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Brauer groups and crepant resolutions

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Abstract

We suggest a twisted version of the categorical McKay correspondence and prove several results related to it.

Keywords: Brauer group; Derived category; McKay correspondence

Contents

1. Introduction

The original McKay correspondence starts with a finite subgroup $G \subset SL(2, \mathbb{C})$ and its natural linear action on $\mathbb{C}^2$. The singular quotient $\mathbb{C}^2/G$ can be resolved by a sequence of blowups of

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singular points. If we request that the canonical class of the resolution is trivial (which ensures that no unnecessary blowups are performed), then such resolution $X \to \mathbb{C}^2/G$ will be unique. It was observed by McKay that the irreducible components of the exceptional divisor of X are in one-to-one correspondence with nontrivial irreducible representations of G. One can reformulate this as a bijection between all irreducible representations of G and a basis in cohomology of X.

A generalization of this statement assumes that a finite group G acts on a smooth irreducible variety U over $\mathbb{C}$ in such a way that

(i) for any $g \in G$ the codimension of the fixed point set U^g is $\geqslant 2$,
(ii) the G-action preserves the canonical bundle of U, and
(iii) the quotient U/G admits a *crepant* resolution X.

Then the cohomology of X has the same dimension as the orbifold cohomology groups

$$H^\bullet_{\mathrm{orb}}(U;G) = \bigoplus_{g\in G} \big(H^\bullet(U^g)\big)^G,$$

where U^g stands for the fixed point set and the action of $h \in G$ sends U^g to $U^{hgh^{-1}}$. When U is a vector space the above expression reduces to a sum of one-dimensional vector spaces over the conjugacy classes, thus recovering the original McKay correspondence at least on the level of dimensions. See [8] for details on orbifold cohomology and [16,20] for the proof of the assertion.

A more general *categorical* version of the McKay correspondence, still largely conjectural, relates G-equivariant vector bundles (or sheaves) on U and usual vector bundles (or sheaves) on X. To formulate the statement one actually needs to consider the corresponding bounded derived categories of sheaves $D^b_G(U)$ and $D^b(X)$, since a vector bundle on X might correspond to a *complex* of G-equivariant sheaves on U, and the other way around. The categorical McKay correspondence conjectures that in this situation there is a derived equivalence

$$D^b_G(U) \xrightarrow{\sim} D^b(X).$$

See [6] for related technical details (most of which will not be used in this paper).

As explained in [3], once such a derived equivalence is established, one can apply Keller's cyclic homology construction, cf. [15], and obtain an isomorphism of $\mathbb{Z}_2$-graded vector spaces

$$H^\bullet_{\mathrm{orb}}(U;G) \simeq H^\bullet(X)$$

recovering the McKay correspondence for cohomology.

The goal of this paper is to describe a conjectural "twisted" version of the categorical McKay correspondence. On one hand, given a class $\alpha \in H^2(G, \mathbb{C}^*)$ one can define the twisted equivariant derived category $D^b_{G,\alpha}(U)$ and the twisted orbifold cohomology $H^\bullet_{\mathrm{orb},\alpha}(U;G)$, cf. [1,19] and Section 2 of this paper. On the other hand, if $\mathcal{A}$ is an Azumaya algebra on X, cf. [12], then we have the corresponding derived category $D^b(X,\mathcal{A})$ and its (co)homology theory $H^\bullet(X,\mathcal{A})$.

By a result of Gabber, a proof of which was recently published by de Jong, cf. [13], any class in the *cohomological* Brauer group $Br(X) = H^2_{\text{ét}}(X, \mathcal{O}^*)_{\mathrm{tors}}$ is represented by an Azumaya algebra, and it follows easily that two Azumaya algebras with the same class in $Br(X)$ lead to equivalent derived categories. Therefore we will denote the above derived category by $D^b(X,\beta)$ where β is the cohomological class of $\mathcal{A}$. The derived category $D^b(X,\beta)$ can also be defined without the reference to Azumaya algebras by using β-twisted coherent sheaves, cf. Section 4 of [7]. One might therefore ask the following

Question. When are the twisted derived categories $D^b(X,\beta)$ and $D^b_{G,\alpha}(U)$ equivalent (respectively when are their homology groups isomorphic)?

Obviously, the classes α and β should be somehow related. It turns out that both $Br(X)$ and $H^2(G,\mathbb{C}^*)$ can be identified with subgroups of $Br(K)$, where $K=\mathbb{C}(X)$ is the field of rational functions on X (cf. discussion after Theorem 2 in Section 3). Hence, for a class in

$$B_G(U) := Br(X) \cap H^2(G,\mathbb{C}^*)$$

one has a twist of $D^b_G(U)$ *and* a twist of $D^b(X)$.

After proving in Section 3 that $Br(X)$ is the same for all resolutions of U/G, not necessarily crepant, we explicitly describe $B_G(U)$ in terms of fixed point subvarieties of G on U, cf. Theorem 4. In the important case when U is a vector space with a linear G-action one has $B_G(U)=Br(X)$. Our proof follows the projective case considered in [5]. The definition and properties of $B_G(U)$ lead to the following

Conjecture *(Twisted McKay correspondence). In the situation described above, let $\alpha \in B_G(U)$. Then there exists a derived equivalence*

$$D^b_{G,\alpha}(U) \xrightarrow{\sim} D^b(X,\alpha).$$

When the G-action is free, we have $B_G(U)=H^2(G,\mathbb{C}^*)$ and the above equivalence reduces to definitions. In Section 4 we give an example of a less trivial case.

