat{y}_{i_1}, \ldots, \widehat{y}_{i_k}) = W^{(i_1,\ldots,i_k)}\big|_{x=0} = \frac{1}{k!} \sum_{\{j \mid \deg W_j = k\}} \frac{\partial^k W_j}{\partial x_{i_1} \cdots \partial x_{i_k}} z_j. \tag{3.15}$$

□

4 Derived equivalences

Recall that C_W is the graded dual coalgebra of S_W. Let $\tau_W : C_W \to E_W$ be a linear map, which sends y_i to $\widehat{y}_i$, extended by zero to the natural complement of the subspace spanned by y_i. Then, τ_W is a generalized twisted cochain (cf. Appendix A.4). The two functors related to it may be modified to give a functor $\mathcal{F}$ from graded S_W-modules to graded E_W-modules and the adjoint functor $\mathcal{G}$ in the opposite direction. Explicitly,

$$\mathcal{F}(N)^{p'}_{q'} = \bigoplus_{\substack{p'=p+q+s \\ q'=t-q}} N^p_q \otimes (E_W)^s_t, \qquad \mathcal{G}(M)^{p'}_{q'} = \bigoplus_{\substack{p'=p+q \\ q'=t-q}} M^p_q \otimes (C_W)_t, \tag{4.1}$$

where the upper index denotes the homological grading and the lower index internal grading, and the A_∞-module structure on $\mathcal{F}(N)$ is given by

$$\mu_k^{\mathcal{F}(N)}\big((m \otimes a) \otimes a_1 \otimes \cdots \otimes a_{n-1}\big) = m \otimes \mu_k\big(a \otimes a_1 \otimes \cdots \otimes a_{n-1}\big). \tag{4.2}$$

Now that C_W is viewed above as S_W-module. The differentials are induced by τ_W via formulas (A.8) and (A.9) of Appendix A.4, respectively. We want to show that $\mathcal{F}$ and $\mathcal{G}$ give mutually inverse equivalences and we begin by defining the categories in which these functors take values a priori.

The algebra S_W is considered with its standard grading in which all generators $x_i \in S_W$ have internal degree 1 and homological degree 0. Let $\mathrm{Mof}-S_W$ be the category of finitely generated graded S_W-modules and $\mathcal{D}^b(S_W)$ its bounded derived category. Following [2] define $C^{\downarrow}(S_W)$ as the category of complexes $M^{\bullet}_{\bullet}$ of S_W-modules such that $M^p_q = 0$ if $p \ll 0$ or $p + q \gg 0$. Its localization at quasi-isomorphisms is denoted by $\mathcal{D}^{\downarrow}(S_W)$. The dual category $(C^b(\mathrm{Mof}-S_W))^{op}$ may be identified with a subcategory of $C^{\downarrow}(S_W)$ by sending a finitely generated module $M = \oplus M_q$ to its graded dual $M^* = \oplus \mathrm{Hom}_k(M_{-q}, k)$ (dualization inverts grading). Define $\mathcal{D}^{\downarrow}_b(S_W) \subset \mathcal{D}^{\downarrow}(S_W)$ as the full subcategory formed by all objects M, for which the bigraded dual $(H^\bullet(N))^*$ of total cohomology is a finitely generated S_W-module (since S_W has homological degree zero, this implies that N has only finitely many nonzero cohomology groups). By taking graded duals in [2, Lemma 2.12.8], we obtain an equivalence $\mathcal{D}^b(S_W)^{op} \simeq \mathcal{D}_b^\downarrow(S_W)$.

