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A UNIVERSAL ENVELOPING FOR L_∞-ALGEBRAS.

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Abstract. For any L_∞-algebra L we construct an A_∞-algebra structure on the symmetric coalgebra $Sym^*_c(L)$ and prove that this structure satisfies properties generalizing those of the usual universal enveloping algebra. These properties follow from an invariant contracting homotopy one the cobar construction of an exterior coalgebra and its relation to combinatorics of permutahedra and semistandard Young tableaux.

1. Introduction

The purpose of this article is to generalize the universal enveloping from Lie to L_∞-algebras. One candidate is well-known, cf. [6]: the cobar construction $\Omega C(L)$ of the Cartan-Chevalley-Eilenberg coalgebra $C(L)$. In fact, for a DG Lie algebra L there exists a surjective quasi-isomorphism of DG algebras $\Omega C(L) \to U(L)$ (and even of DG Hopf algebras). Of course, $\Omega C(L)$ is much larger than $U(L)$: on the level of vector spaces the former is isomorphic to tensor algebra $T^*\Lambda^*(L)$ on the exterior coalgebra $\Lambda^*(L)$, while the latter by PBW theorem is isomorphic to the symmetric coalgebra $Sym^*(L)$.

The DG algebra $\Omega C(L)$ also makes sense for a general L_∞-algebra L and works well enough as a universal enveloping if we deal with DG algebras up to quasi-isomorphism. In other situations, one would like to have some structure on $Sym^*(L)$ generalizing the usual universal enveloping. Since A_∞-algebras relate to associative algebras as L_∞-algebras to Lie algebras, it is natural to expect that $Sym^*(L)$ should be an A_∞-algebra. To construct it, first consider L as a DG vector space (= DG Lie algebra with trivial bracket). Then $C(L)$ turns into the exterior coalgebra $\Lambda^*(L)$ (if we ignore the homological grading) and the universal enveloping into the symmetric algebra $Sym^*_a(L)$. Passing from $\Omega\Lambda^*(L)$ to $\Omega C(L)$ amounts to perturbing the differential on the tensor algebra and the standard techniques of homological perturbation theory, cf. e.g. [5], give an A_∞-structure on $Sym^*(L)$. After the first draft of the present paper has been completed, it was pointed out to the author that a similar strategy (but using filtrations instead of perturbation theory) was used in [13] to prove a PBW-type theorem.

However, the functorial properties of such A_∞-structure will depend on a homotopy contracting $\Omega\Lambda^*(L)$ onto $Sym^*_a(L)$. For example, when L is a finite dimensional vector space in degree zero, one needs the homotopy to be $GL(L)$-invariant.

This motivates a closer study of $\Omega\Lambda^*(V)$ for a DG vector space V. In Section 3 we prove an isomorphism of complexes, cf. Theorem 1:

$$\Omega\Lambda^*(V) \simeq k \oplus \bigoplus_{n\geq 1} \Big(V^{\otimes n} \otimes_{k[\Sigma_n]} C_*(P_n) \Big) \tag{1}$$

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where Σ_n is the symmetric group and $C_*(P_n)$ is the complex computing the cell homology of the n-th *permutahedron* P_n.

Recall, e.g. [15], that P_n may be defined as the convex hull of the orbit of $(1, 2, \ldots, n) \in \mathbb{R}^n$ under the permutation action of Σ_n. The k-dimensional faces of P_n are labeled by *ordered* partitions $\{1, \ldots, n\} = \psi_1 \cup \ldots \cup \psi_{n-k}$ of $\{1, \ldots, n\}$ into a disjoint union of its $(n-k)$ subsets, e.g. the vertices of P_n correspond to permutations in Σ_n. In these terms, an element $(v_1 \otimes \ldots \otimes v_n) \otimes (\psi_1, \ldots, \psi_{n-k})$ on the right hand side of (1), corresponds to $\pm \otimes_{j=1}^{n-k} (\wedge_{i\in\psi_j} v_i)$ on the left hand side. Of course, it is agreement with differentials and contractibility of P_n which are important in (1). Informally, the differentials agree since codimension one sub-faces of a face $(\psi_1, \ldots, \psi_{n-k}) \subset P_n$ are obtained by breaking a single ψ_i into a pair of disjoint subsets, which also corresponds to the standard coproduct in $\Lambda^*(V)$. Permutahedra were studied in [12] (where they are denoted by $C(n-1)$) precisely in relation to the loop spaces, but the algebraic statement (1) appears to be new. It leads to a choice of a contracting homotopy for $\Omega\Lambda^*(V) \to Sym^*(V)$ which is functorial in V - but not quite canonical.

In Section 2 we construct the universal enveloping $U(L)$ and prove that it has several properties generalizing those of the classical universal enveloping. In particular, Theorem 3 shows that $U(L)$ is a sort of "homotopy Hopf algebra" even though the operadic meaning of our construction, e.g. the precise relation to the "operadic" universal enveloping algebras of Lada and Markl, cf. [8], remains unclear at the moment. In particular, we prove that the diagonal of $Sym^*_c(L)$ is a strict morphism of A_∞-algebras if $Sym^*(L) \otimes Sym^*(L)$ is identified with $Sym^*(L \oplus L)$. Although such "A_∞ tensor product" is extremely natural in the present context, its relation to the Saneblidze-Umble diagonal, cf. [15], or the diagonal on the W-construction of the associative operad, cf. [10], still needs to be clarified. The correspondence $L \mapsto U(L)$ falls short of being a functor: we only prove that $U(\psi) \circ U(\phi) = U(\psi \circ \phi)$ if one of the L_∞-morphisms ψ, ϕ is *strict*, and give an example showing that this fails in general. In Theorem 4 we generalize the classical complex $\Lambda^*(L) \otimes U(L)$ and prove a derived equivalence between $C(L)$ and $U(L)$ (i.e. a version of the BGG correspondence). In Theorem 4 we show that appropriate categories of A_∞-modules over $U(L)$ and L_∞-modules over L, are equivalent. While the derived equivalence between $C(L)$ and $U(L)$ and the functor $\mathcal{F}$ from $U(L)$-modules to L-modules are relatively easy to obtain, the inverse functor $\mathcal{G}$ from L-modules to $U(L)$-modules depends on Theorems 1 and 3 in an essential way. By a recent spectacular result of Merkulov, cf. [11], a homotopy Lie bialgebra structure on L induces a homotopy bialgebra structure on $Sym^*(L)$, defined via some non-explicit operadic maps. It would be interesting to see if Theorem 4 allows one to describe the latter structure along the lines of Kazhdan-Etingof.

2. The Universal Enveloping

2.1. Notations and standard definitions. We consider complexes of vector spaces k over a field of characteristic zero. We use cohomological grading, to be denoted by superscripts, in which differentials have degree $+1$. If V is a complex, its suspension sV is defined by $(sV)^p = V^{p+1}$, $d(sv) = -s(dv)$. In particular $\deg(sv) = \deg v - 1$. All tensor products are over k unless indicated otherwise. Throughout this paper we use the Koszul sign rule

$$(F \otimes G)(a \otimes b) = (-1)^{\deg G \cdot \deg a} F(a) \otimes G(b)$$

If V is a graded vector space $Sym^*(V) = \oplus_{k\geq 0} Sym^k(V)$ will stand for its graded symmetric tensors, i.e. $Sym^k(V)$ is the space of vectors in $V^{\otimes k}$ which are invariant with respect to the *graded* action of the symmetric group S_k (i.e. whenever two odd elements are permuted this leads to a change of sign). If we disregard the grading and assume that V has only even vectors (resp. only odd vectors) this will become the usual space of symmetric (resp. antisymmetric) tensors. Note that $Sym^*(V)$ has standard structures of a commutative algebra $Sym^*_a(V)$ and a cocommutative coalgebra $Sym^*_c(V)$. We will also denote $\Lambda^*(V) = Sym^*(sV)$ - observe the *shift of grading* involved.