In Section 5 we consider the cohomological consequence of the twisted McKay correspondence. It turns out that, in characteristic zero, the homology of $D^b(X,\alpha)$ is simply $H^\bullet(X)$. For affine X this was proved essentially by Cortiñas and Weibel, cf. [9], and in Theorem 5 we deduce the general case from their result. On the other hand, generalizing [3], we also prove in Theorem 6 that the periodic cyclic homology of $D^b_{G,\alpha}(X)$ can be identified with the twisted orbifold cohomology $H^\bullet_\alpha(U;G)$ as defined in [19]. Since the computation of $B_G(U)$, cf. Theorem 4, implies that for $\alpha \in B_G(U)$ one has a vector space isomorphism

$$H^\bullet_\alpha(U,G) \simeq H^\bullet(U;G),$$

the twisted McKay correspondence on homological level simply reduces to the untwisted version (not very exciting, but it is hard to expect anything else since the Brauer group captures only torsion information). It is quite possible that homology with finite coefficients can give something different in the twisted case, but we do not pursue this topic here. Finally, in Section 6 we discuss some related open problems.

Remark. Perhaps it is appropriate to mention here two more versions of the cohomological Brauer group:

(i) the analytic Brauer group $Br_{\mathrm{an}}(X)=H^2_{\mathrm{an}}(X,\mathcal{O}^*_{\mathrm{an}})$, and
(ii) the topological Brauer group $Br_{\mathrm{top}}(X)=H^3(X,\mathbb{Z})_{\mathrm{tors}}$.

One can show that $Br(X) = Br_{\mathrm{an}}(X)_{\mathrm{tors}}$ for all X, and that $Br(X) \simeq Br_{\mathrm{top}}(X)$ whenever $H^2_{\mathrm{an}}(X, \mathcal{O}_{\mathrm{an}}) = 0$.

2. Projective cocycles and twisted group algebras

A finite abelian group A which can be generated by (at most) two elements is called *bicyclic*. Thus, either A is itself cyclic, or it is isomorphic to a product of two cyclic groups.

The next theorem deals with the Schur multiplier $H^2(A, \mathbb{C}^*)$ of A in the second case. We assume that 2-cocycles are normalized: $c(1, g) = c(g, 1) = 1$.

Theorem 1. *Let $A \simeq C_1 \times C_2$ with C_1, C_2 cyclic. Then*:

(a) $H^2(A, \mathbb{C}^*) = Hom(C_1 \otimes_{\mathbb{Z}} C_2, \mathbb{C}^*)$;
(b) *A 2-cocycle $c : A \times A \to \mathbb{C}^*$ is a coboundary iff $c(g, h) = c(h, g)$ for all $g, h \in A$.*

Proof. Part (a) follows from the general results in [14]. The "only if" part in (b) follows from the definition of a coboundary and the fact that A is abelian. To prove the "if" part note that the symmetry condition is preserved if we adjust a cocycle by a coboundary, and by part (a) this adjustment can be made in such a way that the value of $c(g, h)$ will depend only on the image of g in C_1 and the image of h in C_2. By symmetry such a cocycle is trivial. $\square$

Let G be a finite group acting on an affine variety $U = Spec(R)$. Fix a 2-cocycle $c : G \times G \to \mathbb{C}^*$ representing a class in $H^2(G, \mathbb{C}^*)$. The *twisted group algebra* $R^c[G]$ is the set of all linear combinations $\sum_{g \in G} r_g \cdot g$ where $r_g \in R$ and multiplication is given by the rule

$$(r_1 \cdot g_1) * (r_2 \cdot g_2) = c(g_1, g_2)\big(r_1 g_1(r_2) \cdot g_1 g_2\big).$$

The cocycle condition for c is equivalent to associativity of $R^c[G]$. Up to isomorphism, $R^c[G]$ depends only on the class α of c in $H^2(G, \mathbb{C}^*)$, hence we can (and will) denote it by $R^\alpha[G]$. Since c is normalized, $1 \in R$ gives a unity in $R^c[G]$.

Note further that $R^c[G]$ is naturally an algebra over the ring of invariants R^G. Moreover, if the G-action is free, $R^c[G]$ gives an Azumaya algebra over R^G. Localizing this construction, for any G acting freely on a quasiprojective variety U, and any class $\alpha \in H^2(G, \mathbb{C}^*)$ we get an Azumaya algebra $\mathcal{A}^\alpha$ on U/G (defined up to isomorphism).

In general, let $U_0 \subset U$ be the open subset on which the action is free. For any resolution of singularities $X \to U/G$ denote by X_0 the preimage of U_0/G. Then by pullback our construction gives an Azumaya algebra $\mathcal{A}^\alpha$ on X_0 for any $\alpha \in H^2(G, \mathbb{C}^*)$.

3. The Brauer group of a resolution

In this paper a *valuation* will always mean a discrete rank one valuation. All varieties are over the field of complex numbers $\mathbb{C}$. Let Y be a reduced irreducible variety with field of rational functions $K = \mathbb{C}(Y)$, and denote by $Val(Y)$ be set of those valuations of K which become divisorial on *some* resolution $Z \to Y$ (i.e., the corresponding map $\nu : K^* \to \mathbb{Z}$ simply computes the order of a rational function along a fixed prime divisor on Z). The next result clarifies the role of $Val(Y)$ in the computation of the cohomological Brauer group $Br(X) = H^2_{\text{ét}}(X, \mathcal{O}^*)$. We send the interested reader to [12] for the relation of $Br(X)$ and the group of equivalence classes of Azumaya algebras. Note that in [12] our group $Br(X)$ is denoted by $Br'(X)$.

