

BGG correspondence for toric complete intersections

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Abstract

We prove a BGG-type correspondence describing coherent sheaves on complete intersections in toric varieties, and a similar assertion for the stable categories of related complete intersection singularities.

1 Introduction

This paper is a continuation of the earlier article on complete intersections in projective spaces, cf. [Ba]. We consider here the case of a complete intersection Y in a toric variety X_Σ over a field k of characteristic zero. In the case when X_Σ has singularities, we actually study the corresponding stacks $\mathbb{Y} \subset \mathbb{X}_\Sigma$ (this point of view is also used, for instance, when toric complete intersections are considered in Mirror Symmetry). Our goal is to give an alternative description for the category of sheaves on such a \mathbb{Y} in the spirit of the one given by Bernstein-Gelfand-Gelfand in [BGG] for projective spaces and by Kapranov in [Ka1] for intersections of projective quadrics. The general approach is modeled on the Koszul duality of Beilinson-Ginzburg-Schechtman, cf. [BGS], but in our case we deal with the higher products on the “Koszul dual” which arise from the fact that the original algebra had non-quadratic relations.

Now we describe the contents in more detail. In the above setting \mathbb{X}_Σ has a “homogenous coordinate ring” S isomorphic to a polynomial algebra graded by a finitely generated abelian group A , cf. [C]. If W_1, \dots, W_m are the defining equations of \mathbb{Y} and J is the ideal of S generated by these equations, then the category $Coh(\mathbb{Y})$ of coherent sheaves on \mathbb{Y} is obtained from the category of finitely generated A -graded modules over $S_W = S/J$ by passing to a certain categorical quotient, see Section 4 for details.

We first study A -graded modules over S_W . In Section 2 we use the polynomials W_1, \dots, W_m to define, more or less tautologically, an L_∞ -algebra L . We further construct an A_∞ -algebra E_W which should be viewed as the “universal enveloping” of L . When W_1, \dots, W_m have no linear terms (which one can always assume replacing S by a quotient polynomial algebra), E_W has zero differential. In the case when all W_j are quadratic E_W becomes the associative graded Clifford algebra considered by Kapranov in [Ka1]. The proof proceeds differently from [Ba] since we do not assume that W_1, \dots, W_m are homogeneous with respect to the usual grading on S (which is necessary for toric applications). Ideally, one would like to characterize E_W as the unique “homotopy bialgebra” of some special sort, such that the restriction of A_∞ -products to $L \subset E_W$ is given by the homogeneous components of W_1, \dots, W_m . However, we leave the task of writing the agreement conditions between the A_∞ -products and the natural coproduct on E_W , to a forthcoming paper.

In Section 3 we prove, see Theorem 7, an equivalence between A_∞ -modules over E_W and L_∞ -modules over L (the latter are viewed as modules over the standard Cartan-Eilenberg-Chevalley coalgebra C of L). The proof follows the general formalism developed in [Le] which we expand slightly to the A_∞ -case. When S_W is graded by A as above and all graded components are finite-dimensional, we obtain an equivalence between A -graded modules over S_W and E_W , see Theorem 8. This result is applied in Section 4 to the derived category of sheaves on a toric complete intersection \mathbb{Y} and to the stable category of the affine complete intersection defined by W_1, \dots, W_m . When \mathbb{Y} has trivial canonical class (with an additional technical assumption always satisfied for intersections in weighted projective spaces) an easy application of a result due to Orlov, cf. [O2], gives an alternative description of the derived category of \mathbb{Y} , cf. Corollary 12.

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2 A universal enveloping algebra

2.1 Differential operators and corrected partial derivatives

Fix a finite dimensional vector space V over k . The symmetric algebras $Sym^\bullet(V)$ and $Sym^\bullet(V^*)$ may be viewed as algebras of differential operators (with constant coefficients) over each other. For any $f \in Sym^\bullet(V)$ let ∂_f be the corresponding operator on $Sym(V^*)$, and similarly for $g \in Sym^\bullet(V^*)$. There is a pairing $\langle \cdot, \cdot \rangle : Sym^\bullet(V) \times Sym^\bullet(V^*) \rightarrow k$ given by

$$\langle f, g \rangle := \partial_f(g)(0) = \partial_g(f)(0)$$

With respect to this pairing, ∂_g is adjoint to multiplication by g on $Sym^\bullet(V^*)$.

We will also need “corrected partial derivatives”: for $v \in V$ let $\widehat{\partial}_v$ be the operator which sends $g \in Sym^k(V^*)$ to $\frac{1}{k}\partial_v(g)$ for $k \geq 1$ and satisfies $\widehat{\partial}_v(1) = 0$.

For a vector space U we view $Sym^\bullet(V)$ as differential operators on $Sym^\bullet(V^*) \otimes U$ extending derivatives (usual or “corrected”) by linearity in the second factor.

2.2 Koszul complex and an L_∞ -algebra.

Choose and fix a regular sequence $W_1, \dots, W_m \in Sym^{\geq 1}(V^*)$. Introducing new variables z_1, \dots, z_m which span a vector space U we can encode the above sequence in a single “total potential”

$$W = W_1 z_1 + \dots + W_m z_m \in Sym^\bullet(V^*) \otimes U$$

Unlike in [Ba], we do not make the assumption that W_j are homogeneous.

Due to the regularity, the quotient $S_W = Sym^\bullet(V^*)/J$ by the ideal generated J by W_j , $j = 1, \dots, m$, admits a Koszul resolution $B = Sym^\bullet(V^*) \otimes \Lambda^\bullet(U^*)$ where the differential δ_B is given by W , if we agree that z_j act on $\Lambda^\bullet(U^*)$ by contraction and W_j on $Sym^\bullet(V^*)$ by multiplication.

The differential δ_C of the dual coalgebra $C = Sym^\bullet(V) \otimes \Lambda(U)$ is also given by W but now we think of W_j as differential operators and z_j act by multiplication (in the natural algebra structure on C). The assumption that the sequence W_1, \dots, W_m is regular will not be needed in this Section.

Introduce an L_∞ -algebra $L = \{0 \rightarrow V \rightarrow U\}$, cf. [LM2], placing V in homological degree 1, U in homological degree 2 and defining the L_∞ -operations as follows. We set

$$l_k(v_1, \dots, v_k) := \partial_{v_1 \dots v_k}(W)(0) = k! \widehat{\partial}_{v_1} \dots \widehat{\partial}_{v_k}(W)(0)$$

whenever all arguments $v_1 \dots, v_k$ are in V , and let $l_k = 0$ otherwise. The L_∞ -identities for L will hold trivially, since every double composition involved in them vanishes. Note that the coproduct of C is independent of W , but its differential contains full information about it. Also, L is isomorphic as a vector space to the space of primitive elements in C . In fact, one has the following lemma which is immediate from definitions

Lemma 1 *The coalgebra C is isomorphic to the cocommutative coalgebra $C(L)$ of L , cf. [LM2].*

2.3 A standard resolution of L

We now describe a resolution $L \rightarrow \mathcal{L}$ in which the bracket does not depend on W . Let \mathcal{L} be the graded vector space with $\mathcal{L}^1 = [Sym^{\geq 1}(V^*) \otimes U] \oplus V$ in homological degree 1 and $\mathcal{L}^2 = Sym^\bullet(V^*) \otimes U$ in homological degree 2. Define the differential

$$\delta_{\mathcal{L}}(f \oplus v) = \widehat{\partial}_v(W) - [f] \quad (1)$$

where for $f \in Sym^{\geq 1}(V^*) \otimes U \subset \mathcal{L}^1$ we denote by $[f]$ its copy in \mathcal{L}^2 . The bracket $\{ , \} : \mathcal{L}^1 \times \mathcal{L}^1 \rightarrow \mathcal{L}^2$ is defined by

$$\{f_1 \oplus v_1, f_2 \oplus v_2\} = \widehat{\partial}_{v_1}(f_2) + \widehat{\partial}_{v_2}(f_1) \quad (2)$$

For any $w \in Sym^\bullet(V^*) \otimes U$ denote by $\overline{w} \in Sym^{\geq 1}(V^*) \otimes U$ its image with respect to the natural projection which has $k \otimes U$ as its kernel. Then the morphism of complexes $G_1 : L \rightarrow \mathcal{L}$

$$G_1(v) = \overline{\partial_v(W)} + v \in [Sym^{\geq 1}(V^*) \otimes U] \oplus V, \quad G_1(u) = u \in U \subset \mathcal{L}^2; \quad v \in V = L^1, u \in U = L^2$$

is a quasi-isomorphism, but not a morphism of DG Lie algebras. However, introducing morphisms

$$G_k : L^k \rightarrow \mathcal{L}, \quad (v_1, \dots, v_k) \mapsto k! \overline{\widehat{\partial}_{v_1} \dots \widehat{\partial}_{v_k}(W)}$$

whenever all v_i are in V , and zero otherwise; we extend G_1 to an L_∞ -morphism $\{G_k\}_{k \geq 1}$, cf. [LM1]. The L_∞ -morphism condition of *loc. cit.* in our case reduces to

$$G_1(l_k(v_1, \dots, v_k)) + \delta_{\mathcal{L}}(G_k(v_1, \dots, v_k)) = \sum_{i=1}^k \{G_{k-1}(v_1, \dots, \widehat{v}_i, \dots, v_k), G_1(v_i)\} \quad (3)$$

when $k \geq 3$; while for $k = 2$ one has

$$G_1(l_2(v_1, v_2)) + \delta_{\mathcal{L}}(G_2(v_1, v_2)) = \{G_1(v_1), G_1(v_2)\}$$

We note here that it is precisely (3) why we use ‘‘corrected partial derivatives’’ in the definitions of l_k and G_k .