Let L be a DG Lie algebra with differential l_1 and the bracket $l_2 : L^{\otimes 2} \to L$. Its *Cartan-Chevalley-Eilenberg construction* $C(L)$ is the DG coalgebra $\Lambda^*_c(L)$ with the differential $\delta_C = c_1 + c_2$ defined as follows. Let $s^{\otimes n} : L^{\otimes n} \to (sL)^{\otimes n}$ be the obvious degree $(-n)$ isomorphism and set

$$c_1 = -s\, l_1 s^{-1} : \Lambda^1(L) \to \Lambda^1(L); \quad c_2 = s\, l_2 (s^{\otimes 2})^{-1} : \Lambda^2(L) \to \Lambda^1(L)$$

extending these maps to $\Lambda^*_c(sL)$ as coderivations. Then $\delta_C^2 = 0$ follows from $l_1^2 = 0$, the Leibniz Rule and the Jacobi Identity for l_2. If δ is a general differential on $\Lambda^*_c(L)$ which is a coderivation, we can consider compositions $c_n : \Lambda^n(L) \to \Lambda^*_c(L) \xrightarrow{\delta} \Lambda^*_c(L) \to \Lambda^1(L)$ and define $l_n : L^{\otimes n} \to L$ via

$$c_n = (-1)^n s\, l_n (s^{\otimes n})^{-1}$$

Then $\{l_n\}_{n\geq 1}$ give L the structure of an L_∞-algebra, cf. [8]. If $\phi : (\Lambda^*_c(L), \delta) \to (\Lambda^*_c(L'), \delta')$ is a degree zero morphism of DG coalgebras, in a similar way we get a sequence of degree $1-i$ maps $\phi_i : \Lambda^i(L) \to \Lambda^1(L')$. The sequence $\{\phi_i\}_{i\geq 1}$ (or, equivalently, the original morphism ϕ) is called an L_∞-morphism from L to L'.

Let $A = k \oplus \overline{A}$ be an augmented DG algebra with differential m_1 and product m_2. Its *reduced cobar construction* $B(A)$ is the tensor coalgebra $T^*_c(s\overline{A})$ with the coproduct

$$\Delta_B[a_1, \dots, a_n] = 1 \boxtimes [a_1, \dots, a_n] + [a_1, \dots, a_n] \boxtimes 1 + \sum_{i=1}^{n-1} [a_1, \dots, a_i] \boxtimes [a_{i+1}, \dots, a_n],$$

and the similar differential $\delta_B = b_1 + b_2$:

$$b_1 = -1 s\, m_1 s^{-1} : s\overline{A} \to s\overline{A}; \quad b_2 = s\, m_2 (s^{\otimes 2})^{-1} : (s\overline{A})^{\otimes 2} \to s\overline{A}.$$

Then b_1 and b_2 extend uniquely to $B(A)$ as coderivations and $\delta_B^2 = 0$ follows from $m_1^2 = 0$, the Leibniz Rule and associativity of m_2.

Again, one can consider a general differential δ_B on $T^*_c(s\overline{A})$ which is a coderivation, and obtain operations $m_n : A^{\otimes n} \to A$ by first considering

$$b_n : (s\overline{A})^{\otimes n} \to BA \xrightarrow{\delta_B} BA \to s\overline{A}$$

and then writing

$$b_n = (-1)^n s\, m_n (s^{\otimes n})^{-1}.$$

The resulting operations $\{m_n\}_{n\geq 1}$ give A a structure of an A_∞-algebra, cf. [7]. Since we use the reduced bar construction, A is automatically *strictly unital*, i.e.

$$m_n(v_1, \ldots, v_n) = 0; \text{ if } n \geq 3 \text{ and } \; v_i = 1 \text{ for some } i$$

and $m_2(v,1) = m_2(1,v) = v$. A DG coalgebra morphism $f : (BA, \delta_B) \to (BA', \delta'_B)$ gives a sequence of degree $(1-i)$ maps $f_i : A^{\otimes i} \to A'$, called an A_∞*-morphism* from A to A'. Again, for reduced bar constructions such a morphism is automatically *strictly unital*: $f_i = 0$ if $i \geq 2$ and one of its arguments is equal to $1 \in A$.

Finally, let $C = k \oplus \overline{C}$ be a coaugmented DG coalgebra. Its *reduced cobar construction* is a DG algebra $\Omega(C) = T^*_a(s^{-1}\overline{C})$ with the differential $\delta_\Omega = \omega_1 + \omega_2$ where ω_1 and ω_2 are obtained from the differential on C and the reduced coproduct $\overline{\Delta} : \overline{C} \otimes \overline{C} \to \overline{C}$, respectively, using the same pattern (w_1 and w_2 are extended from $s^{-1}\overline{C}$ to $\Omega(C)$ as derivations). If C is *cocommutative* the DG algebra $\Omega(C)$ also has a shuffle coproduct $\Delta_\Omega : \Omega(C) \to \Omega(C) \boxtimes \Omega(C)$ defined on $s^{-1}\overline{C} \subset \Omega(C)$ by

$$\Delta_\Omega(u) = u \boxtimes 1 + 1 \boxtimes u$$

and extended to $\Omega(C)$ multiplicatively. Thus, $\Omega(C)$ becomes a DG bialgebra (the fact that δ_Ω is also a coderivation uses cocommutativity of C).

2.2. Universal enveloping: case of Lie algebras and the general plan. Let L be a DG Lie algebra and $U(L)$ its universal enveloping algebra. One way - perhaps a little exotic - to construct $U(L)$ is as follows. The natural projection $C(L) = \Lambda^*(L) \to \Lambda^1(L)$ induces a DG-bialgebra morphism of $\Omega C(L) \to U(L)$. By Theorem 22.9 and the first equality on page 290 in [4], it is also a quasi-isomorphism. In Section 3 we essentially re-prove this assertion.

We can turn this property inside out and use as a definition. First, consider L with the same differential but trivial Lie bracket. The above construction gives a quasi-isomorphism of DG algebras $\Omega\Lambda^*_c(L) \to Sym^*_a(L)$. Bringing back the original bracket on L will deform the differential on $\Lambda^*(L)$, and therefore the differential on $\Omega\Lambda^*_c(L)$. The general machinery of perturbation theory, see [5] and Section 2.3 below, gives a new DG algebra structure on $Sym^*(L)$ and a multiplicative projection from $\Omega C(L)$ onto $Sym^*(L)$ which is still a quasi-isomorphism. In Theorem 3 (v) we prove that the new structure on $Sym^*(L)$ is precisely the universal enveloping $U(L)$ (identified by PBW theorem with $Sym^*(L)$ as a *coalgebra*).

This approach also gives a recipe for a general L_∞-algebra L, since an L_∞-structure also gives a perturbation of the differential on $\Lambda^*(L)$ and we can carry out a similar procedure of adjusting the product on $Sym^*(L)$. By *loc. cit.* such adjustment in general leads to an A_∞-structure on $Sym^*(L)$. As the procedure depends on a choice of homotopy on $\Omega\Lambda^*_c(L)$ our construction will be based on the following result.

Theorem 1. *For a complex V set $A(V) = \Omega\Lambda^*_c(V)$, $E(V) = Sym^*_a(V)$. Let $f_V : A(V) \to E(V)$ be the multiplicative extension of the projection $s^{-1}\Lambda^{\geq 1}(V) \to V$, and $g_V : E(V) \to A(V)$ the map given by composition of natural embeddings*

$$Sym^n(V) \hookrightarrow V^{\otimes n} \hookrightarrow T^*(V) \hookrightarrow T^*(s^{-1}\Lambda^*_c(V)) = \Omega\Lambda^*_c(V)$$

Then $f_V g_V = 1$ *and there exists a contracting homotopy* $h_V : A(V) \to A(V)$ *which satisfies*

$$1 - g_V f_V = dh_V + h_V d; \qquad f_V h_V = 0; \qquad h_V g_V = 0; \qquad h_V h_V = 0$$

and is functorial in the following sense: for every morphism of complexes $\phi : V \to W$ *the natural induced map* $A(V) \to A(W)$ *fits into commutative diagram*

$$\begin{tikzcd} A(V) \arrow[r] \arrow[d, "h_V"'] & A(W) \arrow[d, "h_W"'] \\ A(V) \arrow[r] & A(W) \end{tikzcd}$$

Moreover, one can choose h_V *to commute with the algebra anti-involution* ι_Ω *on* $\Omega\Lambda_c^*(V)$ *which acts by* (-1) *on the space of generators* $s^{-1}\Lambda_c^*(V)$.