Theorem 2. *If $X \to Y$ is a resolution of singularities, then*

$$Br(X) = \bigcap_{\nu \in Val(Y)} Br(\mathcal{O}_\nu) \subset Br(K).$$

In particular, the Brauer group does not depend on the choice of X. Moreover, once X is fixed, in the above intersection it suffices to consider only the divisorial valuations of X.

Proof. Let $\alpha \in Br(X)$ and let $D \subset Z$ be a prime divisor on some resolution Z, giving a valuation ν. After removing a codimension 2 subset $Z' \subset Z$ we can construct a regular birational map $Z \setminus Z' \to X$. The pullback of α gives a class in $Br(Z \setminus Z')$. Localizing at D we get $\alpha \in Br(\mathcal{O}_\nu)$.

Now let α be a class in the right-hand side of the formula. There exists an affine $U_0 \subset X$ such that $\alpha \subset Br(U_0)$. Let $D_1, \ldots, D_r$ be the irreducible components of $X \setminus U_0$ and $\nu_1, \ldots, \nu_r$ the corresponding valuations. Since $\alpha \in Br(\mathcal{O}_{\nu_i})$ for all i, there exist affine open subsets U_i such that $U_i \cap D_i \neq \emptyset$ and $\alpha \in Br(U_i)$. Therefore $\alpha \in Br(\bigcup_{i=0}^r U_i)$ which is equal to $Br(X)$ by the Purity theorem, cf. [12], since $X \setminus (\bigcup_{i=0}^r U_i)$ has codimension at most 2 in X. The same argument shows that the divisorial valuations of X are sufficient to define $Br(X)$. $\square$

Let G be a finite group acting on a smooth variety U *almost freely* (i.e., the action is free on some open dense subset $U_0 \subset U$). If $L = \mathbb{C}(U)$ is the field of rational functions on U, then $K = \mathbb{C}(U/G)$ can be canonically identified with L^G.

By Hilbert Theorem 90 we have an exact sequence

$$1 \to H^2(G, \mathbb{C}^*) \to Br(K) \to Br(L).$$

In terms of the previous section, a class $\alpha \in H^2(G, \mathbb{C}^*)$ gives an Azumaya K-algebra $L^\alpha[G]$, which belongs to a class in the cohomological Brauer group $Br(K)$. Since the Brauer group of U_0/G (respectively U_0) is a subgroup of $Br(K)$ (respectively $Br(L)$) we actually have an exact sequence

$$1 \to H^2(G, \mathbb{C}^*) \to Br(U_0/G) \to Br(U_0).$$

Again, a class α maps to the class representing the Azumaya algebra $\mathcal{A}^\alpha$ defined in the previous section. By the previous result, the Brauer group $Br(X)$ of a resolution $X \to U/G$ does not depend on the choice of X. We can assume that $X \to U/G$ is an isomorphism over U_0/G, then $Br(X)$ naturally becomes a subgroup of $Br(U_0/G)$. Denote

$$B_G(U) := Br(X) \cap H^2(G, \mathbb{C}^*).$$

We view $B_G(U)$ as a subgroup of $H^2(G, \mathbb{C}^*)$.

The next theorem gives a direct computation of $B_G(U)$ in terms of valuations in $Val(U/G)$. First recall some definitions, cf. [18]. If $\nu \in Val(U/G)$ then G acts transitively on the set of extensions of ν from L^G to L and fixing one such extension μ we get a *decomposition group* $D_\nu \subset G$ (the stabilizer of μ) and the *interia subgroup* $I_\nu \subset D_\nu$ (the kernel of the action on the residue field k_ν of ν). In characteristic zero I_ν is cyclic and central in D_ν. We also need the quotient $G_\nu = D_\nu / I_\nu$, isomorphic to the Galois group of the extension $k_\nu \subset k_\mu$.

Theorem 3. *Let $\alpha \in H^2(G, \mathbb{C}^*)$ be a class.*

(1) *$\alpha \in B_G(U)$ if and only if for all $\nu \in Val(U/G)$ the restriction $\alpha|_{D_\nu}$ belongs to the image of the map $H^2(G_\nu, \mathbb{C}^*) \to H^2(D_\nu, \mathbb{C}^*)$.*
(2) *$\alpha \in B_G(U)$ if and only if $\alpha|_A = 0$ for any subgroup $A \subset G$, such that $\exists \nu \in Val(U/G)$ for which $A \subset D_\nu$ and the image of A in G_ν is cyclic.*
(3) *$\alpha \in B_G(U)$ if and only if $\alpha|_A = 0$ for any subgroup $A \subset G$, such that $\exists \eta \in Val(U/A)$ for which $A = D_\eta$ and G_η is cyclic.*

Proof. To prove (1) let $\mathcal{O}_\nu$ be the valuation ring of ν and $\overline{L_G}$, $\overline{\mathcal{O}_\nu}$ be completions with respect to ν. For the fixed extension μ to L we have similar objects $\mathcal{O}_\mu$, $\overline{L}$, $\overline{\mathcal{O}_\mu}$.