The individual maps G_k , $k \geq 1$ can be organized into a single map $G_\infty : C \rightarrow Sym^{\geq 1}(V) \rightarrow \mathcal{L}$. Since C is a cocommutative coalgebra, by Lemma 22.1 in [FHT] there is a unique comultiplicative extension $\tau : C \rightarrow Sym_c^\bullet(\mathcal{L})$ into the symmetric coalgebra of \mathcal{L} . We further use Poincare-Birkhoff-Witt to identify $Sym_c^\bullet(\mathcal{L})$ with the universal enveloping $U(\mathcal{L})$ of \mathcal{L} (as DG coalgebras).

The following lemma deals the multiplicative behavior of τ with respect to the standard universal enveloping product m_2 in of $U(\mathcal{L})$ and the product in the reduced cobar construction $\Omega(C)$. See e.g. [FHT] and [Ka2] regarding the definitions and properties of the cobar construction.

Lemma 2 *The unique comultiplicative extension $\tau : C \rightarrow \text{Sym}_c^\bullet(\mathcal{L}) \simeq U(\mathcal{L})$ satisfies the twisted cochain condition*

$$\tau \circ \delta_C + \delta_{U(\mathcal{L})} \circ \tau + m_2 \circ \tau^{\otimes 2} \circ \Delta = 0.$$

Its own canonical multiplicative extension $\Omega(\tau) : \Omega(C) \rightarrow U(\mathcal{L})$ is a quasi-isomorphism of DG algebras.

Proof. By (3) above the map $G_\infty : C \rightarrow \mathcal{L}$ extends to a morphism of DG-coalgebras $C \rightarrow C(\mathcal{L})$, where $C(\cdot)$ stands for the Cartan-Eilenberg-Chevalley coalgebra of a DG Lie algebra, cf. [FHT]. It is an easy computation that the composition of natural maps $C(\mathcal{L}) \rightarrow \mathcal{L} \rightarrow U(\mathcal{L})$ does satisfy the twisted cochain condition. Since $\tau : C \rightarrow U(\mathcal{L})$ factors as $C \rightarrow C(\mathcal{L}) \rightarrow U(\mathcal{L})$, the first assertion follows.

For the second assertion note that $\Omega(C) \rightarrow U(\mathcal{L})$ commutes with differentials due to the twisted cocycle property of τ .

For any DG coalgebra C' let $\mathbb{L}(C')$ be Quillen's free DG Lie algebra of C' , cf. Section 22(e) of [FHT]. Then $\Omega(C') \simeq U\mathbb{L}(C')$ by the universal properties of the three objects involved. Now decompose $\Omega(\tau)$ as $\Omega(C) \rightarrow \Omega(C(\mathcal{L})) = U\mathbb{L}(C(\mathcal{L})) \rightarrow U(\mathcal{L})$. The first arrow is a quasi-isomorphism because the L_∞ map $G_\infty : C = C(\mathcal{L}) \rightarrow \mathcal{L}$ extends to a quasi-isomorphism of DG coalgebras (this follows from the fact that G_1 is a quasi-isomorphism of complexes). The second arrow is a quasi-isomorphism since it is induced by a quasi-isomorphism of DG Lie algebras $\mathbb{L}(C(\mathcal{L})) \rightarrow \mathcal{L}$, cf. Theorem 22.9 in [FHT]. \square .

2.4 The universal enveloping A_∞ -algebra E_W

By Theorem 22.9 in [FHT] for a DG Lie algebra L' one has a quasi-isomorphism of DG algebras $\Omega(C(L')) \rightarrow U(L')$. We want to use this fact to define an A_∞ -structure on the symmetric coalgebra $\text{Sym}_c(L)$ which should be viewed as the “universal enveloping” algebra of L . An ideal strategy would be as follows: first replace W in the definition of L with the potential $W^{(2)}$ obtained from W by erasing the terms of degree ≥ 3 in the usual homogeneous grading of $\text{Sym}^\bullet(V^*)$. In other words, we forget all higher brackets on L which in our case leads to a DG Lie algebra L_2 and a quasi-isomorphism $\Omega(C(L_2)) \rightarrow U(L_2)$. Bringing back the degree ≥ 3 terms of W amounts to perturbing the differential on $\Omega(C(L_2))$ and the “sum over binary trees” formula of [KS] tells us that this perturbation induces an A_∞ -structure on $U(L_2)$.

However, this formula involves an explicit contracting homotopy on $\Omega(C(L_2))$ which we are not able to write down at the moment. Therefore we replace $\Omega(C)$ by a smaller DG algebra $U(\mathcal{L})$ which is quasi-isomorphic to it by Lemma 2. Moreover, we do not apply the “sum over binary trees” formula but rather the results of [GLS] which, in a sense, stand behind it. In more detail: replacing W by $W^{(2)}$ gives a DG Lie algebra \mathcal{L}_2 and a quasi-isomorphism of DG Lie algebras $G_1 : L_2 \rightarrow \mathcal{L}_2$ (all higher G_k vanish in for $W^{(2)}$). We also denote by \mathcal{L}_1 and L_1 the same objects viewed as complexes with trivial Lie bracket. First we construct a canonical contracting homotopy on $U(\mathcal{L}_1)$ and then take into account the Lie brackets and use [GLS] to define an A_∞ -map $\{F_k\}_{k \geq 1}$ of associative algebras $U(\mathcal{L}_2) \rightarrow U(L_2)$ and a system of higher homotopies $\{H_k\}_{k \geq 1}$ on $U(\mathcal{L}_2)$. Finally, we replace $W^{(2)}$ by W and then the constructed system of homotopies gives an A_∞ -structure on E_W .

The advantage of this approach, which is more complicated than fixing a non-canonical homotopy on $U(\mathcal{L}_2)$, is that the resulting A_∞ structure on $U(L_2)$ only depends on the resolution \mathcal{L}

and, in addition, it has some compatibility with the coproduct (see the remark at the end of this section).

So replace W by $W^{(2)}$ as above and consider the complexes L_1, \mathcal{L}_1 . The map $G_1 : L_1 \rightarrow \mathcal{L}_1$ of the previous section admits a left inverse $F : \mathcal{L}_1 \rightarrow L_1$ which projects $[Sym^{\geq 1}(V^*) \otimes U] \oplus V = (\mathcal{L}_1)^1$ onto $V = (L_1)^1$ in an obvious way (the superscripts denote homological grading), and sends $Sym^\bullet(V^*) \otimes U = (\mathcal{L}_1)^2$ to $U = (L_1)^2$ by evaluating the constant term. Define a homotopy $H : \mathcal{L}_1^2 \rightarrow \mathcal{L}_1^1$ by sending w to $\{\overline{w}\}$ (we use braces to emphasize that an even element w was converted into an odd element). The ‘‘side conditions’’

$$HG_1 = 0, \quad HH = 0, \quad FH = 0$$

follow immediately from the definitions.

Now we consider the symmetric DG bialgebras (in the graded sense) $Sym^\bullet(\mathcal{L}_1)$ and $Sym^\bullet(L_1)$ and the natural extensions of F and G_1 given by multiplicative and comultiplicative maps $F_{sym} : Sym^\bullet(\mathcal{L}_1) \rightarrow Sym^\bullet(L_1)$, and $G_{sym} : Sym^\bullet(L_1) \rightarrow Sym^\bullet(\mathcal{L}_1)$. To define a homotopy H_{sym} we first set $\mathcal{S} = Sym^{\geq 1}(V^* \otimes U)$ and denote by $\{S\}, [S]$ its copies in \mathcal{L}_1^1 and \mathcal{L}_1^2 , respectively. Since $\mathcal{L}_1 = G_1(L_1) \oplus (\{S\} \rightarrow [S])$ as complexes, we have an isomorphism of DG bialgebras

$$Sym^\bullet(\mathcal{L}_1) \simeq Sym^\bullet(\{S\} \rightarrow [S]) \otimes Sym^\bullet(L_1).$$

The graded symmetric bialgebra $Sym^\bullet(\{S\} \rightarrow [S]) \simeq \Lambda^\bullet(S) \otimes Sym^\bullet(S)$ has standard Koszul differential, and therefore a standard homotopy

$$H_{sym} \left(\{f_1\} \cdots \{f_m\} [g_1] \cdots [g_k] \right) = \frac{-1}{k+m} \sum_{t=1}^k \{f_1\} \cdots \{f_m\} \{g_t\} [g_1] \cdots \widehat{[g_t]} \cdots [g_k]. \quad (4)$$

This we extend to $Sym^\bullet(\mathcal{L}_1)$ as $H_{sym} \otimes 1$ denoting the extension again by H_{sym} .