The proof of this theorem is given in Section 3. We will see that such a homotopy h_V (or rather a system of homotopies $V \mapsto h_V$) is not unique but its choice depends on purely combinatorial data that has nothing to do with V.

2.3. Universal enveloping: construction and first properties. *Construction.*

Let $(L, \{l_i\}_{i\geq 1})$ be an L_∞-algebra. First consider the complex (L, l_1) and set $V = L$ in Theorem 1, which gives a contraction (f_L, g_L, h_L) from $A(L) = \Omega\Lambda_c^*(L)$ to $E(L) = Sym_a^*(L)$ and hence a contraction (f'_L, g'_L, h'_L) from $sA(L)$ to $sE(L)$ given by

$$f'_L = sf_L s^{-1}, g'_L = sg_L s^{-1}, h'_L = -sh_L s^{-1}.$$

From this we produce a contraction of the free tensor coalgebra $T_c^*(s\overline{A(L)})$ onto the free tensor coalgebra $T_c^*(s\overline{E(L)})$ (recall that $\overline{(\cdot)}$ denotes the augmentation ideal). On the n-th tensor components set

$$F = (f'_L)^{\otimes n}, \quad G = (g'_L)^{\otimes n}, \quad H = \sum_{t=1}^{n} (g'_L f'_L)^{\otimes(t-1)} \otimes h'_L \otimes 1^{\otimes(n-t)}. \tag{2}$$

It follows from the definitions that

$$1 - GF = dH + Hd; \qquad FH = 0; \qquad HG = 0; \qquad HH = 0$$

Denote by δ_L° and d_L° the differentials of the two tensor coalgebras, respectively. By definition $BA(L)$ differs from $T_c^*(s\overline{A(L)})$ only in its differential, given by

$$\delta_L = \delta_L^\circ + t_\mu + t_L \tag{3}$$

where t_μ is the part that encodes the product on the tensor algebra $A(L)$ and t_L is the perturbation which encodes the L_∞-structure on L, cf. Section 2.1. The contraction (F, G, H) can be adjusted to work with δ_L using the following Basic Perturbation Lemma, cf. [3]:

Lemma 2. *Let* (M, d_M), (N, d_N) *be two complexes and consider a contraction*

$$F : N \to M; \qquad G : M \to N; \qquad H : N \to N$$

which satisfies

$$FG = 1_N, \quad 1_M - GF = d_N H + Hd_N, \quad FH = 0, \quad HH = 0, \quad HG = 0. \tag{4}$$

Given a new differential $d_N + t$ on N such that (tH) is locally nilpotent (i.e. for any element $n \in N$ there is a positive integer $k(n)$ such that $(tH)^{k(n)}(n) = 0$) so that the infinite sum

$$X = t - tHt + tHtHt - \dots$$

is well-defined; the formulas

$$F_t = F(1 - XH); \quad G_t = (1 - HX)G; \quad H_t = H - HXH; \quad (d_M)_t = d_M + FXG$$

give a contraction of the complex $(N, d_N + t)$ to the complex $(M, (d_M)_t)$ which satisfies equations similar to (4).

Applying this result to the perturbation (3) we get a new contracting homotopy

$$F_L = F_{t_\mu + t_L}, \qquad G_L = G_{t_\mu + t_L}, \qquad H_L = H_{t_\mu + t_L}$$

from $B\Omega C(L)$ to $T_c^*(s\overline{E(L)})$ with its new differential $d_L = (d_L^\circ)_{t_\mu + t_L}$. Moreover, the new contraction agrees with the coalgebra structures: since F, G are morphisms of coalgebras, H is a coalgebra homotopy and $t = t_\mu + t_L$ is a coalgebra perturbation, i.e.:

$$\Delta_B H = (H \otimes 1 + GF \otimes H)\Delta_B; \qquad \Delta_B t = (t \otimes 1 + 1 \otimes t)\Delta_B,$$

the new differential d_L is again a coderivation, F_L, G_L are morphisms of DG coalgebras and H_L is a coalgebra homotopy, cf. [5].

Definition.

i. Denote by $U(L)$ the vector space $E(L) = Sym^*(L)$ with the A_∞-structure $\{m_i\}_{i \geq 2}$ given by the above coalgebra differential d_L on $T_c^*(s\overline{E(L)})$. Then $(T_c^*(s\overline{E(L)}), d_L)$ is the cobar construction $BU(L)$ of $U(L)$.

ii. If L, M are two L_∞ algebras and $\phi : C(L) \to C(M)$ is an L_∞ morphism, cf. [8], let $U(\phi) = F_M \; B\Omega(\phi) \; G_L : BU(L) \to BU(M)$.

iii. If $\phi : C(L) \to C(M)$ and $\psi : C(M) \to C(N)$ are two L_∞-morphisms, set $H(\phi, \psi) = F_N \; B\Omega(\psi) \; H_M \; B\Omega(\phi) \; G_L : BU(L) \to BU(N)$.

Theorem 3. *Let $\phi : C(L) \to C(M)$ be an L_∞-morphism of L_∞-algebras L, M and $\phi_1 : L \to M$ be its first component. Then*

i. *$U(\phi)$ is an A_∞-morphism from $U(L)$ to $U(M)$ and its first component $U(\phi)_1 : U(L) = Sym^*(L) \to Sym^*(M) = U(M)$ is given by symmetrization of ϕ_1.*

ii. *If $\phi : L \to M$ is a strict morphism of L_∞-algebras, i.e. $\phi_i = 0$ for $i \geq 2$, then the same holds for $U(\phi)$, i.e. $U(\phi)_i = 0$ for $i \geq 2$.*

iii. *The standard coproduct $\Delta : Sym^*(L) \to Sym^*(L) \otimes Sym^*(L)$ is a strict morphism of A_∞-algebras, if the latter is given an A_∞-structure via the natural isomorphism*

$$Sym^*(L) \otimes Sym^*(L) \simeq Sym^*(L \oplus L).$$

iv. *If $\phi : C(L) \to C(M)$ and $\psi : C(M) \to C(N)$ are two L_∞-morphisms then*

$$U(\psi \circ \phi) - U(\psi) \circ U(\phi) = d_{U(N)} H(\phi, \psi) + H(\phi, \psi) d_{U(L)} : BU(L) \to BU(N).$$

Moreover, if at least one of the morphisms ϕ, ψ is strict, then $H(\phi, \psi) = 0$.

v. *If the 2-truncation (L, l_1, l_2) is a DG Lie algebra then $(U(L), m_1, m_2)$ is a DG algebra isomorphic to the usual universal enveloping of (L, l_1, l_2).*

vi. *Let $\iota : U(L) \to U(L)$ be the linear involution that corresponds to the action of $(-1)^k$ on $Sym^k(L)$. Then*

$$m_n \circ \iota^{\otimes n} = \iota \circ m_n \circ \omega_n$$

where ω_n is the permutation $\{1, \ldots, n\} \to \{n, \ldots, 1\}$. In other words, ι is a strict morphism $U(L) \to U(L)^{op}$, where $(\cdot)^{op}$ is the opposite A_∞-structure.