Recall, cf. Theorem 6.5 in [2], that the quotient map $\overline{\mathcal{O}_\nu} \to k_\nu$ induces an isomorphism of Brauer groups. Then (1) follows from the commutative diagrams:

$$\begin{tikzcd}
Br(\mathcal{O}_\nu) \arrow[r] \arrow[d] & Br(L^G) \arrow[d] & H^2(G, \mathbb{C}^*) \arrow[l] \arrow[d, "Res^G_{D_\nu}"'] \\
Br(\overline{\mathcal{O}_\nu}) \arrow[r] & Br(\overline{L_G}) & H^2(D_\nu, \mathbb{C}^*) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^2(G_\nu) \arrow[r] \arrow[d, equal] & Br(k_\nu) \arrow[r] \arrow[d, equal] & Br(k_\mu) \arrow[d, equal] \\
H^2(G_\nu) \arrow[r] \arrow[d] & Br(\overline{\mathcal{O}_\nu}) \arrow[r] \arrow[d] & Br(\overline{\mathcal{O}_\mu}) \arrow[d] \\
H^2(D_\nu, \mathbb{C}^*) \arrow[r] & Br(\overline{L_G}) \arrow[r] & Br(\overline{L}).
\end{tikzcd}$$

In fact, if $\alpha \in H^2(G, \mathbb{C}^*) \cap Br(\mathcal{O}_\nu)$ then by the first diagram $\alpha|_{D_\nu} \in H^2(D_\nu, \mathbb{C}^*) \cap Br(\overline{\mathcal{O}_\nu})$. Since α has zero image in $Br(L)$, $\alpha|_{D_\nu}$ has zero image in $Br(\overline{L})$ and thus by the second diagram it belongs to the image of the inflation map $H^2(G_\nu, \mathbb{C}^*) \to H^2(D_\nu, \mathbb{C}^*)$.

In the other direction, suppose we have class $\beta \in H^2(G_\nu, \mathbb{C}^*)$ such that its image in $H^2(D_\nu, \mathbb{C}^*)$ is a restriction of some $\alpha \in H^2(G, \mathbb{C}^*)$. By the above β gives a class in $Br(\overline{\mathcal{O}_\nu})$ which maps to the image of α in $B(\overline{L^G})$. Now if $Gal(k_\nu)$ stands for the Galois group of the residue field of ν, then by Theorem III.2.20 and Case (a) in [17] one has the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Br(\mathcal{O}_\nu) \arrow[r] \arrow[d] & Br(L^G) \arrow[r] \arrow[d] & Hom(Gal(k_\nu), \mathbb{C}^*) \arrow[d, equal] \\
0 \arrow[r] & Br(\overline{\mathcal{O}_\nu}) \arrow[r] & Br(\overline{L^G}) \arrow[r] & Hom(Gal(k_\nu), \mathbb{C}^*);
\end{tikzcd}$$

therefore α is in the image of $Br(\mathcal{O}_\nu)$.

To prove (2) use the Hochschild–Serre spectral sequence. Since I_ν is cyclic and central in D_ν, we have $H^2(I_\nu, \mathbb{C}^*) = 0$ and the relevant part of E^2 term boils down to

$$H^2(G_\nu, \mathbb{C}^*) \to H^2(D_\nu, \mathbb{C}^*) \to Hom\big(G_\nu, Hom(I_\nu, \mathbb{C}^*)\big).$$

The second arrow can be described as follows. Let $\alpha \in H^2(G, \mathbb{C}^*)$ be a class. It is easy to show that $\alpha|_{D_\nu}$ can be represented by a cocycle $\bar{\alpha} : D_\nu \times D_\nu \to \mathbb{C}^*$ which descends to a map $\bar{\alpha} : G_\nu \times D_\nu \to \mathbb{C}^*$. Moreover, the restriction $\bar{\alpha} : G_\nu \times I_\nu \to \mathbb{C}^*$ is uniquely determined by the class α and it will be bilinear (multiplicativity in the first argument uses the fact that I_ν is central in D_ν).

Comparing with (1) we see that $\alpha \in B_G(U)$ if and only if for any $\nu \in Val(U/G)$ the homomorphism $\bar{\alpha}$ is trivial. If A is as in (2) then we have a similar spectral sequence for A but now both $A \cap I_\nu$ and the image of A in G_ν are cyclic, so $H^2(A, \mathbb{C}^*)$ is a subgroup of $Hom(A \cap I_\nu, Hom(A/A \cap I_\nu, \mathbb{C}^*))$ and in view of $\bar{\alpha} = 1$ we get $\alpha|_A = 0$.

In the other direction, take any $g \in G_\nu$ and $h \in I_\nu$ and lift g to an element $\hat{g}$ in D_ν. Then the subgroup $A \subset G$ generated by $\hat{g}, h$ satisfies the conditions of (2) and by the preceding argument $\alpha|_A = 0$ implies $\bar{\alpha}(g, h) = 1$. Since this holds for all g and h we conclude that $\bar{\alpha}$ is trivial, i.e. $\alpha|_{D_\nu}$ comes from a class in $H^2(G_\nu, \mathbb{C}^*)$. Since this holds for all ν, by part (1) we get $\alpha \in B_G(U)$.

Finally we need to show that the conditions of (2) and (3) are equivalent. In general let $A \subset G$ be a subgroup and η the restriction of μ from L to L^A. It follows from the definitions that $A \cap D_\nu = D_\eta$ and $A \cap I_\nu = I_\eta$. The first equality implies that $A \subset D_\nu$ iff $A = D_\eta$. Assuming this, we see that G_η can be identified with the image of A in G_ν, thus (2) and (3) are equivalent. □

Finally, we relate the algebraic property of A in the previous theorem to fixed point subvarieties of G in U. Note that A as in (3) is automatically *bicyclic*, i.e. abelian with at most two generators, since both I_η and $G_\eta = A/I_\eta$ are cyclic and I_η is central in A.