The contraction $(F_{sym}, G_{sym}, H_{sym})$ induces a similar contraction (F'_B, G'_B, H'_B) on the reduced bar constructions (see Section 19 of [FHT] and [Ka2] for definitions and properties). Here F'_B, G'_B are defined in an obvious way and

$$H'_B|_{(Sym^{\geq 1}(\mathcal{L}_1))^{\otimes k}} = \sum_{s=0}^{k-1} 1^{\otimes s} \otimes H_{sym} \otimes (G_{sym} F_{sym})^{\otimes (k-s-1)}$$

Then F'_B and G'_B are maps of DG coalgebras and H'_B is a coalgebra homotopy:

$$\Delta H'_B = (1 \otimes H'_B + H'_B \otimes G'_B F'_B) \Delta \quad (5)$$

The side conditions for $(F_{sym}, G_{sym}, H_{sym})$ and (F'_B, G'_B, H'_B) follow from those for (F, G_1, H) .

Next we replace (\mathcal{L}_1, L_1) by (\mathcal{L}_2, L_2) , taking into account the Lie structures. The symmetric DG bialgebras of \mathcal{L}_1 and L_1 turn into the universal enveloping DG bialgebras $U(\mathcal{L}_2), U(L_2)$, respectively. Denote by $\rho : Sym^\bullet(\cdot) \rightarrow U(\cdot)$ the Poincare-Birkhoff-Witt isomorphism which identifies the two spaces as DG coalgebras, cf. Propositions 21.2 and 22.6 in [FHT]. Denote by $*$ the product in the universal enveloping and by \cdot the product in the symmetric bialgebra.

Since both \mathcal{L}_2 and L_2 are 2-step nilpotent (i.e. all double brackets vanish) it is easy to track the multiplicative behavior of ρ . In fact, let K be a general graded Lie algebra with bracket l_2 such that $l_2(l_2(a, b), c) = 0$ for all $a, b, c \in K$. Then for odd elements $v_i, w_j \in K$

$$\rho(v_1 \cdot \dots \cdot v_n) * \rho(w_1 \cdot \dots \cdot w_m) = \tag{6}$$

$$\sum_{k \geq 0} \sum_{\substack{I=\{i_1, \dots, i_k\} \subset \{1, \dots, n\} \\ J=\{j_1, \dots, j_k\} \subset \{1, \dots, m\}}} (-1)^{(I, J)} \det \left| \frac{1}{2} l_2(v_{i_p}, w_{j_q}) \right|_{p, q=1, \dots, k} * \rho(v_{r_1} \cdot \dots \cdot v_{r_{n-k}} \cdot w_{l_1} \cdot \dots \cdot w_{l_{m-k}})_{r_t \notin I, l_s \notin J}$$

where I, J are subsets of equal cardinality with the induced natural ordering and $(-1)^{(I, J)}$ is defined by the equality in $Sym^\bullet(K)$:

$$v_1 \cdot \dots \cdot v_n \cdot w_1 \cdot \dots \cdot w_m = (-1)^{(I, J)} v_{r_1} \cdot \dots \cdot v_{r_{n-k}} \cdot v_{i_k} \cdot \dots \cdot v_{i_1} \cdot w_{j_1} \cdot \dots \cdot w_{j_k} \cdot w_{l_1} \cdot \dots \cdot w_{l_{m-k}}$$

This formula is proved by first considering the case $n = 1$ where it reduces to an easy computation, and then iterating and (anti)symmetrizing in v_1, \dots, v_n .

Using the PBW isomorphism we can view $(F_{sym}, G_{sym}, H_{sym})$ as a contracting homotopy between $U(\mathcal{L}_2)$ and $U(L_2)$ but now F_{sym} will not be multiplicative since $F : \mathcal{L}_2 \rightarrow L_2$ is not a Lie map. To “repair” this we adjust the homotopy on the bar construction.

Let D_B, d_B be the canonical differentials of $BU(\mathcal{L}_2), BU(L_2)$ and denote by D'_B, d'_B another pair of differentials on the same spaces, arising from their PBW isomorphism with $BSym^\bullet(\mathcal{L}_2)$ and $BSym^\bullet(L_2)$, respectively. The contracting homotopy (F'_B, G'_B, H'_B) may be viewed as a homotopy between $(BU(\mathcal{L}_2), D'_B)$ and $(BU(L_2), d'_B)$. Using the Basic Perturbation Lemma we can adjust it to work with D_B, d_B as follows.

Let $\delta_B = D_B - D'_B$. Explicitly, δ_B is obtained by considering the map

$$\delta_U : U(\mathcal{L}_2) \otimes U(\mathcal{L}_2) \rightarrow U(\mathcal{L}_2), \quad \rho(a) \otimes \rho(b) \mapsto \rho(a) * \rho(b) - \rho(a \cdot b)$$

and then extending to $BU(\mathcal{L}_2)$ as a coderivation. Now set

$$X = \delta_B - \delta_B H'_B \delta_B + \delta_B H'_B \delta_B H'_B \delta_B - \dots$$

This infinite expression is well defined since δ_B decreases the tensor degree in $BU(\mathcal{L}_2)$ by 1 and H'_B preserves this degree. By Basic Perturbation Lemma, cf. e.g. [GLS], the formulas

$$F_B = F'_B(1 - XH'_B); \quad H_B = H'_B(1 - XH'_B); \quad G_B = (1 - H'_B X)G'_B$$

define a contracting homotopy between complexes $(BU(\mathcal{L}_2), D_B)$ and $(BU(L_2), d'_B + F'_B X G'_B)$

Proposition 3 *The following properties hold*

1. $\delta_B G'_B = G'_B(d_B - d'_B)$, $G_B = G'_B$, $d_B = d'_B + F'_B X G'_B$;
2. F_B is a coalgebra map and H_B is a coalgebra homotopy, see (5);
3. F_B and H_B are uniquely determined by the compositions

$$F_{B,k} : (U^{\geq 1}(\mathcal{L}_2))^{\otimes k} \rightarrow BU(\mathcal{L}_2) \rightarrow BU(L_2) \rightarrow U^{\geq 1}(L_2);$$

$$H_{B,k} : (U^{\geq 1}(\mathcal{L}_2))^{\otimes k} \rightarrow BU(\mathcal{L}_2) \rightarrow BU(L_2) \rightarrow U^{\geq 1}(L_2).$$

Proof. The identity $\delta_B G'_B = G'_B(d_B - d'_B)$ follows from the fact that $G_1 : L_2 \rightarrow \mathcal{L}_2$ commutes with brackets. The other two identities follow from it and a side condition $H'_B G'_B = 0$.

Part (2) is proved in [GLS]. Part (3) is an easy consequence: for F_B it is well-known, cf. e.g. [Ka2], while for H_B one has an explicit formula

$$H_B|_{(U^{\geq 1}(\mathcal{L}_2))^{\otimes k}} = \sum_{s+p+q=k} 1^{\otimes s} \otimes H_{B,p} \otimes (G_B F_B)^{\otimes q}. \quad \square$$

To summarize the above: we have defined a map of DG bialgebras $G_{sym} : U(L_2) \rightarrow U(\mathcal{L}_2)$, an A_∞ -map of associative algebras $\{F_{B,k} : U(\mathcal{L}_2)^{\otimes k} \rightarrow U(L_2)\}_{k \geq 1}$, and a system of ‘‘higher homotopies’’ $\{H_{B,k} : U(\mathcal{L}_2)^{\otimes k} \rightarrow U(\mathcal{L}_2)\}_{k \geq 1}$ which are encoded in a coalgebra contraction (F_B, G_B, H_B) from $BU(\mathcal{L}_2)$ to $BU(L_2)$.

It is easy to see that $F_{B,k}$ and $H_{B,k}$ are given by (restrictions of) $(-1)^{k-1} F_{sym}(\delta_B H'_B)^{k-1}$ and $(-1)^{k-1} H_{sym}(\delta_B H'_B)^{k-1}$, respectively.