As for E_W, let $\widehat{y}_p$ have homological degree 0 and internal degree 1, while z_j have homological degree $2-d_j$ and internal degree d_j. Note that $(E_W)^p_q = 0$, for $p > 0$ or $p+q < 0$. Define $\mathrm{Mod}-E_W$ as the category of all strictly unital right A_∞-modules equipped with internal grading preserved by μ_k^M (see Appendix A.1). The morphisms in $\mathrm{Mod}-E_W$ are strictly unital A_∞-module homomorphisms preserving the internal grading, see Appendix A.1. Let $C^\uparrow(E_W)$ be the full subcategory of $\mathrm{Mod}-E_W$ formed by modules $M = \oplus M^p_q$ with $M^p_q = 0$, if $p \gg 0$ or $p+q \ll 0$. Let $\mathcal{D}^\uparrow(E_W)$ denote the localizations of $C^\uparrow(E_W)$ at quasi-isomorphisms (those maps $f = \{f_n\}$ for which $f_1 : M \to N$ is a quasi-isomorphism of complexes). Since E_W has trivial differential, the total cohomology $H^\bullet(M)$ of any A_∞-module M is naturally a module over the associative algebra (E, μ_2). Define $\mathcal{D}_b^\uparrow(E_W) \subset \mathcal{D}^\uparrow(E_W)$ as the full subcategory of all objects M for which $H^\bullet(M)$ is finitely generated over (E_W, μ_2). By a slight abuse of notation, we also denote $\mathcal{D}_b^\uparrow(E_W)$ by $\mathcal{D}^b(E_W)$.

Proposition 4.1. The above functors $\mathcal{F}$ and $\mathcal{G}$ induce mutually inverse equivalences between $\mathcal{D}^\downarrow(S_W)$ and $\mathcal{D}^\uparrow(E_W)$. Moreover, they restrict to mutually inverse equivalences between $\mathcal{D}_b^\downarrow(S_W)$ and $\mathcal{D}_b^\uparrow(E_W)$. □

Proof. It follows from definitions that $\mathcal{F}$ sends $C^\downarrow(S_W)$ to $C^\uparrow(E_W)$ and $\mathcal{G}$ sends $C^\uparrow(E_W)$ to $C^\downarrow(S_W)$. As in [2], using a spectral sequence one can show that the functors descend to derived categories.

To show that it is an equivalence, we use intermediate algebras $\Omega(C_W)$ and A. First, note that by [13] the quasi-isomorphism of complexes $G = G_1 : E_W \to A$ can be completed to a quasi-isomorphism $\{G_i\}_{i\geq 1} : E_W \to A$ of A_∞-algebras. Firstly, this gives a twisted cochain $B(E_W) \to A$, where $B(\cdots)$ is the reduced bar construction (cf. [9]) and secondly, any A-module Q becomes an E_W-module by composing $E_W \to A$ with the DG-algebra map $A \to \mathrm{End}(Q)^{op}$ (cf. [9, Sections 3.4 and 4.2]).

Consider the natural functors $\mathcal{F}'''$, $\mathcal{F}''$, and $\mathcal{F}'$ taking S_W-modules to $\Omega(C_W)$-modules, $\Omega(C_W)$-modules to A-modules and A-modules to E_W-modules, respectively. On the level of vector spaces, we have $\mathcal{F}'''(N) = N \otimes \Omega(C_W)$, $\mathcal{F}''(P) = P \otimes_{\Omega(C_W)} A$, $\mathcal{F}'(Q) = Q$. Since for any S_W-module N the maps $G_i : E_W^{\otimes i} \to A$ give a quasi-isomorphism $N \otimes E_W \to N \otimes A$ of E_W-modules, we have a canonical quasi-isomorphism of functors $\mathcal{F} \to \mathcal{F}' \circ \mathcal{F}'' \circ \mathcal{F}'''$.

Similarly, a canonical quasi-isomorphism of complexes $B(E_W) \otimes A \to k$ gives rise to a canonical quasi-isomorphism $\mathcal{G}''' \circ \mathcal{G}'' \circ \mathcal{G}' \to \mathcal{G}$, where $\mathcal{G}$s act in the direction opposite to $\mathcal{F}$s, and on the level of vector spaces we have $\mathcal{G}'(M) = M \otimes B(E_W) \otimes A$, $\mathcal{G}''(Q) = Q$, and $\mathcal{G}'''(P) = P \otimes C_W$.

By [1], the compositions $\mathcal{F}''' \circ \mathcal{G}'''$ and $\mathcal{G}''' \circ \mathcal{F}'''$ are canonically quasi-isomorphic to identity. Since $\Omega(C_W) \to A$ and $E_W \to A$ are quasi-isomorphisms of DG- or A_∞-algebras, the same holds for pairs $(\mathcal{F}'', \mathcal{G}'')$, $(\mathcal{F}', \mathcal{G}')$. Therefore, $\mathcal{F}$ and $\mathcal{G}$ are mutually inverse equivalences.