We say that a bicyclic subgroup $A \subset G$ acts *cyclically* on a subvariety $W \subset U$ if W is A-invariant and A acts on W via some cyclic quotient of A.

Theorem 4. *Let G be a finite with an almost free action on a smooth variety U*

$$B_G(U) = \bigcap_{A \in Cyc(G,U)} Ker\big(H^2(G, \mathbb{C}^*) \to H^2(A, \mathbb{C}^*)\big),$$

where the intersection is taken over the set $Cyc(G, U)$ of all bicyclic subgroups $A \subset G$ which act cyclically on a closed irreducible subvariety $W \subset U$.

Proof. In view of the previous theorem it suffices to show that for a bicyclic subgroup A of G the following conditions are equivalent:

(i) A acts cyclically on a closed irreducible subvariety $W \subset U$, and
(ii) for some $\eta \in Val(U/A)$ we have $A = D_\eta$ and G_η is cyclic.

To prove (ii) $\Rightarrow$ (i) choose a resolution $Z \to U/A$ and a prime divisor E corresponding to η. There exists a A-equivariant birational map $Y \to U$ with smooth Y, and a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ U & \longrightarrow & U/A. \end{array}$$

Denote the map $Y \to Z$ by ρ. Those irreducible components of $\rho^{-1}(E)$ which dominate E correspond to extensions of η from L^A to L. The particular extension μ chosen to define D_η and I_η, gives an irreducible component $F \subset \rho^{-1}(E)$. Since $A = D_\eta$, F is A-invariant and A acts on F via a cyclic quotient ($A/I_\eta = G_\eta$ is cyclic). The image W of F in U is a closed irreducible subvariety on which A acts cyclically.

To prove (i) $\Rightarrow$ (ii) write $A = B \oplus C$ where B, C are cyclic and B acts on W trivially (if A itself is cyclic then B is trivial).

First we find a smooth A-variety $\widetilde{U}$, an A-equivariant birational map $\widetilde{U} \to U$ (not necessarily proper) and an A-invariant divisor $E \subset \widetilde{U}$ on which B acts trivially. To that end, let b be a generator of B. We can replace W by an irreducible component of the fixed point set U^b and assume that $W = U^B$. Next, we can shrink U and assume that W is smooth, irreducible and defined in U by vanishing of a regular sequence of A-eigenfunctions $f_1, \ldots, f_k$. Fix a generator ε in the group of characters $B^\vee$ and assume that B acts on f_i via ε^{m_i} where $m_i \in \{1, \ldots, n-1\}$ (since $W = U^B$ the smooth case of Luna Slice Theorem implies that $m_i \neq 0$). To construct $\widetilde{U}$ and E we proceed by double induction on $k = \operatorname{codim}_U W$ and $\max\{m_1, \ldots, m_k\}$.

If $\operatorname{codim}_U W = 1$ set $\widetilde{U} = U$ and $E = W$. Otherwise consider the blowup U' of U at the regular sequence $f_1, \ldots, f_k$. By definition $U' \subset U \times \mathbb{P}^{k-1}$ is given by equations $f_i \xi_j = f_j \xi_i$ where $[\xi_1 : \cdots : \xi_k]$ are homogeneous coordinates on $\mathbb{P}^{k-1}$. Assume that $m_1 = \min\{m_1, \ldots, m_k\}$ and pass to the open subset $U'' \subset U'$ given by $\xi_1 \neq 0$. Setting $u_i = \xi_i/\xi_1$ for $i = 2, \ldots, k$ we see that $U'' \subset U \times \mathbb{C}^{k-1}$ is given by $f_i = f_1 u_i$, with $2 \leqslant i \leqslant k$. Then U'' has obvious A-action, all u_i are A-eigenfunctions and B acts on u_i via the character $\varepsilon^{m_i - m_1}$. Note that each $m_i'' = m_i - m_1$ is now in $\{0, \ldots, n-2\}$. Define $W'' \subset U''$ by the equations

$$f_1 = u_i = 0, \quad \text{for all } i \in \{2, \ldots, k\} \text{ such that } m_i'' \neq 0.$$

It is easy to see that W'' is an affine bundle over W, hence smooth and irreducible. Also, $W'' \subset U''$ is A-invariant and B acts trivially on W''. Set $m_1'' = m_1$ and note that either $m_i'' \in \{1, \ldots, n-2\}$ for $1 \leqslant i \leqslant k$ and in this case $\max\{m_1'', \ldots, m_k''\} < \max\{m_1, \ldots, m_k\}$; or $\operatorname{codim}_{U''} W'' < \operatorname{codim}_U W$. In both cases we can replace (U, W) by (U'', W'') and find $(\widetilde{U}, E)$ by inductive assumption.

To finish the proof, note that $X = \widetilde{U}/A$ is smooth at the generic point of $E/A = E/C$ and therefore E/A defines a valuation η of the field of rational functions on A. To see that $\eta \in \mathit{Val}(U/A)$ note $\widetilde{U}$ will be proper over U if we do not pass to open subsets but blow up the ideal sheaves of (the closures of) subvarieties W in the inductive proof above. This may lead to a singular $\widetilde{U}$ but X is still smooth at the generic point of E/A hence resolving the singularities of X we get $\eta \in \mathit{Val}(U/A)$. Also, $A = D_\eta$ since E is irreducible and A-invariant, and G_η is cyclic since B acts trivially on E and therefore G_η is a quotient of the cyclic group $C \simeq A/B$. This finishes the proof. $\square$