Now we want to pass from $W^{(2)}$ to the full potential W . This means that the differential D_B on $BU(\mathcal{L}_2)$ will be replaced by $\tilde{D}_B = D_B + \tilde{\delta}_B$. To describe $\tilde{\delta}_B$ we first define a differential $\tilde{\delta}_L$ on \mathcal{L}_2 which sends $v \in V \subset \mathcal{L}_2$ to $\tilde{\delta}_v(W - W^{(2)})$ (and vanishes on the natural complement to V); then extend $\tilde{\delta}_L$ to a derivation $\tilde{\delta}_U$ on $U(\mathcal{L}_2)$; and finally extend $\tilde{\delta}_U$ to a coderivation $\tilde{\delta}_B$ on $BU(\mathcal{L}_2)$. Using the Basic Perturbation Lemma again, we set

$$\tilde{X} = \tilde{\delta}_B - \tilde{\delta}_B H_B \tilde{\delta}_B + \tilde{\delta}_B H_B \tilde{\delta}_B H_B \tilde{\delta}_B - \dots;$$

which is well-defined since $\tilde{\delta}_B$ decreases by 1 the number of occurrences of elements in $V \subset \mathcal{L}_2 \subset U(\mathcal{L}_2)$; and define

$$\tilde{F}_B = F_B(1 - \tilde{X}H_B); \quad H_B = H_B(1 - \tilde{X}H_B); \quad \tilde{G}_B = (1 - H_B\tilde{X})G_B; \quad \tilde{d}_B = d_B + F_B\tilde{X}G_B.$$

Then $(\tilde{F}_B, \tilde{G}_B, \tilde{H}_B)$ is a contraction of $(BU(\mathcal{L}_2), \tilde{D}_B)$ to $(BU(L_2), \tilde{d}_B)$. As in the previous Proposition we conclude that \tilde{F}_B and \tilde{G}_B and maps of DG coalgebras, \tilde{H}_B is a coalgebra homotopy and \tilde{d}_B is a coderivation.

In particular, the coderivation \tilde{d}_B defines an A_∞ -structure on $U(L_2)$, cf. [Le], i.e a series of higher products $\mu_n : U(L_2)^{\otimes n} \rightarrow U(L_2)$ given by the composition of natural maps

$$U(L_2)^{\otimes n} \rightarrow U^{\geq 1}(L_2)^{\otimes n} \hookrightarrow BU(L_2) \xrightarrow{\tilde{d}_B} BU(L_2) \rightarrow U^{\geq 1}(L_2) \hookrightarrow U(L_2).$$

Writing out the definitions and using $F'_B \tilde{\delta}_B = 0$, $H'_B \delta_B G'_B = 0$ we see that for $n \geq 3$, μ_n is given by the expression

$$\sum_{k \geq (n-2); a_1, \dots, a_k} (-1)^{k-1} F_{sym} \delta_U H'_B(a_1 H'_B) \dots (a_k H'_B) \tilde{\delta}_B G_{sym}^{\otimes n} \quad (7)$$

where each a_i is either δ_B or $\tilde{\delta}_B$ and the first possibility occurs precisely $(n-2)$ times.

Alternatively, one can write a formula in the spirit of [KS]: μ_n is given by the sum over all planar trees with n leaves, one root and internal vertices of valency 2 or 3. Similarly to *loc. cit* we place G_{sym} on each leaf, F_{sym} on the root, δ_B on each internal vertex of valency 3, $\tilde{\delta}_B$ on each

internal vertex of valency 2, and H_{sym} in the middle of each internal edge. A tree marked in this way is viewed as a “flowchart” of operations applied to the arguments of μ_n . Note that due to the valency 2 vertices each μ_n becomes an infinite sum over trees, but on each particular set of n arguments only finitely many give nonzero contributions.

Proposition 4 *The product μ_2 is the usual universal enveloping algebra product in $U(L_2)$. The higher products μ_n for $n \geq 3$ have the following properties:*

- (1) *Each μ_n is multilinear in $R = Sym^\bullet(U) \subset U(L_2)$.*
- (2) *$\mu_n(a_1, \dots, a_n) = 0$ if $a_i = 1$ for some i . Thus, the A_∞ -structure is strictly unital.*
- (3) *$\mu_n(v_1, \dots, v_n) = \frac{1}{n!} l_n(v_1, \dots, v_n)$ if $v_i \in V \subset L \subset U(L)$ for all i .*

Proof. To prove the assertion about μ_2 first note that the “correction” to the product on $U(L_2)$ introduced by $\tilde{\delta}_B - \delta_B$ is given by the formula similar to as above expression for μ_n , $n \geq 3$:

$$\sum_{k \geq 1} (-1)^k F_{sym} \delta_U (H'_B \tilde{\delta}_B)^k G_{sym}^{\otimes 2}$$

but a single application of $H'_B \tilde{\delta}_B$ will produce terms in $Sym^{\geq 1}(\{S\}) \subset U(L_2)$. Since such terms are central in $U(L_2)$, δ_U is multilinear with respect to them. But F_{sym} vanishes on $Sym^{\geq 1}(\{S\})$ therefore the correction to the product on $U(L_2)$ vanishes.

Part (1) follows from (7) (or better, the sum over trees presentation) and the fact that the operators F_{sym} , H_{sym} and G_{sym} involved in it, are all R -linear. Part (2) follows from the fact that we are using the *reduced* bar construction hence by definition all higher products factor through $U^{\geq 1}(L_2)^{\otimes n}$. To prove part (3) use the formula (7) to compute $\mu(v_1, \dots, v_n)$. The only non-zero contributions come from the terms with $k = (n - 2)$, i.e. for which all $a_i = \delta_B$. In fact, if a term in (7) contains $\tilde{\delta}_B$ at least twice, its evaluation at $v_1 \otimes \dots \otimes v_n$ will necessarily contain $\delta_U(a \otimes b)$ with $a, b \in \{S\} \subset U(L_2)$. To explain that in terms of trees: if we connect the two occurrences of $\tilde{\delta}_B$ on a tree with its root by shortest paths, the point at which the two paths merge will correspond to the $\delta_U(a \otimes b)$ above. Since $\{S\}$ is central, $\delta_U(a \otimes b) = 0$. Therefore

$$\mu_n(v_1, \dots, v_n) = (-1)^{n-1} F_{sym} (\delta_B H'_B)^{n-1} \tilde{\delta}_B G_{sym}^{\otimes n} (v_1 \otimes \dots \otimes v_n)$$

Now an easy induction involving (6) finishes the proof. \square

Remark. The properties stated in the previous proposition do not determine the A_∞ -structure uniquely. In the case of projective complete intersections, cf. [Ba], an additional formula allows to compute all higher operations recursively. Such a formula can be proved in this case as well but this will not be done here.

By a recent work of Merkulov, cf. [Me], the L_∞ -structure on L deforms the commutative and cocommutative bialgebra structure of $U(L_1)$ to a structure of a homotopy bialgebra. However, this structure depends on the choice of a minimal model of the bialgebra PROP, and a certain lift of a morphism of PROPs (see [Me] for more detail).

Comparing our construction with the standard bialgebra $\Omega C(L)$, one can show that in the situation of this paper the higher products on $U(L_2)$ extend to a homotopy bialgebra structure and in fact determine it uniquely. We plan to return to this matter in a forthcoming work.

Notation. From now on we denote by E_W the universal enveloping $U(L_2)$ equipped with the A_∞ -algebra structure of this section.

3 An equivalence of categories

3.1 A generalized twisted cochain.

Let $C_W = Ker(\delta_C) \cap Sym^\bullet(V) \subset C$ be the “dual coalgebra” of the polynomial quotient S_W defined in Section 2.2. Write $C_W = k \oplus \overline{C}$ where $\overline{C} = Ker(\delta_C) \cap Sym^{\geq 1}(V)$. If $\overline{\Delta} : \overline{C} \rightarrow \overline{C} \otimes \overline{C}$ is the reduced coproduct $(\Delta - Id \otimes 1 - 1 \otimes Id)$, and $\overline{\Delta}^{(k)} : \overline{C} \rightarrow \overline{C}^{\otimes k}$ are its iterations, then $\overline{C} = \cup Ker(\overline{\Delta}^{(k)})$, i.e. C_W is *cocomplete*. In fact, this property holds for the free cocommutative coalgebra C and C_W is its subcoalgebra.

Denote by $\Delta^{(k)} : C \rightarrow C^{\otimes k}$ similar iterations for $k \geq 2$ and set $\Delta^{(1)}$ to identity. Consider the composition $\tau_W : C_W \hookrightarrow C \rightarrow L \hookrightarrow U(L) = E_W$. In other words, we compose the projection $C_W \rightarrow C_W \cap V$ with the embedding $V \subset L \subset E_W$.

Lemma 5 *The map τ_W satisfies the generalized twisted cochain condition, cf. Section 4.1 of [Le], which reads in our case:*

$$\sum_{s \geq 1} \mu_s \circ \tau_W^{\otimes s} \circ \Delta^{(s)} = 0$$

Proof. Note that the infinite sum is well defined since $\tau_W|_k = 0$ and C_W is cocomplete. First consider $C \rightarrow L \hookrightarrow E_W$. Then by the last part of Proposition 4 one has

$$\tau \circ \delta_C + \sum_{s \geq 1} \mu_s \circ \tau^{\otimes s} \circ \Delta^{(s)} = 0.$$

Since $C_W \hookrightarrow C$ is a morphism of coalgebras, the assertion for C_W follows trivially. \square

3.2 A pair of adjoint functors.

The previous lemma allows to apply the general formalism outlined in Sections 2.2.1 and 4.3.1 of [Le]. Since some of the formulas are given in [Le] only for DG-algebras we give the definitions here for reader’s convenience. See [Le] for definitions and properties of A_∞ -algebras and modules over them.