vii. *Let $n \geq 2$ and $v_1, \ldots, v_n \in L \subset U(L)$. Let $Alt(v_1 \otimes \ldots \otimes v_n)$ be the graded antisymmetrization of $v_1 \otimes \ldots \otimes v_n$. Then*

$$m_n(Alt(v_1 \otimes \ldots \otimes v_n)) = l_n(v_1, \ldots, v_n).$$

Proof. Parts (i) - (iv). In this proof we deal with several L_∞-algebras and it helps to re-denote the operators in (2) by $F_L^\circ, G_L^\circ, H_L^\circ$ and similarly for M, N. To prove (i) first observe that F_M and G_L are DG coalgebra morphisms by [5] and $B\Omega(\phi)$ is a DG coalgebra morphism since ϕ itself is a DG coalgebra morphism. Therefore $U(\phi) : BU(L) \to BU(M)$ is a DG coalgebra morphism encoding an A_∞-morphism $U(L) \to U(M)$. To compute the first component we need to evaluate $U(\phi)$ on $v \in \overline{U(L)} \subset BU(L)$. But, for such an element, all terms in F_M, G_L which involve perturbation of the differentials on $B\Omega C(L)$, $B\Omega C(M)$, are identically zero, therefore $U(\phi)(v) = F_M^\circ B\Omega(\phi) G_L^\circ(v)$ and the latter map is precisely given by the symmetrization $Sym(\phi)$ of ϕ. To prove (ii) we observe that for a strict morphism ϕ one has $H_M^\circ B\Omega(\phi) = B\Omega(\phi) H_L^\circ$ by Theorem 1. Using the explicit formulas of the Basic Perturbation Lemma,

$$F_M = F_M^\circ(1 - X_M H_M^\circ); \quad G_L = (1 - H_L^\circ X_L) G_L^\circ; \quad H_M = H_M^\circ(1 - X_M H_M^\circ)$$

and the side conditions $H_M^\circ H_M^\circ = 0, F_M^\circ H_M^\circ = 0, H_L^\circ G_L^\circ = 0$ we obtain

$$F_M \circ B\Omega(\phi) \circ G_L = F_M^\circ \circ B\Omega(\phi) \circ G_L^\circ = BSym(\phi)$$

Part (iii) is an immediate application of (ii) to the diagonal map $L \to L \oplus L$, $x \mapsto x \oplus x$ which is a strict morphism of L_∞-algebras. Finally, the left hand side in (iv) by definition is equal to

$$F_N B\Omega(\psi)(1 - G_M F_M) B\Omega(\phi) G_L = F_N B\Omega(\psi)(\delta_N H_M + H_M \delta_N) B\Omega(\phi) G_L$$

and the assertion follows since F_N, $B\Omega(\psi)$, $B\Omega(\psi)$ and G_L are morphisms of complexes. To prove the vanishing we observe that, by Theorem 1, $H_M^\circ B\Omega(\phi) = B\Omega(\phi) H_L^\circ$ if ϕ is strict, and similarly for ψ. Now the side conditions and the formulas for F, G, H finish the proof.

Part (v). First we assume that L is a Lie algebra, i.e. all l_i vanish for $i \geq 3$. The A_∞-structure on $E(L) = Sym^*(L)$ is given by the following differential on $T_c^*(sE(L))$:

$$d_L = d_L^\circ + F_L^\circ \Big(\sum_{i \geq 0} (-1)^i \big((t_\mu + t_L) H_L^\circ \big)^i \Big) (t_\mu + t_L) G_L^\circ$$

To simplify this expression we first introduce a "geometric grading" on $\Omega\Lambda_c^*(L)$ by declaring that elements of $s^{-1}\Lambda^k(L)$ have degree $(k-1)$, and extending to $\Omega\Lambda_c^*(L)$ multiplicatively (we can declare that $k \subset \Omega\Lambda_c^*(L)$ has degree (-1) but that will not be used in the proof). From the point of view Lemma 6 in Section 3, this grading corresponds to dimension of the cells of permutahedra. We extend it to $B\Omega\Lambda_c^*(L)$ in the obvious way (again, setting to (-1) on the constants).

Then t_L vanishes on elements of geometric degree 0 since those elements are products of linear symmetric tensors, and the bracket l_2 encoded by t_L needs two inputs. Since the image of G_L° belongs to the degree 0 part we will have $t_L G_L^\circ = 0$. Also, the proof of Theorem 1, cf. Section 3.2, implies that H_L° increases the geometric degree by 1, t_L decreases by 1, t_μ preserves it, while F_L° vanishes on elements of positive degree. Consequently, the above formula for the deformed differential simplifies to

$$d_L = d_L^\circ + F_L^\circ\Big(\sum_{i\geq 0}(-1)^i\big(t_L H_L^\circ\big)^i\Big)t_\mu G_L^\circ$$

The terms responsible for a multiple product $m_n : U(L)^{\otimes n} \to U(L)$ contain t_μ exactly $(n-1)$ times, thus the differential on $U(L)$ is the same on $Sym^*(L)$ and all m_n with $n \geq 3$ vanish. So $U(L)$ is a DG algebra. Denoting the usual symmetric product in $E(L) = Sym^*(L)$ by $*$, we also see that for $x, y \in Sym^*(L)$ homogeneous in the geometric grading:

$$m_2(x,y) = x * y + (\text{terms of lower geometric degree}).$$

Therefore, the subspace $L \subset U(L)$ generates $U(L)$ as an algebra. For $v, u \in L$ an explicit computation shows

$$m_2(v,u) = v * u + \frac{1}{2}l_2(v,u).$$

Denote for a moment by $U^{cl}(L)$ the classical universal enveloping. The last formula gives a surjective DG algebra morphism $U^{cl}(L) \to U(L)$ which is easily seen to be an isomorphism by an inductive argument involving natural filtrations on both algebras.

Next, we assume that the higher products l_i, $i \geq 3$ of L are not necessarily zero. Then the pertrubation $\delta_L = \delta_L^\circ + t_\mu + t_L$ can be split as $(\delta_L^\circ + t_\mu + t_L^{(2)}) + (t_L - t_L^{(2)})$ where $t_L^{(2)}$ is the term coming from the bracket l_2. The expression in the first parenthesis has square zero since by assumption (L, l_1, l_2) is a DG Lie algebra. Thus, setting $t_1 = t_\mu + t_L^{(2)}$, $t_2 = t_L - t_L^2$ we see that both $\delta_L^\circ + t_1$ and $\delta_L^\circ + t_1 + t_2$ satisfy the condition of Lemma 2. By direct computation one can show that in such a setting one always has $(d_M)_{t_1+t_2} = ((d_M)_{t_1})_{t_2}$ and similarly for F, G and H.

Thus the A_∞-structure of $U(L)$ corresponds to the perturbation of $F_{t_1}, G_{t_1}, H_{t_1}$ and d_{t_1} by t_2. In particular, the differential of $BU(L)$ is given by

$$d_{t_1} + F_{t_1}\big(\sum_{i\geq 0}(-1)^i(t_2 H_{t_1})^i\big)t_2 G_{t_1}.$$

Evaluating the second term on $s\overline{U(L)} \subset BU(L)$ and $s\overline{U(L)} \otimes s\overline{U(L)} \subset BU(L)$ will give zero for the following reasons. Firstly, for $x \in s\overline{U(L)}$ we have $G_{t_1}(x) = G_L^\circ(x)$ since $t_1 G_L^\circ(x) = 0$. But then $t_2 G_{t_1}(x) = t_2 G_L^\circ(x) = 0$ since t_2 vanishes on terms of geometric degree ≤ 1. Secondly, for $x_1, x_2 \in s\overline{U(L)}$ by a similar computation

$$G_{t_1}(x_1 \otimes x_2) = \Big[\sum_{i\geq 0}(-1)^i(H_L^\circ t_L^{(2)})^i\Big]H_L^\circ(G_L^\circ(x_1) \otimes G_L^\circ(x_2))$$

Since H increases the geometric degree by 1 and $t_L^{(2)}$ decreases it by 1, the above expression has degree 1, hence t_2 vanishes on it. So the differential and the product of $U(L)$ are the same as for the 2-truncation (L, l_1, l_2), which proves (v).