4. An example

Consider an almost free action of G on a vector space V and assume that for all $g \in G$ we have $\operatorname{codim}(V^g) \geqslant 2$. Take U to be the complement of a G-invariant closed subset Z of codimension $\geqslant 2$. Then $Pic(U) = 0$, $Br(U) = 0$; hence the Brauer group of any resolution $X \to U/G$ is equal to the subgroup $B_G(U) \subset H^2(G, \mathbb{C}^*)$. The condition of Theorem 4 may be reformulated as follows: $\alpha \in B_G(U)$ iff $\alpha|_A = 0$ for all subgroups A generated by a pair of commuting elements (h, g) such that U^h is nonempty and g preserves an irreducible component of U^h. In our case U^h is an open subset of a linear subspace in V hence irreducible, therefore

$$B_G(U) = \{\alpha \in H^2(G, \mathbb{C}^*) \mid \alpha(g,h) = \alpha(h,g) \text{ whenever } U^h \neq \emptyset \text{ and } gh = hg\}.$$

Note that the condition on the right-hand side is preserved when a cocycle is multiplied by a coboundary. For instance, when $U = U_0$ is the subset of all vectors in V with trivial stabilizers, we have $B_G(U) = H^2(G, \mathbb{C}^*)$; when $U = V$ we get the subgroup $B_0(G)$ of classes in $H^2(G, \mathbb{C}^*)$ which restrict to zero on any bicyclic abelian subgroup of G (all fixed point sets are nonempty since they contain the origin of V). This subgroup, known to coincide with *unramified Brauer group of* G, was studied extensively in [4]. Observe, that $B_0(G)$ does not depend on the choice of V.

Groups with $B_0(G) \neq 0$ are relatively rare and the condition that V/G admits a *crepant* resolution puts a further restriction on the pair (V, G) (see the last section of this paper). However, it is possible to find a group G with $B_0(G) \neq 0$ and an open subset U in a representation V such that U/G is not smooth but admits a crepant resolution (then automatically $0 \neq B_0(G) \leqslant B_G(U)$). We now proceed to describe such an example.

Let p be a prime and consider a central extension of the form

$$1 \to \mathbb{Z}_p^3 \to G \to \mathbb{Z}_p^4 \to 1.$$

If (a, b, c) is a basis of $\mathbb{Z}_p^3$ and (x_1, x_2, x_3, x_4) a lift of a basis from $\mathbb{Z}_p^4$ to G, it was proved in [4] (cf. Example 3 before Lemma 5.5) that the relations

$$[x_1, x_2] = [x_3, x_4] = a; \qquad [x_1, x_3] = [x_1, x_4] = 1; \qquad [x_2, x_4] = b; \qquad [x_2, x_3] = c$$

(where $[x, y] = xyx^{-1}y^{-1}$), imply that $B_0(G) \simeq \mathbb{Z}_p$. To describe an exact representation of G let $\varepsilon = \exp(2\pi i/p)$ and choose a pair of $p \times p$ matrices P, Q such that $[P, G] = \varepsilon I$. For $p = 2$ we can take the Pauli matrices

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};$$

and for odd p we can define P as a matrix of the cyclic permutation of basis vectors $(v_1, \ldots, v_p): v_i \mapsto v_{i+1}$ for $i = 1, \ldots, p-1$, and $v_p \mapsto v_1$; then Q will be the diagonal matrix $diag(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1})$.

Let $V \simeq \mathbb{C}^{p^2+2p} = (\mathbb{C}^p \otimes \mathbb{C}^p) \oplus \mathbb{C}^p \oplus \mathbb{C}^p$ be the representation given by

$$\begin{aligned}
&x_1 \mapsto (P \otimes I) \oplus I \oplus I; && x_2 \mapsto (Q \otimes I) \oplus P \oplus P; \\
&x_3 \mapsto (I \otimes P) \oplus I \oplus Q; && x_4 \mapsto (1 \otimes Q) \oplus Q \oplus I; \\
&a \mapsto \varepsilon(I \otimes I) \oplus I \oplus I; && b \mapsto (I \otimes I) \oplus \varepsilon I \oplus I; \qquad c \mapsto (I \otimes I) \oplus I \oplus \varepsilon I.
\end{aligned}$$

Let H_1 be the subgroup of order p^3 in $GL_p(\mathbb{C})$ generated by P and Q. One can check directly, that non-scalar elements in H_1 have p distinct eigenvectors with eigenvalues $1, \varepsilon, \dots, \varepsilon^{p-1}$. For each of these eigenvectors, the stabilizer in H_1 is isomorphic to $\mathbb{Z}_p$. Similarly, all non-scalar elements in the group $H_2 \subset GL_{p^2}(\mathbb{C})$ of order p^5 generated by $H_1 \otimes 1$ and $1 \otimes H_1$ have p eigenspaces of dimension p, with the same eigenvalues. Again, for each of the eigenspaces its stabilizer in H_2 is isomorphic to $\mathbb{Z}_p$.

It follows that for each $g \in G$ the fixed point subspace V^g has codimension $\geqslant p$ and for odd p the fixed subspaces of codimension precisely p are

$$V^b = (\mathbb{C}^p \otimes \mathbb{C}^p) \oplus \mathbb{C}^p \oplus 0 \quad \text{and} \quad V^c = (\mathbb{C}^p \otimes \mathbb{C}^p) \oplus 0 \oplus \mathbb{C}^p.$$

For $p = 2$ in addition to V^b and V^c one also has the fixed point subspace $V^{x_1} = V' \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ where V' is the $(+1)$-eigenspace of $P \otimes 1$. Now define

$$Z := \bigcup_{\{g \in G \mid \operatorname{codim}(V^g) \geqslant (p+1)\}} V^g$$

and set $U = V \setminus Z$. Then the singularities of U/G are the images of V^b, V^c (and V^{x_1} if $p = 2$). A single canonical blowup gives a crepant resolution $X \to U/G$. By the Purity Theorem, cf. [12], and $\operatorname{codim}(Z) \geqslant 2$ we conclude that $Br(X) = B_G(U)$ and this group is non-zero since it contains the subgroup $B_0(G) \simeq \mathbb{Z}_p$.