Consider a general cocomplete coaugmented DG-coalgebra (C, δ_C) , a strictly unital A_∞ -algebra E with $\delta_E = \mu_1^E$ and a generalized twisted cochain $\tau : C \rightarrow E$ satisfying

$$\delta_C \circ \tau + \tau \circ \delta_E + \sum_{s \geq 2} \mu_s \circ \tau^{\otimes s} \circ \Delta^{(s)} = 0$$

Let (N, δ_N) be a counital DG comodule over C with the reduced coaction map $\overline{\Delta}_N : N \rightarrow N \otimes \overline{C}$. We assume that N is also cocomplete, i.e. $N = \bigcup_{s \geq 1} Ker \overline{\Delta}_N^{(s)}$, where $\overline{\Delta}_N^{(s)} : N \rightarrow N \otimes \overline{C}^{\otimes(s-1)}$ is the reduction of the iterated coaction map $\Delta_N^{(s)} : N \rightarrow N \otimes C^{\otimes(s-1)}$. Whenever we speak of a *filtered* morphism of cocomplete comodules, we always have in mind the filtration by $Ker \overline{\Delta}_N^{(s)}$.

Denote by $\mathcal{F}(N)$ the tensor product $N \otimes E$ with the differential

$$\delta_{\mathcal{F}(N)} = \delta_N \otimes 1 + 1 \otimes \delta_E + \sum_{s \geq 2} (1 \otimes \mu_s^E)(1 \otimes \tau^{\otimes(s-1)} \otimes 1)(\Delta_N^{(s)} \otimes 1)$$

which is well-defined since N is cocomplete and E is strictly unital. Then $\delta_{\mathcal{F}(N)}^2 = 0$ by the generalized twisted cochain condition. Also, $\mathcal{F}(N)$ is an A_∞ -module over E with the action maps

$$\mu_k^{\mathcal{F}(N)} : \mathcal{F}(N) \otimes E^{\otimes(k-1)} \rightarrow \mathcal{F}(N); \quad (n \otimes a) \otimes a_1 \otimes \dots \otimes a_{k-1} \mapsto n \otimes \mu_k^E(a, a_1, \dots, a_{k-1})$$

for $k \geq 2$. This module structure is *strictly unital*: $\mu_2^{\mathcal{F}(N)}(x, 1_E) = x$ and $\mu_k^{\mathcal{F}(N)}(x, a_1, \dots, a_{k-1}) = 0$ if $k \geq 3$ and $a_i = 1_E$ for some i , since the same property was assumed about E . If $\psi : N_1 \rightarrow N_2$ is a morphism of C_W -comodules then $\mathcal{F}(\psi) = \psi \otimes 1 : N_1 \otimes E \rightarrow N_2 \otimes E$ is a strict morphism of E -modules (i.e. commutes with all higher products).

In the other direction, take a strictly unital A_∞ -module (M, δ_M, μ_k^M) over E , where $\mu_k^M : M \otimes E^{\otimes(k-1)} \rightarrow M$ are the action maps for $k \geq 2$, and consider the C -comodule $\mathcal{G}(M) = M \otimes C$, with the differential

$$\delta_{\mathcal{G}(M)} = \delta_M \otimes 1 + 1 \otimes \delta_C + \sum_{k \geq 2} (\mu_k^M \otimes 1)(1 \otimes \tau_W^{\otimes(k-1)} \otimes 1)(1 \otimes \Delta^{(k)}).$$

Again the differential is well-defined since C is cocomplete and E is strictly unital. A morphism of A_∞ -modules M_1, M_2 is given by degree $(1-k)$ maps $f_k : M_1 \otimes E^{\otimes(k-1)} \rightarrow M_2$ for $k \geq 1$, which satisfy some quadratic identities, cf. Chapter 2 of [Le]. Such a morphism $f = \{f_k\}$ is called *strictly unital* if $f_k(m, a_1, \dots, a_{k-1}) = 0$ whenever $k \geq 2$ and $a_i = 1_E$ for some i . For every such morphism define a morphism of C -comodules $\mathcal{G}(f) : M_1 \otimes C \rightarrow M_2 \otimes C$ by the formula

$$\mathcal{G}(f) = \sum_{k \geq 1} (f_k \otimes 1)(1 \otimes \tau^{\otimes(k-1)} \otimes 1)(1 \otimes \Delta^{(k)}).$$

This is well-defined for the same reason as before.

Thus we obtain a pair of functors \mathcal{F}, \mathcal{G} between the category $Comodc(C)$ of cocomplete counital DG-comodules over C and the category $Mod_\infty(E)$ of strictly unital A_∞ -modules over E_W and strictly unital morphisms. These functors are adjoint:

$$Hom_{Mod_\infty(E)}(\mathcal{F}(N), M) = Hom_{Comodc(C)}(N, \mathcal{G}(M))$$

since both spaces may be identified with

$$\{\phi \in Hom_k(N, M) \mid \phi \delta_N - \delta_M \phi = \sum_{k \geq 2} \mu_k^M(\phi \otimes \tau^{\otimes(k-1)}) \Delta^{(k)}\}.$$

More explicitly, given such ϕ one defines a morphism of C -comodules $\Phi : N \xrightarrow{\Delta_N} N \otimes C \xrightarrow{\phi \otimes 1} M \otimes C$, and a morphism of A_∞ -modules $\Psi : N \otimes E \rightarrow M$

$$\Psi_k = \sum_{s \geq 1} \mu_{k+s}^M(\phi \otimes \tau^{\otimes(s-1)} \otimes 1^{\otimes k})(\Delta_N^{(s)} \otimes 1^{\otimes k}) : N \otimes E^{\otimes k} \rightarrow M.$$

The map ϕ may be recovered from Ψ . as $\Psi_1|_{N \otimes 1}$, or from Φ as its composition with the projection $\eta_M : M \otimes C \rightarrow M \otimes k = M$ coming from the counit of C . The fact that the above formulas indeed

define morphisms, and that every Φ, Ψ is given by a certain ϕ , is proved by a straightforward (but tedious) induction using the filtration of N by $\text{Ker}(\overline{\Delta}^{(k)} : N \rightarrow N \otimes \overline{C}^{\otimes(k-1)})$.

Below we need an explicit formula for the adjunction morphism $\Psi : \mathcal{FG}(M) \rightarrow M$. The component $\Psi_k : M \otimes C \otimes E^{\otimes k} \rightarrow M$ is given by

$$\sum_{s \geq 1} \mu_{k+s}(\eta_M \otimes \tau^{\otimes(s-1)} \otimes 1)(1 \otimes \Delta^{(s)} \otimes 1)$$

3.3 A coalgebra equivalence.

The generalized twisted cochain $\tau_W : C_W \rightarrow E_W$ extends to a coalgebra map $C_W \rightarrow B(E_W)$ by the standard formula $\sum_k \tau_W^{\otimes k} \Delta^{(k)}$, cf. [Ka2]. The condition of Lemma 5 is equivalent to the fact that this extension commutes with differentials.

Lemma 6 *The canonical coalgebra extension $C_W \rightarrow B(E_W)$ defined by τ_W , is a weak equivalence of coalgebras, i.e. induces a quasi-isomorphism of DG algebras $\Omega(C_W) \rightarrow \Omega B(E_W)$.*

Proof. Recall that the A_∞ -structure on E_W is encoded in the differential d_B on $B(E_W)$. In the previous section we have also constructed a quasi-isomorphism of DG coalgebras $\tilde{F}_B : BU(\mathcal{L}) \rightarrow BE_W$ which naturally induces a quasi-isomorphism of DG algebras $\Omega BU(\mathcal{L}) \rightarrow \Omega B(E_W)$. It follows from the definitions that the algebra homomorphism $\Omega(C_W) \rightarrow \Omega B(E_W)$ factors as

$$\Omega(C_W) \rightarrow \Omega(C) \rightarrow \Omega BU(\mathcal{L}) \rightarrow \Omega B(E_W)$$

where the first and the last arrows are quasi-isomorphisms. Therefore it suffices to check that $\Omega(C) \rightarrow \Omega BU(\mathcal{L})$ is a quasi-isomorphism. To that end, we note that the composition

$$\Omega(C) \rightarrow \Omega BU(\mathcal{L}) \rightarrow U(\mathcal{L}),$$

is a quasi-isomorphism by Lemma 3, and the second arrow is a quasi-isomorphism by a standard result in homotopical algebra (see e.g. page 272 of [FHT]). Therefore the first arrow is also a quasi-isomorphism, which finishes the proof. \square

Let $\mathcal{D}(E_W)$ be the localization $\text{Mod}_\infty(E_W)$ at quasi-isomorphisms. To get a derived category $\mathcal{D}(C_W)$ we must localize $\text{Comodc}(C_W)$ at weak equivalences, i.e. such maps that induce quasi-isomorphism on cobar construction, cf. [Le]. In general, a weak equivalence of comodules is a stronger condition than quasi-isomorphism.