To prove (vi) for $n \geq 3$ consider a similar anti-involution $\iota_\Omega : \Omega C(L) \to \Omega C(L)^{op}$ of Theorem 1. Let $\widehat{\omega}$ be a linear involution on $BU(L)$ which acts by ω_n on $(s\overline{U(L)})^{\otimes n}$ and use the same notation for the corresponding involution on $B\Omega C(L)$. Denote by $\pi : BU(L) \to \overline{U(L)}$ projection onto the first component. Also, let $B\iota$, $B\iota_\Omega$ be the linear involutions on the bar constructions which act by $s^{\otimes n}\iota^{\otimes n}(s^{\otimes n})^{-1}$, $s^{\otimes n}\iota_\Omega^{\otimes n}(s^{\otimes n})^{-1}$ on the n-th tensor components, respectively. Since $\omega_n s^{\otimes n} = (-1)^{\frac{n(n-1)}{2}} s^{\otimes n}\omega_n : (\overline{U(L)})^{\otimes n} \to (s\overline{U(L)})^{\otimes n}$, we need to show that

$$(-1)^{\frac{n(n-1)}{2}}\pi(F_L^\circ X_L G_L^\circ)(B\iota\,\widehat{\omega}) = (B\iota\,\widehat{\omega})\pi(F_L^\circ X_L G_L^\circ)$$

on $(s\overline{U(L)})^{\otimes n}$. By Section 5.2 in the appendix X_L is a sum of several terms of the form

$$(-1)^s a_1 \dots a_s t_\mu$$

where each a_i is either $(t_L H_L^\circ)$ or $(t_\mu H_L^\circ)$. If such a term is to give a nonzero contribution to the expression above, the operator t_μ should be used exactly $(n-1)$ times, since we need to get from $(s\overline{U(L)})^{\otimes n}$ to $s\overline{U(L)}$. It is easy to see that

$$(B\iota\,\widehat{\omega})F_L^\circ = F_L^\circ(B\iota_\Omega\,\widehat{\omega}); \quad (B\iota_\Omega\,\widehat{\omega})G_L^\circ = G_L^\circ(B\iota\,\widehat{\omega})$$

and that $(B\iota_\Omega\,\widehat{\omega})$ commutes with the operators t_L and H_L°. Now what we need to prove follows from the following formula, easily checked by direct computation:

$$(B\iota_\Omega\,\widehat{\omega})t_\mu = (-1)^{i-1}t_\mu(B\iota_\Omega\,\widehat{\omega}) : (s\overline{U(L)})^{\otimes i} \to (s\overline{U(L)})^{\otimes(i-1)}.$$

For $n = 2$ the same argument works for $(m_2 - *)$ where $*$ is the usual product on $Sym^*(L)$. Since $*$ is commutative, the assertion holds for m_2 as well. For $n = 1$, the differential on $U(L)$ is the same as on $Sym^*(L)$ and the statement holds again.

Finally, (vii) is a restatement of Theorem 3 (i) below and its proof will be given there. $\square$

Example. We give an example when $U(\psi \circ \phi) \neq U(\psi) \circ U(\phi)$ regardless of the choice of h_V in Theorem 1. To that end, assume that L, M, N are vector spaces placed in homological degrees $0, -1, -3$, respectively and that they have have trivial differentials and brackets (i.e. $C(L)$, $C(M)$, $C(N)$ reduce to graded exterior algebras with trivial differentials). Then ϕ is a sequence of degree zero linear maps $\phi_i : \Lambda^i(L) \to (sM)$ and for degree reasons only ϕ_2 can be non-zero. Since differentials and brackets vanish, any degree zero linear map $\phi_2 : \Lambda^2(L) \to sM$ gives an L_∞-morphism. Similarly, for ψ only the component $\psi_2 : \Lambda^2(M) \to sN$ can be non-zero and any such degree zero linear map will do. Note that ϕ_2 is skew-symmetric and ψ_2 is symmetric (since M is an *odd* vector space). Then $\rho = \psi \circ \phi$ can only have a non-trivial component $\rho_4 : \Lambda^4(L) \to N$ given, up to normalizing factor, by

$$\rho(v_1, v_2, v_3, v_4) =$$
$$= \psi(\phi(v_1, v_2), \phi(v_3, v_4)) - \psi(\phi(v_1, v_3), \phi(v_2, v_4)) + \psi(\phi(v_1, v_4), \phi(v_2, v_3)).$$

In notation of the proof of Theorem 2(iv), we want to show that $F_N B\Omega(\psi)(1 - G_M F_M)B\Omega(\phi)G_L \neq 0$ which would imply that $U(\rho) \neq U(\psi) \circ U(\phi)$. To distinguish between the tensor products in $B(\dots)$ and $\Omega(\dots)$ we denote the first one by $\boxtimes$ and the second by $\otimes$. Observe that in our case $F_N = F_N^\circ$ and $F_M = F_M^\circ$.

Consider now $Z = v_1 \boxtimes v_2 \boxtimes v_3 \boxtimes v_4 \in (sL)^{\boxtimes 4} \subset BU(L)$. Then by Basic Perturbation Lemma, $G_L(Z) = (1 - (H_L^\circ t) + (H_L^\circ t)^2 - (H_L^\circ t)^3) G_L^\circ$ where t is the coderivation of $B\Omega C(L)$ which encodes the product structure of $\Omega C(L)$. By explicit computation, the terms of $G_L(Z)$ which can give a non-zero contribution into $B\Omega(\phi) G_L(Z)$ are

$$-\frac{1}{4}(v_1 \wedge v_2) \boxtimes (v_3 \wedge v_4) + c\,(v_1 \wedge v_2 \wedge v_3 \wedge v_4) \in B\Omega C(L)$$

where c is a certain constant and $\star$ is the graded symmetric product in $C(L)$. All other terms will involve a $\boxtimes$-factor of either v_i or $v_i \wedge v_j \wedge v_k$, and $B\Omega(\phi)$ applied to them is zero since ϕ_2 must have two inputs. Up to a factor,

$$B\Omega\phi(v_1 \wedge v_2 \wedge v_3 \wedge v_4) =$$

$$= \phi_2(v_1, v_2) \star \phi_2(v_3, v_4) - \phi_2(v_1, v_3) \star \phi_2(v_2, v_4) + \phi_2(v_1, v_4) \star \phi_2(v_2, v_3)$$

where $\star$ is the product in $C(M)$. The last expression is in the kernel of F_M therefore applying $F_N B\Omega(\psi)(1 - G_M F_M)$ will give (a multiple of) $\rho(v_1, v_2, v_3, v_4)$. On the other hand, applying $(1 - G_M F_M) B\Omega(\phi)$ to $(v_1 \wedge v_2) \boxtimes (v_3 \wedge v_4)$ we get

$$(1 - G_M F_M)(\phi_2(v_1, v_2) \boxtimes \phi_2(v_3, v_4)) = (H_M^\circ t')(\phi_2(v_1, v_2) \boxtimes \phi_2(v_3, v_4))$$

where t' is the coderivation of $B\Omega C(M)$ which encodes the product of $\Omega C(M)$. Then

$$F_N B\Omega(\psi)(1 - G_M F_M) B\Omega(\phi) G_L(Z) = c' \psi_2(\phi_2(v_1, v_2), \phi_2(v_3, v_4)) + c'' \rho(v_1, v_2, v_3, v_4)$$

where c', c'' are constants, and $c' \neq 0$. Since the first term is not antisymmetric in v_1, v_2, v_3, v_4, the sum cannot be zero and $U(\psi \circ \phi) \neq U(\psi) \circ U(\phi)$ on $v_1 \boxtimes v_2 \boxtimes v_3 \boxtimes v_4$. We have even proved that $U(\psi) \circ U(\phi) : U(L) \to U(N)$ does not arise from any L_∞-morphism $L \to N$.

2.4. Universal enveloping: categories of modules. Recall that $U(L)$ denotes the vector space $Sym^*(L)$ with the A_∞-structure constructed in the previous subsection. The next theorem deals with the notion of a generalized twisted cochain and the functors defined by it, see appendix. Part (iii) asserts a BGG-type equivalence to two derived categories, $\mathcal{D}U(L)$ and $\mathcal{D}C(L)$. The derived category $\mathcal{D}U(L)$ is obtained by localizing the category $Mod_\infty(U(L))$ of strictly unital A_∞-modules over $U(L)$ and strictly unital morphisms (= the full subcategory of DG-comodules over $BU(L)$ which are free as comodules), at the class of quasi-isomorphisms. The derived category $\mathcal{D}C(L)$ is obtained by localizing the category $Comodc(C(L))$ of cocomplete counital DG-comodules over $C(L)$, by the class of weak equivalences (i.e. morphisms which induce a quasi-isomorphism on the bar construction). See Chapter 2 in [9] and Section 3.2 in [2] for more details.