Since the cohomology of $\mathcal{F}(N)$ is simply $\mathrm{Ext}^\bullet(N, k)$, the fact that $\mathcal{F}$ perserves the finiteness condition follows from [6, Section 3].

To prove $\mathcal{G}(\mathcal{D}^{\uparrow}_b(E_W)) \subset \mathcal{D}^{\downarrow}_b(S_W)$, use the original BGG correspondence between the symmetric algebra $S = \mathrm{Sym}^\bullet(V^*)$ and the exterior algebra $\Lambda = \Lambda^\bullet(V)$. Let N be an E_W-module with finitely generated total cohomology. We have seen before that $\mathcal{F}\mathcal{G}(N) = N \otimes C_W \otimes E_W$ is quasi-isomorphic to N. Since $N \otimes C_W \otimes E_W$ is also a complex of free modules over the associative algebra (E_W, μ_2), we can apply $\otimes_{(E_W, \mu_2)}\Lambda$ and obtain a complex of free Λ-modules $N \otimes C_W \otimes \Lambda$ with finitely generated total cohomology. Then, by the original BGG-correspondence $N \otimes C_W \otimes \Lambda \otimes S$ is a complex of S-modules with finitely generated bigraded dual of total cohomology. But $N \otimes C_W$ and $N \otimes C_W \otimes \Lambda \otimes S$ are quasi-isomorphic as S-modules, hence the cohomology of $N \otimes C_W$ satisfies the required finiteness condition over S and therefore over S_W, since the S-module structure on $N \otimes C_W$ is obtained by restriction of scalars from $S \to S_W$. ∎

Proof of Theorem 1.1. The properties (a)–(c) of E_W are established in Proposition 3.1. Equivalence $(\mathcal{D}^b(S_W))^{op} \simeq \mathcal{D}^b(E_W)$ follows from Proposition 4.1 since $(\mathcal{D}^b(S_W))^{op} \simeq \mathcal{D}^{\downarrow}_b(S_W)$ and $\mathcal{D}^b(E_W) = \mathcal{D}^{\uparrow}_b(E_W)$ by definition. Equivalence $\mathcal{D}^b(\mathrm{Coh}(X))^{op} \simeq \mathcal{D}^b(E_W)/I$ follows as in [8]. ∎

Remark 4.2. Alternatively, we could define $\mathcal{D}^b(E_W)$ by considering the category $C^b(E_W)$ of all strictly unital A_∞-modules N over E_W, which are also modules over the associative algebra (E_W, μ_2) (this happens precisely when $\mu_3^N(\cdot, a_1, a_2)$ commutes with the differential on N, for all $a_1, a_2 \in E_W$). The morphisms in $C^b(E_W)$ are still strictly unital A_∞-module homomorphisms. Then we can set $\mathcal{D}^b(E_W)$ to be the localization at quasi-isomorphisms, which leads to a category equivalent to the one used above. Note that both notations $\mathcal{D}^b$ and C^b are somewhat deceptive here, since E_W itself has nontrivial homological grading and even free E_W-modules are only bounded above as complexes of abelian groups. For a different homological grading on E_W in which $\deg_h(\widehat{y}_i) = 1$, $\deg_h(z_j) = 2$, free E_W-modules will be bounded below. However, one cannot work with complexes which are bounded above *and* below since the (generally nonzero) operations μ_k have degrees $(2 - k)$.

Appendix

Some differential homological algebra

A.1 A_∞-algebras, modules, and derived categories [9, 12]

An A_∞-algebra is a graded vector space E equipped with a system of products $\mu_k : E^{\otimes k} \to E$ of degrees $(2-k)$, which satisfy "higher associativity identities" for $m \geq 1$,

$$\sum_{j+k+l=m} (-1)^{jk+l} \mu_{j+1+l}\left(1^{\otimes j} \otimes \mu_k \otimes 1^{\otimes l}\right) = 0. \tag{A.1}$$

The first identity simply says that $\delta_E = \mu_1$ is a differential. If $\mu_1 = 0$ (i.e., E is *minimal*, as is the algebra E_W in this paper), the first two identities become trivial while the third states that μ_2 is an associative product. However, the higher operations μ_k can still be nontrivial.