Further similar examples can be obtained with other finite p-groups listed in [4].

5. Homology of categories

Even if the G-action on U is not free, for every G-invariant affine open subset $U' \subset U$ with algebra of functions R we can still consider $R^\alpha[G]$, cf. Section 2, and modules over this algebra. Localizing at G-invariant affine open subsets of U we get the notion of an *α-twisted equivariant sheaf* $\mathcal{F}$: this is a sheaf of $\mathcal{O}$-modules on U such that for any G-invariant open subset $V \subset U$, the $\mathcal{O}$-module structure on $\mathcal{F}(V)$ is extended to an $\mathcal{O}(V)^\alpha[G]$-module structure, and for different invariant open subsets such structures agree with restriction of sections. Morphisms of α-twisted equivariant sheaves are given by those morphisms of $\mathcal{O}$-modules which commute with the $\mathcal{O}(V)^\alpha[G]$-action for every G-invariant V. Considering the bounded complexes of coherent α-twisted equivariant sheaves and localizing at quasi-isomorphisms, we get the bounded derived category $D^b_{G,\alpha}(U)$ of α-twisted G-equivariant sheaves on U.

Alternatively, denote by the same letter α a cocycle representing the cohomology class. There exists a central group extension

$$1 \to \mathbb{Z}_n \to \widetilde{G} \xrightarrow{\pi} G \to 1$$

and a character $\psi : \mathbb{Z}_n \to \mathbb{C}^*$ such that $\alpha(g,h) = \psi(\tilde{g}\tilde{h}\widetilde{gh}^{-1})$, where $\tilde{g}$, etc. denotes some lift of $g \in G$ to $\widetilde{G}$. The group $\widetilde{G}$ acts on U via its homomorphism to G. Since the subgroup $\mathbb{Z}_n \subset \widetilde{G}$ acts trivially on U, the stalk of a $\widetilde{G}$-equivariant sheaf on U at any point has a natural structure of a $\mathbb{Z}_n$-module. The derived category of equivariant sheaves $D^b_{\widetilde{G}}(U)$ splits into orthogonal direct sum of subcategories corresponding to different characters of $\mathbb{Z}_n$. It follows from the above definition that $D^b_{G,\alpha}(U)$ is equivalent to the subcategory of $D^b_{\widetilde{G}}(U)$ corresponding to the character ψ.

For affine U this reduces to the statement that $R^\alpha[G]$ is isomorphic to a quotient of $R[\widetilde{G}]$ by the ideal J generated by $(t - \psi(t)1)$ with $t \in \mathbb{Z}_n$.

If now $\alpha \in Br(X)$ and α corresponds to an Azumaya algebra $\mathcal{A}$ on X we define $D^b(X, \alpha)$ to be the bounded derived category of finitely generated modules over $\mathcal{A}$.

Suppose that we have a derived equivalence

$$D^b_{G,\alpha}(U) \simeq D^b(X, \alpha). \tag{1}$$

By a construction explained in [15] both derived categories have a series of homology theories, including Hochschild homology, cyclic homology and its variants. We fix any of these homology theories and denote it by H. To simplify notation we disregard the grading on H as we will only need the total dimension of this vector space. Since derived equivalences induce isomorphisms on H, cf. [15], (this result requires an additional assumption, always satisfied in a geometric situation such as ours), the above equivalence (1) should imply an isomorphism of homology.

Let $H^\alpha(X)$ be the homology of $D^b(X, \alpha)$ and $H^\alpha_G(U)$ the homology of $D^b_{G,\alpha}(U)$. For $\alpha = 0$ we drop the superscript α. First, we show that the definition of $H^\alpha(X)$ does not give anything new.

Theorem 5. *The natural inclusion of algebras $\mathcal{O} \hookrightarrow \mathcal{A}$ induces an isomorphism $H(X) \simeq H^\alpha(X)$.*

Proof. In suffices to prove the claim for Hochschild homology ($H = HH_*$), the other cases being a consequence by Proposition 2.4 of [11].

In the affine case the derived category homology coincides with the usual Hochschild homology of rings, hence the result is proved in [9].

In general, we cover X with affine open subsets $\{U_i\}_{i\in I}$ and recall that by a result of Gabber $\alpha|_{U_i}$ does come from an Azumaya algebra. Therefore, applying the Mayer–Vietoris sequence and Noetherian induction we finish as in Proposition 3.3 in [3]. □

The computation of $H^\alpha_G(U)$ is given by a theorem parallel to Theorem 7.4 in [1]. For any $g \in G$ denote by Z_g the centralizer of g and observe that the fixed point subvariety U^g is Z_g-invariant. Following [1], we denote by L^α_g the one-dimensional representation of Z_g on which $h \in Z_g$ acts by $\alpha(g, h)\alpha(h, g)^{-1}$.

Theorem 6. *Let U be a smooth complex variety with an action of a finite group G and let $\alpha \in H^2(G, \mathbb{C}^*)$. Then*

$$H^\alpha_G(U) = \bigoplus_{(g)} \big(H\big(U^g\big) \otimes L^\alpha_g\big)^{Z_g}$$

where the sum is taken over all conjugacy classes of G.