Theorem 4. *The universal enveloping $U(L)$ has the following properties:*

i. *the composition $\tau : C(L) \to L \to U(L)$ is a generalized twisted cochain;*
ii. *the complex $C(L) \otimes_\tau U(L)$ is quasi-isomorphic to k and the DG algebra morphism $\Omega C(L) \to \Omega BU(L)$ induced by τ, is a quasi-isomorphism;*
iii. *the functors $M \mapsto M \otimes_\tau C(L)$ and $N \mapsto N \otimes_\tau U(L)$ induce mutually inverse equivalences of the derived categories $\mathcal{D}C(L)$ and $\mathcal{D}U(L)$.*

Proof. To prove (i), start with the composition

$$C(L) \to B\Omega C(L) \xrightarrow{F_L} BU(L).$$

Since it is a DG coalgebra morphism, by 5.3 in the appendix, its projection onto $U(L)$ is a generalized twisted cochain $C(L) \to U(L)$. It is easy to check that it coincides with τ. Part (ii) is known when L is an abelian and the general case follows by perturbation lemma as in the construction before Theorem 3. Alternatively, for the fist assertion we could first replace $U(L)$ by $\Omega C(L)$ where the corresponding results are again well known, cf. [4], and then pass from $\Omega C(L)$ to $U(L)$ using the strategy of [1]; while the second assertion is entirely similar to the case of Lemma 6 in [2]. Part (iii) follows from (ii) as in Section 3.3. of [2], see also [9] for the associative case. □

We can also construct a pair of functors relating L-modules to $U(L)$-modules, see appendix for the definitions. Let $Mod(L)$ be the category of L_∞-modules over L and L_∞-morphisms (= the category of DG comodules over $C(L)$ which are free as $C(L)$-comodules). By the appendix, we can also view an L-module structure on M as a twisted cochain $\tau : C(L) \to End(M)$. The corresponding DG coalgebra map $C(L) \to BEnd(M)$ admits a canonical factorization

$$C(L) \to B\Omega C(L) \to BEnd(M)$$

since we can extend τ to a DG algebra map $\Omega C(L) \to End(M)$ and then apply the bar construction. Therefore, composing with $G_L : BU(L) \to B\Omega C(L)$ we get a DG-coalgebra map $BU(L) \to BEnd(M)$, i.e. a strictly unital A_∞-module structure on M. This defines a functor

$$\mathcal{G} : Mod(L) \to Mod_\infty(U(L))$$

In the other direction, we start with a DG coalgebra morphism $BU(L) \to BEnd(M)$ and then composing with the canonical map $C(L) \to B\Omega C(L)$ and $F_L : B\Omega C(L) \to BU(L)$ we get a DG coalgebra map $C(L) \to BEnd(M)$, i.e. a twisted cochain $C(L) \to End(M)$ which gives M a structure of an L_∞-module over L. This defines a functor

$$\mathcal{F} : Mod_\infty(U(L)) \to Mod(L).$$

Observe that in both cases the underlying vector space does not change.

Theorem 5. *The above functors $\mathcal{G}$, $\mathcal{F}$ are mutually inverse equivalences.*

Proof. In one direction, suppose we start with an A_∞-module structure on M given by $BU(L) \to BEnd(M)$. Applying $\mathcal{GF}$ amounts to considering the composition

$$BU(L) \xrightarrow{G_L} B\Omega C(L) \xrightarrow{F_L} BU(L) \to BEnd(M).$$

Since the composition of the first two arrows is identity, we conclude that the identity map on M gives an isomorphism of A_∞-modules $\mathcal{GF}(M)$ and M.

In the other direction, suppose we start with a twisted cochain $C(L) \to End(M)$ and construct $B\Omega C(L) \to BEnd(M)$ as above. The L_∞-module corresponding to $\mathcal{FG}(M)$ is obtained from a DG coalgebra morphism

$$C(L) \to B\Omega C(L) \xrightarrow{F_L} BU(L) \xrightarrow{G_L} B\Omega C(L) \to BEnd(M)$$

In view of $G_L F_L = 1 - \delta_L H_L - H_L \delta_L$ it suffices to show that the composition

$$C(L) \to B\Omega C(L) \xrightarrow{\delta_L H_L + H_L \delta_L} B\Omega C(L) \to BEnd(M)$$

is zero. That in its turn would follow from the vanishing of

$$C(L) \to B\Omega C(L) \xrightarrow{H_L} B\Omega C(L).$$

But the latter holds since h_L vanishes on $s^{-1}\overline{C}(L) \subset \Omega C(L)$ by its construction, see Section 3.2 (the homotopy $\mathcal{H}_n$ vanishes on the top-dimensional cell of the permutahedron P_n). Thus, the identity on M also gives an isomorphism of L_∞-modules M and $\mathcal{FG}(M)$, which finishes the proof. $\square$

2.5. An example: toric complete intersections. The following example had originally motivated our study of L_∞-algebras. See [2] for details. Let $X \subset \mathbb{P}^\Sigma$ be a complete intersection in a toric variety defined by a fan Σ. Then X has a "homogeneous coordinate ring" $S(X) = Sym^*(V)/J$, a quotient of a polynomial ring by an ideal generated by a regular sequence of polynomials $W_1, \ldots, W_m$. For a general toric variety $S(X)$ will be graded by a finitely generated abelian group $A(X)$ and $W_1, \ldots, W_m$ will be homogeneous in this grading (but not the usual grading of $Sym^*(V)$). One can always assume that $W_1, \ldots, W_m$ have no constant or linear terms. In this setting, define the "Koszul dual" of $S(X)$ as the Yoneda algebra $E(X) = Ext^*_{S(X)}(k, k)$ with its natural A_∞-structure, cf. [7], defined in general up to A_∞-homotopy.

Introducing formal degree 2 variables $z_1, \ldots, z_m$ which span a vector space U we can define an L_∞-algebra $L = s^{-1}V^\vee \oplus U$ by viewing the formal sum $W = \sum W_i(sz_i)$ as a differential on $C(L) \simeq Sym^*_c(V^\vee) \otimes \Lambda^*_c(U)$, if we agree that W_j act by differential operators on $Sym^*(V^\vee)$.

It was shown in [2] that the Koszul dual $E(X)$ may be identified with the universal enveloping $U(L)$ (*loc. cit.* uses Koszul type-resolutions instead of $\Omega C(L)$ which still leads to the same A_∞-structure, perhaps after an adjustment of a contracting homotopy). The interpretation in terms of *Ext* groups also follows from Theorem 3 (ii).

3. A homotopy on the cobar construction

3.1. Permutahedra. Let $n \geq 1$ and $1 \leq d \leq n$ and set $P(n, d)$ to be the set of ordered partitions of $\{1, \ldots, n\}$ which have d parts. Equivalently, any such partition can be viewed as a surjective map $\psi : \{1, \ldots, n\} \to \{1, \ldots, d\}$: setting $\psi_i = \psi^{-1}(i) \subset \{1, \ldots, n\}$, $1 \leq i \leq d$ we get an ordered partition $\psi_1 \cup \ldots \cup \psi_d$. As mentioned in the introduction, $P(n, d)$ labels the set of d-dimensional faces of the $(n-1)$-dimensional polytope P_n. To consider the homology complex of P_n define an orientation of

$$\psi : \{1, \ldots, n\} \to \{1, \ldots, d\}$$

as an equivalence class of orderings on each subset ψ_i, such that two orderings are equivalent if they differ by an even permutation of $\{1, \ldots, n\}$. We choose the orientation given by the natural increasing ordering on ψ_j and denote by $[\psi_1|\ldots|\psi_d]$ the corresponding oriented cell.

Let $C_*(P_n)$ be the *homology* complex of P_n with grading inverted to ensure that differential has degree $+1$ (thus, $C_*(P_n)$ is concentrated in degrees $-n+1, \ldots, 0$).

The notation $\psi = [\psi_1|\ldots|\psi_d]$ allows to reduce most of the signs below to the Koszul sign rule if we assume that the symbol $|$ has degree $(+1)$ and each of the elements in ψ_i degree (-1).