A (right) A_∞-module over an A_∞-algebra E is a graded vector space M together with a system of operations $\mu_k^M : M \otimes A^{\otimes k-1} \to M$, satisfying essentially similar identities (terms with $j \geq 0$ are interpreted as $\mu^M_{j+1+l}(1^{\otimes j} \otimes \mu_k \otimes 1^{\otimes l})$, and terms with $j = 0$ as $\mu^M_{j+1+l}(\mu^M_k \otimes 1^{\otimes l})$). An A_∞-morphism between E-modules M, N is a family of maps $f_k : M \otimes A^{\otimes k-1} \to N$, such that

$$\sum_{j+k+l=m} (-1)^{jk+l} \mu^M_{j+1+l}\left(1^{\otimes j} \otimes \mu_k \otimes 1^{\otimes l}\right) = \sum_{r+s=m} \mu^N_{s+1}\left(f_r \otimes 1^{\otimes s}\right) \tag{A.2}$$

(for $j = 0$, one uses μ^M_k instead of μ_k). For $g : N \to T$ and $f : M \to N$, define the composition $f \circ g$ by setting $(f \circ g)_i = \sum_{k+l=i} f_{1+l}(g_k \otimes 1^{\otimes l})$.

A *strictly unital* A_∞-algebra E is equipped with a unit morphism $\eta : k \to E$ such that $\mu_i(1 \cdots 1 \otimes \eta \otimes 1 \cdots 1) = 0$, for $i \neq 2$ and $\mu_2(1 \otimes \eta) = \mu_2(\eta \otimes 1) = 1$. A module M over such E is *strictly unital* if $\mu^M_i(1_M \otimes 1 \cdots 1 \otimes \eta \otimes 1 \cdots 1) = 0$, for $i \geq 3$ and $\mu^M_2(1_M \otimes \eta) = 1_M$. Finally, a morphism $f : M \to N$ of two such modules is called *strictly unital* if $f_i(1_M \otimes 1 \cdots 1 \otimes \eta \otimes 1 \cdots 1) = 0$, for $i \geq 2$.

A.2 Basic perturbation lemma [4]

Lemma A.1. Let (C_1, δ_1) and (C_2, δ_2) and $F_0 : C_1 \to C_2$, $G_0 : C_2 \to C_1$ be maps of complexes such that $F_0 G_0 = 1_{C_2}$ and $1_{C_1} - G_0 F_0 = \delta_1 H_0 + H_0 \delta_1$, where $H_0 : C_1 \to C_1$ is a homotopy. Suppose further that the following "side conditions" are satisfied:

$$F_0 H_0 = 0, \qquad H_0 G_0 = 0, \qquad H_0^2 = 0. \tag{A.3}$$

Then, given a "perturbation" $\widehat{\delta}_1 = \delta_1 + \partial$ of the differential δ_1 (i.e., $\widehat{\delta}_1^2 = 0$) such that the operator ∂H_0 is locally nilpotent, there exist a new differential $\widehat{\delta}_2 = \delta_2 + \widehat{\partial}$ on C_2, maps of complexes $F : (C_1, \widehat{\delta}_1) \to (C_2, \widehat{\delta}_2)$, $G : (C_2, \widehat{\delta}_2) \to (C_1, \widehat{\delta}_1)$, and a homotopy $H : C_1 \to C_1$ such that

$$FG = 1_{C_2}, \qquad 1_{C_1} - GF = \widehat{\delta}_1 H + H\widehat{\delta}_1, \qquad FH = 0, \qquad HG = 0, \qquad H^2 = 0. \tag{A.4}$$

Explicitly, setting $X = (\partial - \partial H_0 \partial + \partial H_0 \partial H_0 \partial - \cdots)$, one can choose

$$F = F_0\big(1 - XH_0\big), \qquad G = \big(1 - H_0 X\big)G_0, \qquad H = H_0 - H_0 X H_0; \qquad \widehat{\partial} = F_0 X G_0. \tag{A.5}$$

□

A.3 Transferred A_∞-structures [7, 13]