Corollary 7 *The functors \mathcal{F}, \mathcal{G} induce mutually inverse derived equivalences $\mathcal{D}(C_W) \simeq \mathcal{D}(E_W)$ between the derived category $\mathcal{D}(C_W)$ of cocomplete comodules over C_W and the derived category $\mathcal{D}(E_W)$ of strictly unital A_∞ -modules over E_W .*

Proof. We factorize \mathcal{F} and \mathcal{G} as follows

$$\begin{array}{ccc} \text{Comodc}(C_W) & \xrightarrow{\mathcal{F}_1} & \text{Comodc}(B(E_W)) & \xrightarrow{\mathcal{F}_0} & \text{Mod}_\infty(E_W) \\ & \xleftarrow{\mathcal{G}_1} & & \xleftarrow{\mathcal{G}_0} & \end{array}$$

Here \mathcal{F}_0 and \mathcal{G}_0 are induced by the universal generalized twisted cochain $B(E_W) \rightarrow \overline{E}_W \hookrightarrow E_W$, where \overline{E}_W is the kernel of the augmentation map. The functor \mathcal{F}_1 is given by corestriction (i.e. every C_W -comodule is automatically a $B(E_W)$ -comodule); and \mathcal{G}_1 by coinduction:

$$\mathcal{G}_1(N) = \text{Ker}\{\Delta_N \otimes 1 - (1 \otimes \tau \otimes 1)(1 \otimes \Delta_{C_W}) : N \otimes C_W \rightarrow N \otimes B(E_W) \otimes C_W\}.$$

It follows from definitions that $\mathcal{F} = \mathcal{F}_0\mathcal{F}_1$ and $\mathcal{G} = \mathcal{G}_1\mathcal{G}_0$. Therefore, to prove that for any object M of $\text{Mod}_\infty(E_W)$ the first component of the canonical A_∞ -morphism $\mathcal{F}\mathcal{G}(M) \rightarrow M$ is a quasi-isomorphism, one needs to show that

- 1) For any object L of $\text{Comodc}(B(E_W))$ the canonical morphism $\mathcal{F}_1\mathcal{G}_1(L) \rightarrow L$ is a weak equivalence;
- 2) For any M as above the canonical morphism $\mathcal{F}_0\mathcal{G}_0(M) \rightarrow M$ is a quasi-isomorphism;
- 3) \mathcal{F}_0 sends weak equivalences to quasi-isomorphisms.

Similarly, to prove that for any object N of $\text{Comodc}(C_W)$ the canonical morphism $N \rightarrow \mathcal{G}\mathcal{F}(N)$ is a weak equivalence, one needs to show that

- 1') For any N as above the canonical morphism $N \rightarrow \mathcal{G}_1\mathcal{F}_1(N)$ is a weak equivalence;
- 2') For any L as above $L \rightarrow \mathcal{G}_0\mathcal{F}_0(L)$ is a weak equivalence;
- 3') \mathcal{G}_1 sends weak equivalences to weak equivalences.

In addition, to prove that the functors descend to derived categories one needs to show that

- 4) \mathcal{G}_0 sends quasi-isomorphisms to weak equivalences;
- 5) \mathcal{F}_1 sends weak equivalences to weak equivalences.

The assertions 1), 1'), 3') and 5) follow from the previous Lemma, since the morphism $\Omega(C_W) \rightarrow \Omega B(E_W)$ gives restriction-induction functors which descend to equivalences of derived categories; and $\Omega\mathcal{F}_1(N)$ is canonically isomorphic to the $\Omega(B(E_W))$ -module induced from the $\Omega(C_W)$ -module $\Omega(N)$, while for $\Omega\mathcal{G}_1(N')$ there is a canonical quasi-isomorphism to the restriction of $\Omega(N')$ from $\Omega(C')$ to $\Omega(C)$.

Statements 2) and 2') are proved by first setting formally $W = 0$ where they both reduce to standard facts about the bar construction of an associative algebra, and then applying the Basic Perturbation Lemma to derive the case of general W (recall, cf. Chapter 2 of [Le], that in 2') it suffices to prove a filtered quasi-isomorphism).

Finally, to prove 3) and 4) we first note that by definition \mathcal{G}_0 sends quasi-isomorphisms to filtered quasi-isomorphisms, which are automatically weak-equivalences (the proof is as in Lemma 1.3.2.2 of [Le]). To prove the assertion for \mathcal{F}_0 observe that the adjunction morphism $B(E_W) \rightarrow B\Omega B(E_W)$ defines an A_∞ -morphism $H. : E_W \rightarrow \Omega B(E_W)$. Therefore, every module M over $\Omega B(E_W)$ becomes an A_∞ -module over E_W if we set $\mu_k^M : M \otimes E_W^{\otimes(k-1)} \rightarrow M$ to be the obvious composition $M \otimes E_W^{\otimes(k-1)} \xrightarrow{1 \otimes H_{k-1}} M \otimes \Omega B(E_W) \rightarrow M$.

In particular, for any L as in 1) the cobar construction $\Omega(L)$ is an A_∞ -module over E_W . Now the maps $1_L \otimes H_i$ for $i \geq 1$ give an A_∞ morphism $\mathcal{F}_0(L) \rightarrow \Omega(L)$ of modules over E_W which is easily seen to be a quasi-isomorphism (since $H.$ is a quasi-isomorphism). If $L \rightarrow L'$ is a weak equivalence in $\text{Comodc}(B(E_W))$ then $\Omega(L) \rightarrow \Omega(L')$ is a quasi-isomorphism of modules over $\Omega B(E_W)$ (and also over E_W), therefore $\mathcal{F}_0(L) \rightarrow \mathcal{F}_0(L')$ is also a quasi-isomorphism. \square

3.4 The graded case.

Now suppose that A is an abelian group (not necessarily torsion-free) generated by elements $\alpha_1, \dots, \alpha_n$ where $n = \dim_k V$. We fix a basis x_1, \dots, x_n in V^* and consider the A -grading $Sym^\bullet(V^*) = \bigoplus_{\alpha \in A} Sym_\alpha^\bullet(V^*)$ such that $\deg_A(x_i) = \alpha_i$. Assume that all $Sym_\alpha^\bullet(V^*)$ are finite dimensional. In this case, if A_+ denotes the semigroup generated in A by $\alpha_1, \dots, \alpha_n$ then $A_+ \cap (-A_+) = \emptyset$ since otherwise there is a non-trivial monomial x^I with $\deg_A(x^I) = 0$ and all its powers will satisfy the same condition too.

Assume further that polynomials W_1, \dots, W_m are A -homogeneous of degrees β_1, \dots, β_m (see the next section for examples) and that neither of them has terms linear in x_1, \dots, x_n (this is mostly to simplify notation; if this condition is not satisfied, in the argument below one can either pass to a smaller polynomial quotient of $Sym^\bullet(V^*)$ or adjust some definitions). The last assumption implies that L and E_W have zero differentials.

Under these assumptions the quotient ring S_W is also A -graded with finite dimensional components, and the same holds for $Sym^\bullet(V)$ (since the space V will have the dual basis with A -degrees $-\alpha_1, \dots, -\alpha_n$). The action of $Sym^\bullet(V^*)$ by differential operators on $Sym^\bullet(V)$ agrees with the A -grading and by assumption on W_1, \dots, W_m the coalgebra C_W will inherit the A -grading as well. It follows immediately from the definitions that the pairing $\langle \cdot, \cdot \rangle : Sym^\bullet(V^*) \times Sym^\bullet(V) \rightarrow k$ descends to $\langle \cdot, \cdot \rangle : S_W \times C_W \rightarrow k$.

The A_∞ -algebra $E_W \simeq \Lambda^\bullet(V) \otimes Sym^\bullet(U)$ has A -grading in which z_1, \dots, z_m have degrees $-\beta_1, \dots, -\beta_m$, respectively, (this ensures that $\deg_A \mu_k = 0$ for $k \geq 1$). Assuming that the upper indices refer to homological grading and lower indices to A -grading we see that $(E_W)_\alpha^i = 0$ unless (α, i) is in

$$\mathcal{A} = \{\text{subsemigroup of } A \times \mathbb{Z} \text{ generated by } (-A_+, 1), (-A_+, 0) \text{ and } 0\}.$$

Here we use the assumption that W_1, \dots, W_m have no linear terms.

Let $C^\clubsuit(S_W)$ be the category of complexes $N = (\dots \rightarrow N^{-1} \rightarrow N^0 \rightarrow N^1 \rightarrow \dots)$ of A -graded modules $N^i = \bigoplus_{\alpha \in A} N_\alpha^i$ over S_W , such that $N_\alpha^i = 0$ unless $(\alpha, i) \in \alpha + \mathcal{A}$ for some $\alpha \in A \times \mathbb{Z}$ depending on N . Note that objects in the category $C^b(S_W)$ of finite complexes of finitely generated S_W -modules will not in general satisfy this condition. However, the *opposite* category $(C^b(S_W))^{opp}$ (with inverted arrows) can be embedded into $C^\clubsuit(S_W)$ if for any $N \in Ob(C^b(S_W))$ we consider the graded dual N^* with $(N^*)_{-\alpha}^{-i} = (N_\alpha^i)^*$.