The differential of $C_*(P)$, cf. [15], is given by:

$$\partial[\psi_1|\ldots|\psi_d] = \sum_{\substack{1\le k\le d\\ M\subsetneq\psi_k}} (-1)^{\psi,M}[\psi_1|\ldots|\psi_{k-1}|M|\psi_k\setminus M|\psi_{k+1}|\ldots|\psi_d].$$

The sign is

$$(-1)^{\psi,M} = (-1)^{m_1+\ldots+m_{k-1}+(k-1)+\#M}(-1)^{\sigma_M}$$

where $m_i = \#\psi_i$ and σ_M is the unshuffle that takes ψ_k to $[M|\psi_k\setminus M]$ (again, taken with the natural increasing ordering). The symmetric group Σ_n acts from the left on each $P(n,d)$ and on $C_*(P_n)$:

$$\sigma[\psi_1|\ldots|\psi_d] = \pm[\sigma(\psi_1)|\ldots|\sigma(\psi_d)].$$

where the sign is $(+1)$ if the ordering induced from ψ by σ is equivalent to the increasing ordering, and (-1) otherwise. In addition, $C_*(P_n)$ has an involution

$$\nu_n[\psi_1|\psi_2|\ldots|\psi_{d-1}|\psi_d] = -(-1)^{n(d-1)+\frac{(d-1)(d-2)}{2}+\sum_{i<j}m_im_j}[\psi_d|\psi_{d-1}|\ldots|\psi_2|\psi_1]$$

which commutes with the differential and the Σ_n-action. Therefore, we actually have a $\Sigma_n\times\mathbb{Z}_2$-action on $C_*(P)$.

Define a bilinear map $\Theta : V^{\otimes n}\times C_*(P_n)\to\Omega\Lambda_c^*(sV)$ by

$$\Theta(v_1\otimes\ldots\otimes v_n, [\psi_1|\ldots|\psi_d]) = (-1)^{(n-d)(\sum_i \deg v_i)}\bigotimes_{i=1}^{d} s^{-1}(s^{\otimes m_i})\big[(v_1\otimes\ldots\otimes v_n)\cdot\sigma_\psi\big]$$

where σ_ψ is the permutation $\{1,\ldots,n\}\to[\psi_1|\ldots|\psi_d]$ and each $s^{-1}(s^{\otimes m})$ is viewed as a map $V^{\otimes m}\to s^{-1}\Lambda^m(V)$, $u_1\otimes\ldots\otimes u_m\mapsto \pm s^{-1}(su_1\wedge\ldots\wedge su_m)$ with the sign determined by the Koszul rule.

Lemma 6. *The map Θ induces an isomorphism of complexes*

$$\Omega\Lambda_c^*(V)\simeq k\oplus\bigoplus_{n\ge1}\Big(V^{\otimes n}\otimes_{k[\Sigma_n]}C_*(P_n)\Big)$$

which takes ι_Ω to $1\oplus\bigoplus_{n\ge1}(1\otimes\nu_n)$.

Proof. Since Θ maps $(v_1\otimes\ldots\otimes v_n)\otimes[\psi_1|\ldots|\psi_d]$ to $\pm\bigotimes_{j=1}^d s^{-1}(\bigwedge_{i\in\psi_j} sv_i)$, it is clearly well-defined, i.e. indeed descends to a tensor product over $k[\Sigma_n]$, and surjective.

To prove injectivity, consider all faces of P_n which correspond to ordered partitions $\psi = [\psi_1|\ldots|\psi_d]$ with fixed $m_i = \#\psi_i$. Denoting $\mathbf{m} = (m_1,\ldots,m_d)$ we see that the set of such faces is a single Σ_n-orbit of

$$\psi_{\mathbf{m}} = [1,\ldots,m_1|(m_1+1),\ldots,(m_2+m_1)|\ldots|(m_1+\ldots+m_{d-1}+1),\ldots,n]$$

If orientations are taken into account, it becomes clear that the line $k\cdot\psi_{\mathbf{m}}\subset C_*(P_n)$ is isomorphic to the sign representation $\rho_{\mathbf{m}}$ of the stabilizer $\Sigma_{\mathbf{m}} = \Sigma_{m_1}\times\ldots\times\Sigma_{m_d}\subset\Sigma_n$. Therefore, the Σ_n-submodule

$$M_{\mathbf{m}} = \bigoplus_{\{\psi|\#\psi_i=m_i\ \forall i\}} k\cdot\psi\subset C_*(P_n) \qquad (5)$$

is the induced representation $\rho\uparrow_{\Sigma_{\mathbf{m}}}^{\Sigma_n}$. Therefore, $V^{\otimes n}\otimes_{k[\Sigma_n]}\rho_{\mathbf{m}}\uparrow_{\Sigma_{\mathbf{m}}}^{\Sigma_n}$ can be identified with the set of vectors in $V^{\otimes n}$ on which $\Sigma_{\mathbf{m}}$ acts by the sign representation, i.e. precisely with $\bigotimes_{i=1}^{d} s^{-1}\Lambda^{m_i}(V)\subset\Omega\Lambda_c^*(V)$, so Θ is indeed a bijection.

The assertion about involutions follows from the definitions.

To prove that Θ commutes with the differentials, we work up to signs (leaving the signs to the motivated reader). To simplify notation also assume that V has zero differential (the general case is quite similar). Then denoting by $v(\psi)=s^{-1}(\bigwedge_{i\in\psi} sv_i)\in s^{-1}\Lambda^*(V)$ for any $\psi\subset\{1,\dots,n\}$ and $v_1,\dots,v_n\in V$, we have

$$\Theta(1\otimes\partial)\Big[(v_1\otimes\ldots\otimes v_n)\otimes[\psi_1|\ldots|\psi_d]\Big]=\Theta\Big(\sum_{\substack{1\le k\le d\\ M\subsetneq\psi_k}}\pm[\psi_1|\ldots|M|\psi_k\setminus M|\ldots|\psi_d]\Big)=$$

$$=\sum_{\substack{1\le k\le d\\ M\subsetneq\psi_k}}\pm v(\psi_1)\otimes\ldots\otimes v(M)\otimes v(\psi_k\setminus M)\otimes\ldots\otimes v(\psi_d)=$$

$$=\delta_\Omega\Theta\Big[(v_1\otimes\ldots\otimes v_n)\otimes[\psi_1|\ldots|\psi_d]\Big]$$

This finishes the proof. □

3.2. Proof of Theorem 1.

Proof. Since P_n is a convex polyhedron, the complex $C_*(P_n)$ has cohomology k in degree 0, and zero everywhere else. Let $\mathcal{F}_n : C_*(P_n)\to k$, $\mathcal{G}_n : k\to C_*(P_n)$, be the natural $\Sigma_n\times\mathbb{Z}_2$-equivariant projection and embedding, respectively (where k is viewed as a trivial $\Sigma_n\times\mathbb{Z}_2$-module). Since $char\ k=0$, we can find a $\Sigma_n\times\mathbb{Z}_2$-equivariant contracting homotopy $\mathcal{H}_n : C_*(P_n)\to C_*(P_n)$. It is well known that we can also assume the side conditions:

$$\mathcal{H}_n\mathcal{G}_n=0,\quad \mathcal{F}_n\mathcal{H}_n=0,\quad \mathcal{H}_n\mathcal{H}_n=0$$

(if the first two identities are not satisfied then replace $\mathcal{H}_n$ by $\mathcal{H}'_n=(1-\mathcal{G}_n\mathcal{F}_n)\mathcal{H}_n(1-\mathcal{G}_n\mathcal{F}_n)$, then if the last identity is not satisfied, replace $\mathcal{H}'_n$ by $\mathcal{H}''_n=\mathcal{H}'_n\partial\mathcal{H}'_n$). Using the decomposition Lemma 5, set

$$h_V=0\oplus\bigoplus_{n\ge1}(1\otimes\mathcal{G}_n)$$

By $\Sigma_n\times\mathbb{Z}_2$-equivariance of $\mathcal{H}_n$, it follows that h_V is a homotopy contracting $\Omega\Lambda_c^*(V)$ to

$$k\oplus\bigoplus_{n\ge1}\Big(V^{\otimes n}\otimes_{k[\Sigma_n]}k\Big)=Sym^*(V)$$

and that h_V commutes with the anti-involution ι_Ω as well. □

3.3. Relation with semistandard tableaux. Our original approach to Theorem 1 was based on the equivalent language of semistandard tableaux. The main advantage of using permutahedra is better compatibility with the involution ι_Ω on $\Omega\Lambda_c^*(V)$. On the other hand, semi-standard tableax give an explicit decomposition of $\Omega\Lambda_c^*(V)$ into irreducible $GL(V)$-modules (e.g. when V is a finite dimensional vector space in homological degree 0). These results (perhaps known to experts in combinatorics) are not used in this paper, and the proof is left to the interested reader.