Let A be a DG-algebra and E a complex. Consider maps of complexes $F : A \to E$, $G : E \to A$, and a homotopy $H : A \to A$ such that $1_A - GF = d_A H + H d_A$. This data defines an A_∞-structure on E as follows. First, define degree $(n-2)$ "p-kernels" $\mathbf{p}_n : A^{\otimes n} \to A$, $n \geq 2$ with $\mathbf{p}_2 = m_2$, and

$$\begin{aligned} \mathbf{p}_n = (-1)^n m_2\big(1 \otimes H\mathbf{p}_{n-1}\big) + \sum_{k=2}^{n-2} (-1)^{kn} m_2\big(H\mathbf{p}_k \otimes H\mathbf{p}_{n-k}\big) \\ + m_2\big(H\mathbf{p}_{n-1} \otimes 1\big), \quad n \geq 3. \end{aligned} \tag{A.6}$$

Then compositions $\mu_n = F \circ \mathbf{p}_n \circ G^{\otimes n} : E^n \to E$ give an A_∞-structure on E.

A.4 Twisted cochains and functors between (co)modules [10, 12]

Let $C = k \oplus \overline{C}$ be a coaugmented DG-coalgebra, N a comodule over it, and $A = k \oplus \overline{A}$ an augmented DG-algebra. Let $\Delta^{(k)} : C \to C^{\otimes k}$ be the iteration of the coproduct, and $\Delta_N^{(k)} : N \to N \otimes C^{\otimes(k-1)}$ the iteration of the comodule structure map. Then, C (resp., N) is called *cocomplete* if $C = \bigcup_{n\geq 2} \ker(\Delta^{(k)})$ (resp., $N = \bigcup_{n\geq 2} \ker(\Delta_N^{(k)})$). Assume that both hold for C, N.

Let E be an augmented A_∞, algebra, then a degree $+1$ linear map $\tau : C \to E$ is called a *generalized twisted cochain* if τ vanishes on the coaugmentation of C, takes values in $\overline{A}$, and satisfies

$$\tau \circ \delta_C + \delta_E \circ \tau + \sum_{k\geq 2} \mu_k \circ \tau^{\otimes k} \circ \Delta^{(k)} = 0, \tag{A.7}$$

where μ_k are the products on E. Note that the sum is finite on each particular element since C is cocomplete. If $E = A$ is an associative algebra, the sum on the left has only one term corresponding to μ_2, and then τ is called a *twisted cochain*.

If E is strictly unital, a generalized twisted cochain τ gives rise to functors $\mathcal{G}_\tau, \mathcal{F}_\tau$ between the categories of cocomplete C-comodules and strictly unital E-modules, respectively, see [10] and [12, Section 2.2.1]. For a strictly unital A_∞-module M over E, let $\mathcal{G}_\tau(M) = M \otimes C$ with the differential

$$\delta_{\mathcal{G}_\tau(M)} := 1 \otimes \delta_C + \delta_M \otimes 1 + \sum_{k\geq 2} \left(\mu_k^M \otimes 1\right)\left(1 \otimes \tau^{\otimes(k-1)} \otimes 1\right)\left(1 \otimes \Delta^{(k)}\right). \tag{A.8}$$

Similarly, for a right DG-comodule N over C, let $\mathcal{F}_\tau(N) = N \otimes E$ with the differential

$$\delta_{\mathcal{F}_\tau(N)} := 1 \otimes \delta_E + \delta_N \otimes 1 - \sum_{k\geq 2} \left(1 \otimes \mu_k\right)\left(1 \otimes \tau^{\otimes(k-1)} \otimes 1\right)\left(\Delta_N^{(k)} \otimes 1\right). \tag{A.9}$$

This is well defined by cocompleteness. Then, $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ are adjoint: both $\mathrm{Hom}_E(\mathcal{F}_\tau(N), M)$ and $\mathrm{Hom}_C(N, \mathcal{G}_\tau(M))$ are isomorphic to the space of graded k-linear maps $\phi : N \to M$, satisfying

$$\delta_M \phi - \phi \delta_N + \sum_{k\geq 2} \mu_{k,M}\left(\phi \otimes \tau^{\otimes(k-1)}\right)\Delta_N^{(k)} = 0. \tag{A.10}$$

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