Similarly we define $C^\clubsuit(E_W)$ as the category of strictly unital A -graded A_∞ -modules M over E_W which satisfy $M_\alpha^i = 0$ unless $(\alpha, i) \in \gamma + \mathcal{A}$ for some $\gamma \in A \times \mathbb{Z}$ depending on M .

Any S_W -module $N \in Ob(C^\clubsuit(S_W))$ is also a C_W -comodule with the reduced coaction map

$$\overline{\Delta}_N : N \rightarrow N \otimes \overline{C}_W; \quad n \mapsto \sum_{\alpha \in A_+, y_{\alpha, i}} (ny_{\alpha, i}) \otimes x_{\alpha, i}$$

where the sum is taken over dual bases $\{x_{\alpha, i}\}_{1 \leq i \leq \dim(C_W)_\alpha}$ and $\{y_{\alpha, i}\}_{1 \leq i \leq \dim(S_W)_\alpha}$ of $(C_W)_\alpha$ and $(S_W)_\alpha$, respectively. The condition imposed on the grading of N , together with $(-A_+) \cap A_+ = \emptyset$, ensure that the sum in the definition of $\overline{\Delta}_N$ is finite on every $n \in N$.

Thus we can still define a functor $\mathcal{F} : C^\clubsuit(S_W) \rightarrow C^\clubsuit(E_W)$ sending N to $N \otimes E_W$. As for $\mathcal{G} : C^\clubsuit(E_W) \rightarrow C^\clubsuit(S_W)$, note that $\mathcal{G}(M) = M \otimes C_W$ is not only a C_W -comodule but also an

S_W -module (since C_W itself is a graded dual to the free rank one module over S_W). These functors descend to the corresponding derived categories $D^\clubsuit(S_W)$, $D^\clubsuit(E_W)$.

To formulate the next theorem we define the “bounded” derived category $D^b(E_W)$ as the full triangulated subcategory of $D^\clubsuit(E_W)$ formed by all objects for which the total cohomology (= direct sum of all cohomology groups) is a finitely generated module over the associative algebra (E_W, μ_2) . Note that the E_W itself is only bounded from the left, so many objects in $D^b(E_W)$ will be unbounded in the usual sense. Let also $D^b(S_W)$ be the usual bounded derived category of finitely generated A -graded S_W -modules, embedded contravariantly into $D^\clubsuit(S_W)$ by the above.

Theorem 8 *The functors \mathcal{F} , \mathcal{G} give mutually inverse derived equivalences*

$$D^\clubsuit(S_W) \rightarrow D^\clubsuit(E_W)$$

Moreover, their restrictions induce a derived equivalence

$$D^b(S_W)^{opp} \simeq D^b(E_W)$$

Proof. From the proof of Corollary 7 we already know that the adjunction morphisms $\mathcal{F}\mathcal{G}(M) \rightarrow M$ and $N \rightarrow \mathcal{G}\mathcal{F}(N)$ are quasi-isomorphisms. To establish the first claim it remains to show that \mathcal{F} and \mathcal{G} send quasi-isomorphisms to quasi-isomorphisms. This is obvious for \mathcal{G} since it sends quasi-isomorphisms to weak equivalences of C_W -comodules (again by proof of Corollary 7) and every weak equivalence is a quasi-isomorphism. To prove the assertion for \mathcal{F} first assume that $N \rightarrow N'$ is a quasi-isomorphism in $C^\clubsuit(S_W)$. Then by the above $\mathcal{G}\mathcal{F}(N) \rightarrow \mathcal{G}\mathcal{F}(N')$ is also a quasi-isomorphism. It follows from the definitions that the last map can be viewed as filtered quasi-isomorphism (hence a weak equivalence) of C_W -comodules. Apply the proof of Corollary 7 again we see that $\mathcal{F}\mathcal{G}\mathcal{F}(N) \rightarrow \mathcal{F}\mathcal{G}\mathcal{F}(N')$ is a quasi-isomorphism and therefore $\mathcal{F}(N) \rightarrow \mathcal{F}(N')$ is a quasi-isomorphism.

The proof of the second assertion proceeds exactly as in of Proposition 4.1 in [Ba]: first we note that for an object N in $D^b(S_W)$ the cohomology of $\mathcal{F}(N)$ is simply $Ext^\bullet(N, k)$ therefore by Section 3 of [G] it is finitely generated over $(E_W, \mu_2) \simeq Ext_{S_W}(k, k)$. Therefore, \mathcal{F} sends $D^b(S_W)^{opp}$ to $D^b(E_W)$. In the other direction, if M is an object of $D^b(E_W)$ we can first replace it by $\mathcal{F}\mathcal{G}(M)$ which is a complex of free E_W -modules. It suffices to check that $\mathcal{G}\mathcal{F}\mathcal{G}(M)$ gives a finitely generated module over $S = Sym^\bullet(V^*)$ but that module may be computed using the equivalence functors \mathcal{F}_0 and \mathcal{G}_0 for S and $E = \Lambda^\bullet(V)$, respectively. Note that E is a quotient of E_W (the quotient map sends all $z_j \in U$ to zero). Inspecting the definitions of the functors involved we see that $\mathcal{G}\mathcal{F}\mathcal{G}(M)$ is isomorphic to $\mathcal{G}_0(\mathcal{F}\mathcal{G}(M) \otimes_{E_W} E)$, as objects of $D^b(S)^{opp}$. Therefore the finite generation of the graded dual follows from the original BGG correspondence for S and E (adjusted to the case of A -graded modules). \square

Remark. When $A = \mathbb{Z}$ and all W_i are quadratic one can adjust the grading to ensure that E_W is in homological degree zero. In the general case this will not be possible since by Proposition 4 the higher products μ_k , $k \geq 3$ on E_W will be nontrivial and they have homological degree $2 - k$.

4 Toric complete intersections and Landau-Ginzburg models.

4.1 Homogeneous coordinates on toric varieties

We fix notation by recalling some facts about toric varieties. Let $T \simeq (k^*)^n$ be an algebraic torus over k . Associated to T are the character lattice $M = \text{Hom}_{\text{alg}}(T, k^*)$ and the dual lattice of one-parametric subgroups $N = \text{Hom}_{\text{alg}}(k^*, T)$ (the subscript *alg* means homomorphisms in the category of algebraic groups over k). Set also $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and consider a fan $\Sigma \subset N_{\mathbb{R}}$ defining a toric variety X_{Σ} , cf. [F]. We assume that the support of Σ is equal to $N_{\mathbb{R}}$, i.e. the variety X_{Σ} is complete. If $\Sigma(1)$ is the set of all 1-dimensional cones in Σ let S be the polynomial algebra over k generated by variables x_{ρ} , $\rho \in \Sigma(1)$.

Recall, cf. [F], that $\Sigma(1)$ is in bijective correspondence with the codimension 1 orbits of the T -action on X_{Σ} . For any $\rho \in \Sigma(1)$ let D_{ρ} be the closure of the corresponding orbit. Define the group A by the exact sequence

$$0 \rightarrow M \xrightarrow{\alpha} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_{\rho} \xrightarrow{\beta} A \rightarrow 0$$

where $\alpha(m) = \sum_{\rho} \langle m, n_{\rho} \rangle D_{\rho}$, β is the quotient map and $n_{\rho} \in N$ is the primitive generator of $\rho \in N_{\mathbb{R}}$. If we view the elements of M as rational functions on $X_{\Sigma} \supset T$ then α computes the orders of poles and zeros along the divisors D_{ρ} . One can show that A is isomorphic to the Chow group $A_{n-1}(X_{\Sigma})$. In general A will have torsion.

Note that the map β gives an A -grading on S . For projective spaces this reduces to the usual \mathbb{Z} -grading on polynomials. Denoting

$$G = \text{Hom}_{\mathbb{Z}}(A, k^*)$$

we get a G -action on the space $V = k^{\Sigma(1)}$ dual to the vector space spanned by x_{ρ} , $\rho \in \Sigma(1)$. To obtain X_{Σ} from this action first denote by $\Sigma(\text{max})$ the set of *maximal* cones in Σ (i.e. those which are not contained in a larger cone) and then for $\sigma \in \Sigma(\text{max})$ define $x^{\hat{\sigma}} \in S$ as $\prod_{\rho \in \Sigma(1) \setminus \sigma(1)} x_{\rho}$. The monomials $x^{\hat{\sigma}}$, $\sigma \in \Sigma(\text{max})$ generate an ideal $B \subset S$ which corresponds to a closed subvariety $V(B) \subset V$. To describe $V(B)$ more explicitly, let $\Gamma \subset \Sigma(1)$ be a subset and $V(\Gamma)$ the coordinate subspace of V defined by vanishing of the coordinates in Γ . Then $V(\Gamma) \subset V(B)$ iff Γ is not contained in the closure of any maximal cone $\sigma \in \Sigma(\text{max})$. It is easy to see that $V(B)$ is the union of all such $V(\Gamma)$ and we can restrict this union to those Γ which are minimal (with respect to inclusion) among all subsets with the above property.