Denote by $\mathcal{S}^\lambda$ the irreducible Specht module corresponding to a partition λ on n, cf. e.g. [14], and recall that its multiplicity in $M_{\mathbf{m}} \simeq \rho_{\mathbf{m}} \uparrow_{\Sigma_{\mathbf{m}}}^{\Sigma_n}$, cf. (5), can be computed as the number of column-semistandard tableaux T with content $\mathbf{m}$, cf. Theorem 2.11.2 in *loc. cit.* Thus, Lemma 5 above will give a decomposition of $\Omega\Lambda_c^*(V)$ in terms of Schur complexes. It takes some additional effort to make it compatible with the differential. We use the same notation λ for the corresponding Young diagram. Choose a λ-tableau T, i.e. a bijective map $\{\lambda\} \to \{1,\dots,n\}$ where $\{\lambda\}$ is the set of cells in λ. Let $C_T, R_T \subset \Sigma_n$ be the column stabilizer and row stabilizer, respectively, i.e. those permutations which preserve values in the columns, resp. rows of T. Setting

$$c_T = \sum_{\sigma\in C_T} \sigma; \qquad r_T^- = \sum_{\sigma\in R_T} (-1)^\sigma \sigma; \qquad e_T = c_T r_T^-$$

we can define the Schur complex $S^T(V) = (V^{\otimes n})e_T$ for any complex of vector spaces V.

Now suppose that T is standard, i.e. the values increase in rows and columns. Set

$$J_T = \{i \mid 1 \le i \le n-1, \text{ and } T^{-1}(i) \text{ is strictly above } T^{-1}(i+1)\} \subset \{1,\dots,n\}$$

For any subset $J \subset J_T$ with p elements consider the unique weakly increasing surjective map

$$\zeta_J : \{1,\dots,n\} \to \{1,\dots,n-p\}$$

such that $J = \{i \mid \zeta_J(i) = \zeta_J(i+1)\}$. Then the composition

$$T_J : \{\lambda\} \xrightarrow{T} \{1,\dots,n\} \xrightarrow{\zeta_J} \{1,\dots,n-p\}$$

is a *column-semistandard tableaux*, i.e. the values increase weakly in the columns and strictly in the rows. It is easy to see that any column-semistandard tableau $U : \{\lambda\} \to \{1,\dots,n-p\}$ has the form T_J for unique T and $J \subset J_T$.

Theorem 7. *One has a direct sum decomposition*

$$\Omega\Lambda_c^*(V) \simeq k \oplus \bigoplus_{\lambda} \bigoplus_{\substack{T \text{ is a standard} \\ \lambda-\text{tableau}}} \big(C_T \otimes S^T(V)\big)$$

where C_T is a combinatorial complex spanned in degree $(-p)$ by T_J with $J \subset J_T, \#J = p$ and differential given by

$$\partial(T_J) = \sum_{j\in J} (-1)^{\#X(J,j)}\, T_{(J\setminus j)}; \qquad X(J,j) = \{i \mid 1 \le i \le j-1, i \notin J\}$$

To write the isomorphism explicitly, for $J \subset J_T$ let $\mathbf{m}(J) = (m(J)_1, \dots, m(J)_{n-p})$ with $m(J)_i = \zeta_J^{-1}(i)$ and $\sigma_{\mathbf{m}(J)} \in k[\Sigma_n]$ the average of all elements in the corresponding subgroup $\Sigma_{\mathbf{m}(J)} \subset \Sigma_n$. Then for $u \in S^T(V) = V^{\otimes n}e_T$ we set

$$(T_J \otimes u) \mapsto \frac{1}{m(J)_1! \dots m(J)_{n-p}!} \pi_J(u\sigma_{\mathbf{m}(J_T)}) \in \Omega\Lambda_c^*(V)$$

where π_J is induced by the projection (combined with (de)suspensions)

$$V^{\otimes n} \to s^{-1}\Lambda^{m(J)_1}(V) \otimes \dots \otimes s^{-1}\Lambda^{m(J)_{n-p}}(V)$$

The complex C_T is isomorphic to the standard Koszul complex on the vector space spanned by elements of J_T, hence it admits a homotopy

$$h_T(T_J) = \frac{1}{\# J_T} \sum_{j \in (J_T \setminus J)} (-1)^{\# X(J,j)} T_{(J \cup j)}.$$

Setting $h_V = \sum_T h_T \otimes 1$ in terms of the decomposition of Theorem 7, we get an explicit functorial homotopy as in Theorem 1. But to ensure that h_V commutes with ι_Ω we may have to replace it by $h'_V = \frac{1}{2}(h_V + \iota_\Omega h_V \iota_\Omega)$ and this has no apparent meaning in terms of semistandard tableaux.

4. Appendix: twisted cochains and L_∞, A_∞-modules

A degree +1 map $\tau : \overline{C} \to \overline{A}$ for a coagumented DG coalgebra C and an augmented DG algebra A, is called a *twisted cochain* if τ satisfies

$$\tau d_C + d_A \tau = \mu \circ (\tau \otimes \tau) \circ \Delta$$

where $\Delta : \overline{C} \to \overline{C} \otimes \overline{C}$ is the *reduced* coproduct of C and μ is the product in A. This conditions guarantees that both the canonical coalgebra morphism $C \to BA$ and the canonical algebra morphism $\Omega C \to A$, induced by τ, commute with differentials. For a general strictly unital A_∞-algebra A, the canonical coalgebra morphism $C \to BA$ induced by τ, commutes with differentials precisely when the following *generalized twisted cochain* condition holds, cf. [1]:

$$\tau d_C + d_A \tau = \sum_{i \geq 2} \mu_i \circ \tau^{\otimes i} \circ \Delta^{(i)}$$

where μ_i are the products in A and $\Delta^{(i)} : \overline{C} \to \overline{C}^{\otimes i}$ is the iteration of the reduced coproduct.

If L is an L_∞-algebra then an L_∞*-module* structure on a vector space M is defined by choosing a differential d on $C(L) \otimes M$ which makes it a DG-comodule over $C(L)$. This differential encodes maps $\Lambda^k(L) \otimes M \to M$ which satisfy a series of quadratic identities arising from $d^2 = 0$. It follows from the definitions that the same structure is also encoded by a twisted cochain $C(L) \to End(M)$. Similarly, A_∞*-modules* over an A_∞-algebra A are encoded either by comodule differentials on $BA \otimes M$ or twisted cochains $BA \to End(M)$.

If τ is a generalized twisted cochain and N is a DG comodule over C, denote by $N \otimes_\tau A$ the tensor product $N \otimes A$ with the differential

$$\delta = \delta_N \otimes 1 + 1 \otimes \delta_A + \sum_{s \geq 2} (1 \otimes m_s)(1 \otimes \otimes \tau^{\otimes (s-1)} \otimes 1)(\Delta_N^{(s)} \otimes 1)$$

where m_s is the s-th product in A and $\Delta_N^{(s)} : N \to N \otimes \overline{C}^{\otimes (s-1)}$ is the iterated reduced coaction map. The infinite sum makes sense if N is *cocomplete*, i.e. $N = \cup_i Ker(\Delta_N^{(i)})$.

On the other hand, is M is an A_∞-module over A with action maps $m_s^M : M \otimes A^{\otimes (s-1)} \to M$ then denote by $M \otimes_\tau C$ the tensor product $M \otimes C$ with the differential

$$\delta = \delta_M \otimes 1 + 1 \otimes \delta_C + \sum_{s \geq 2} (m_s^M \otimes 1)(1 \otimes \tau^{\otimes (s-1)} \otimes 1)(1 \otimes \Delta^{(s)}).$$

See Section 3 in [2] on how to define these functors on morphisms, and other details.

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