Proof. Let $\widetilde{G}$, π and ψ be as in the beginning of this section. By the main result of [3] the homology of $D^b_{\widetilde{G}}(U)$ can be identified with $(\bigoplus_{f\in\widetilde{G}} H(U^f))^{\widetilde{G}}$ where an element $t \in \widetilde{G}$ sends

$$U^f \mapsto U^{tft^{-1}}$$

inducing an action on homology. Since the derived category $D^b_{\widetilde{G}}(U)$ splits into orthogonal direct sum of subcategories labeled by characters of $\mathbb{Z}_n \subset \widetilde{G}$, we just have to extract from the above expression the component corresponding to ψ.

It follows from Step 2 after the proof of Proposition 3.2 in [3], that the induced $\mathbb{Z}_n$-action on $\bigoplus_{f\in\widetilde{G}} H(U^f)$ can be describe as follows: an element $h \in \mathbb{Z}_n$ sends $H(U^f)$ to $H(U^{hf})$ (both fixed point spaces are the same, but h permutes different copies of the same homology group in the direct sum). Since $\mathbb{Z}_n$ is central in $\widetilde{G}$ and acts trivially on U, this action commutes with the earlier $\widetilde{G}$-action.

To compute the component of ψ in $(\bigoplus_{f\in\widetilde{G}} H(U^f))^{\widetilde{G}}$ we split the direct sum by grouping together those f which map to the same conjugacy class in G. For a conjugacy class $C \subset G$ consider

$$W_C = \bigoplus_{\pi(f)\in C} H\big(U^f\big).$$

Denote $\widetilde{Z}_g = \pi^{-1}(Z_g) \subset \widetilde{G}$, then the $\widetilde{G}$-module W_C is induced from the $\widetilde{Z}_g$-module

$$W_g = \bigoplus_{\pi(f)=g} H\big(U^f\big).$$

As a $\mathbb{Z}_n$-module the latter space is just a multiple of the regular representation of $\mathbb{Z}_n$. By definition of $\widetilde{G}$ the component of ψ in the latter sum, viewed as a Z_g-module, is simply $H(U^g) \otimes L_g^\alpha$. Taking the invariants and summing over all conjugacy classes of G we obtain the right-hand side of the formula stated in the theorem. □

In our last result we specialize to periodic cyclic homology, which is equal to the usual topological cohomology by a result of Feigin–Tsygan, cf. [10]. This result provides an indirect evidence for the twisted McKay correspondence conjectured in this paper.

Corollary 7. *Let $X \to U/G$ be a crepant resolution. For any $\alpha \in B_G(U)$ the derived categories $D^b(X,\alpha)$ and $D^b_{G,\alpha}$ have periodic cyclic homology of the same dimension.*

Proof. One one hand, any homology theory of $D^b(X,\alpha)$ is isomorphic to that of $D^b(X)$. On the other hand, by definition of $B_G(U)$ the character L_g^α vanishes whenever U^g is non-empty. Therefore the previous theorem implies that also $D^b_{G,\alpha}(U)$ and $D^b_G(U)$ have the same cyclic homology theories. Applying periodic cyclic homology to $D^b_G(U)$ (respectively $D^b(X)$) we get orbifold cohomology of U (respectively usual cohomology of X). But these have the same dimension by [16,20]. □

6. Open problems

In conclusion we state the following open problems:

(i) If would be interesting to construct an example of a finite group G with a linear action on a vector space V, such that $B_G(V) = B_0(V) \neq 0$ and V/G admits a crepant resolution. Such examples should be relatively rare; for instance the standard symplectic example $V =$

$W \oplus W^*$ will definitely not work, for in this case V/G admits a crepant resolution iff G acts on W by complex reflections which implies $B_0(G) = 0$ (this is because $B_0(G)$ does not depend on the choice of V and W/G is isomorphic to an affine space).

(ii) The second problem refers to the subgroup $B_G(U) \subset H^2(G, \mathbb{C}^*)$. Assume for simplicity that $Br(U) = 0$ then $Br(X) = B_G(U)$ for any resolution $X \to U/G$. Is it possible, however, to define a "derived Brauer group" purely in terms of the (enhanced) derived category $D^b(X)$? Such a group should contain the full Schur multiplier $H^2(G, \mathbb{C}^*)$ in the simple case above, and in general it should contain $Br(X)$ as a subgroup. We ask this question by analogy with the *derived Picard group*, which is a natural extension of the usual Picard group. The Merkurjev–Suslin theorem suggests that some answer may perhaps be obtained from K_2, but for practical purposes it should be more computable than K_2.

(iii) The third problem is related to the above two. Suppose we have an action of G on a vector space V and V/G admits a crepant resolution X. As we have seen in this paper, not all Brauer classes of $\mathbb{C}(V)^G$ extend to X. For example, when $G = S_N$ is the symmetric group acting of $V = (\mathbb{C}^2)^{\oplus n}$, the Hilbert scheme $Hilb^n(\mathbb{C}^2)$ of points on $\mathbb{C}^2$ provides a crepant resolution of V/G and it is easy to check that $Br(Hilb^n(\mathbb{C}^2)) = 0$. In general, if $\alpha \in H^2(G, \mathbb{C}^*) \setminus B_0(G)$ it would be interesting to find an interpretation of the orbifold cohomology $H^*_{G,\alpha}(V)$ in terms of X. To state the question differently: what type of geometric objects on X will correspond to projective representations of G?

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