Since the support of Σ is $N_{\mathbb{R}}$, every Γ which is not in the closure of a maximal cone must contain at least 2 elements. Therefore $\text{codim}_V V(B) \geq 2$.

According to the main result of [C] we have

$$X_{\Sigma} = [V \setminus V(B)]/G$$

The right hand side is usually understood as the universal categorical quotient or geometric quotient, if Σ is simplicial (i.e. all cones in Σ are simplices). However, in this paper we will view it as a *stack* and use the above equality to *define* the quotient stack \mathbb{X}_{Σ} . Thus, in general \mathbb{X}_{Σ} will be an Artin stack; when Σ is simplicial, a Deligne-Mumford stack, cf. [LM2], [F].

4.2 BGG correspondence for toric complete intersections

Since \mathbb{X}_Σ was explicitly defined as a global quotient it is easy to deal with bundles and sheaves on \mathbb{X}_Σ : these are just G -equivariant objects on $V^\circ = V \setminus V(B)$. For instance, line bundles on \mathbb{X}_Σ are just G -equivariant line bundles on V° .

Since $V(B)$ has codimension at least two, $Pic(V^\circ)$ is trivial and $H^0(V^\circ, \mathcal{O}) = k$. It follows that the Picard group of \mathbb{X}_Σ may be identified with $Hom_{alg}(G, k^*) = A$. Note that for X_Σ the Picard group is in general only a subgroup of A (some line bundles on \mathbb{X}_Σ give only torsion-free sheaves on X_Σ). Thus, for any $\alpha \in A$ we have a line bundle $\mathcal{O}(\alpha)$ on \mathbb{X}_Σ . If $S_\alpha \subset S$ is the graded component corresponding to α then

$$H^0(\mathbb{X}_\Sigma, \mathcal{O}(\alpha)) = S_\alpha$$

and $\dim_k S_\alpha < \infty$, cf. [C].

Now let W_1, \dots, W_m be a regular sequence of elements in S which are A -homogeneous of A -degrees β_1, \dots, β_m , respectively. Since these can be viewed as sections of line bundles $\mathcal{O}(\beta_1), \dots, \mathcal{O}(\beta_m)$, respectively, they define a complete intersection substack $\mathbb{Y} \subset \mathbb{X}_\Sigma$. In other words,

$$\mathbb{Y} = [V^\circ \cap Z(J)]/G,$$

where J stands for the ideal generated in S by W_1, \dots, W_m and $Z(J)$ is the zero set. Denote S/J by S_W (note that the notation for J, S_W and α_i, β_j is consistent with that of Sections 2 and 3, respectively).

Recall that a full triangulated subcategory I of a triangulated category D is called *thick* if it is closed with respect to the operation of taking direct summands.

Lemma 9 *The category $Coh(\mathbb{X}_\Sigma)$ of coherent sheaves on \mathbb{X}_Σ is equivalent to the quotient of the category $mod_A(S)$ of A -graded finitely generated modules over S , by the subcategory of modules supported on $V(B)$.*

Similarly, the category $Coh(\mathbb{Y})$ of coherent sheaves on \mathbb{Y} is equivalent to the quotient of the category $mod_A(S_W)$ of A -graded finitely generated modules over S_W , by the subcategory of modules supported on $V(B) \cap Z(J)$. The derived category $D^b(Coh(\mathbb{Y}))$ is equivalent to the quotient $D^b(S_W)/I$ where I is the thick subcategory of all complexes with cohomology supported on $V(B) \cap Z(J)$.

Proof. For $Coh(\mathbb{X}_\Sigma), Coh(\mathbb{Y})$ everything follows easily from definitions since a coherent sheaf on \mathbb{X}_Σ , resp. \mathbb{Y} , is simply a G -equivariant coherent sheaf on V° , resp. $V^\circ \cap Z(J)$ which can always be extended to V , resp. $Z(J)$. Thus, $Coh(\mathbb{Y})$ is a quotient of $mod_A(S/J)$ and the kernel is easily seen to be the subcategory of modules supported on $V(B) \cap Z(J)$. The derived category statement is similar. \square

Since the graded components of S are finite-dimensional, by Theorem 8, the derived category $D^b(\mathbb{Y})$ is equivalent to a quotient of the $D^b(E_W)^{opp}$. To describe this quotient, suppose that $\Gamma \subset \Sigma(1)$ defines an irreducible component of $V(B)$, i.e. that Γ is a minimal subset of $\Sigma(1)$ which is not contained in the closure of a maximal cone of Σ . Set L_Γ to be the subspace of V^* spanned by x_σ with $\sigma \in \Gamma$. Let also V_Γ be the annihilator of L_Γ in V . Obviously, $L_\Gamma^* \simeq V/V_\Gamma$.

Recalling the quotient $E_W \rightarrow E = \Lambda^\bullet(V)$ from the proof of Theorem 8 we see that every E -module automatically becomes an E_W -module (with vanishing higher products).

Proposition 10 *The derived category $D^b(\mathbb{Y})$ is equivalent to the $(D^b(E_W)/T)^{opp}$ where T is the thick subcategory generated by the A -shifts of E_W -modules $\Lambda^\bullet(V/V_\Gamma)$ and Γ runs through the collection of all minimal subsets in $\Sigma(1)$ which are not contained in the closure of any maximal cone of Σ .*

Proof. Fix a Γ as above and consider the Koszul complex $(\Lambda^\bullet(L_\Gamma) \otimes S_W, d_{Kos})$ of the (not necessarily regular) sequence of elements in S_W given by the images of x_σ , $\sigma \in \Gamma$. Since it is exact on $Z(J) \setminus (Z(J) \cap V(B))$ and its zero cohomology is the algebra of functions on the scheme intersection $Z_J \cap V(B)$, by Lemma 1.2 in [N] the thick subcategory of $D^b(S_W)$ generated by the A -shifts of $(\Lambda^\bullet(L_\Gamma) \otimes S_W, d_{Kos})$ is precisely the category formed by objects with cohomology supported on $Z(J) \cap V(B)$.

By Theorem 1.5 in [N] the thick subcategory I of the previous lemma is generated by the A -shifts of $(\Lambda^\bullet(L_\Gamma) \otimes S_W, d_{Kos})$ for all Γ as in the statement of the proposition.

It remains to prove that the functor \mathcal{F} of Theorem 8 takes $(\Lambda^\bullet(L_\Gamma) \otimes S_W, d_{Kos})$ to the E_W -module $\Lambda(V/V_\Gamma)$. By the same Theorem 8 it suffices to show instead that \mathcal{G} takes $\Lambda(V/V_\Gamma)$ to $(\Lambda^\bullet(L_\Gamma) \otimes S_W, d_{Kos})$ but that follows from the definitions. \square

4.3 Categories of singularities and BGG correspondence

Let $Z = Z(J)$ be the complete intersection in V as before and $D^{perf}(Z) \subset D^b(Z)$ the triangulated subcategory of perfect complexes (in this case, complexes quasi-isomorphic to finite complexes of finitely generated A -graded projective S_W -modules). The quotient $D_{sg}(Z) = D^b(Z)/D^{perf}(Z)$ has been studied in [O1] and [O2] in relation to Landau-Ginzburg models. See *loc. cit.* for more details and motivation. The following proposition is a direct consequence of Theorem 8.

Proposition 11 *The A -graded category $D_{sg}(Z)$ of singularities on Z is equivalent to the quotient $D^b(E_W)/R$ where R is the thick subcategory generated by the A -shifts of k .*

Proof. Recall, that the functor \mathcal{F} simply sends an S_W -module M to $Ext^\bullet(M, k)$ (viewed as an A_∞ -module over the Yoneda algebra $Ext^\bullet(k, k)$). Therefore, if M is projective, \mathcal{F} is a direct sum of several copies of k with their A -grading shifted. The assertion follows. \square

4.4 The Calabi-Yau case

Suppose that \mathbb{Y} has trivial canonical class. By the canonical class formula in Section 4.3 of [F] combined with the adjunction formula, this condition is equivalent to

$$\beta_1 + \dots + \beta_m = \alpha_1 + \dots + \alpha_n$$

Suppose further that the set-theoretic intersection $Z(J) \cap V(B)$ consists only of the origin. This is automatically the case when \mathbb{X}_Σ is a weighted projective space (since then $V(B)$ itself reduces to the origin). In this case we have an alternative description of $D^b(\mathbb{Y})$ as $D_{sg}(S_W)$, cf. Theorem 2.5 in [O2].

Corollary 12 *If \mathbb{Y} has trivial canonical class and $Z(J) \cap V(B)$ is supported at the origin, there exists a derived equivalence*

$$D^b(\mathbb{Y}) \simeq D^b(E_W)/R$$

where R is the thick subcategory generated by k . \square

When \mathbb{Y} is a complete intersection of quadrics in a projective space, after a slight adjustment of grading on E_W (see the Remark at the end of Section 3) this reduces to the result of Bondal and Orlov, [BO].

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