Algebraization of bundles on non-proper schemes.

Vladimir Baranovsky

March 6, 2008

Abstract

We study the algebraization problem for principal bundles with reductive structure groups on a non-proper formal scheme. When the formal scheme can be compactified by adding a closed subset of codimension at least 3, we show that any such bundle admits an algebraization. For codimension 2 we provide a necessary and sufficient condition.

1 Introduction

This work is a contribution toward an algebraic understanding of the Uhlenbeck compactification. Recall, cf. [DK] that for a complex projective surface $S$ the moduli space $M_n$ of semistable vector bundles with fixed rank, determinant and $c_2 = n$ is non-compact, but the union $Uhl_n = \coprod_{s \geq 0} M_{n-s} \times Sym^s S$ can be given a topology of a compact space (since one deals with semistable bundles for $s \gg 0$ the space $M_{n-s}$ will be empty). We will call $Uhl_n$ the Uhlenbeck moduli space although sometimes this name is reserved for the closure of $M_n$ in $Uhl_n$.

Some time ago, see e.g. [Li], [BFG], [FGK], the Uhlenbeck moduli space started to appear in algebraic geometry and higher dimensional Langlands Program. For instance, it is a convenient tool for the study of higher versions of Hecke correspondences which modify a vector bundle on $S$ (more generally, a principal bundle) along a divisor, obtaining a new bundle. For several reasons, we would like to have a definition of $Uhl_n$ as a “functor”, i.e. we want to be able to describe in geometric terms the set of maps $F(T)$ (actually, a category of maps) from any test scheme $T = \text{Spec}(A)$ to $Uhl_n$. Firstly, that would allow to define $Uhl_n$ over any field $k$ and not to require stability. Secondly, in the study of the cohomology of $Uhl_n$ and the action of Hecke correspondences on it, one needs to deal with the phenomenon of unexpected dimension of $Uhl_n$. A possible approach involves defining a “derived moduli space” $DUhl_n$ in the sense of [Lu] which would amount to considering more general “spaces” $T$. Thus, defining $Uhl_n$ as a functor is a necessary preliminary step to constructing $DUhl_n$.

Very roughly, it is expected that a map $T \to Uhl_n$ should be described by a vector bundle $F$ on an open subset $U \subset T \times S$ such that its complement $Z$ is finite over $T$ plus an agreement condition between $\xi$ and $F$. Such a definition gives a “reasonable space” $Uhl_n$ if it satisfies a criterion due to Artin, cf. [Ar], or its “derived” generalization proved in [Lu]. The most difficult part of Artin’s criterion is the effectiveness condition: if $A$ is a complete noetherian local $k$-algebra with maximal ideal $m$ and $A_p = A/m^{p+1}$ one needs to show that $F(\text{Spec}(A)) = \lim_{\leftarrow} F(\text{Spec}(A_p))$. Ignoring the family of zero cycles $\xi$ (as will be done in this paper), if $X = \text{Spec}(A) \times_k S$ and $\hat{X}$ is its formal completion along the fiber over the closed point of $\text{Spec}(A)$, we are trying to find whether a bundle $\mathcal{F}$ on an open subset $\hat{U} \subset \hat{X}$ comes from a bundle $\mathcal{F}$ on an open subset $U \subset X$. Such $\mathcal{F}$ is called an algebraization of $\mathcal{F}$. 
In this paper we prove that, when $S$ has arbitrary dimension and $\hat{U}$ has complement of codimension $\geq 3$, algebraization always exists (for vector bundles and principal bundles over reductive groups). If $\hat{U}$ has complement of codimension $\geq 2$ then algebraization exists only under an additional condition (which, in the Uhlenbeck functor case, guaranteed by the presence of $\xi$).

Earlier similar questions were studied for coherent sheaves on proper schemes by Grothendieck, see [EGAIII], and in the case of Lefschetz type theorems by Grothendieck and Raynaud in [SGA2], and [R]. Although these results do not apply to our case directly, our proof is based on the tools developed in [EGAIII], [SGA2].

In Section 2 we fix the notation, give examples illustrating some issues to be encountered, and prove algebraization results for vector bundles, summarized in Corollary 8. In Section 3 we formulate an algebraization criterion for principal bundles over reductive groups, see Theorem 9. Finally, Section 4 provides a categorical restatement of our results, see Theorem 13.

**Acknowledgements.** The author thanks V. Ginzburg who first formulated the problem of defining the Uhlenbeck functor and whose unpublished preprint on it (written jointly with the present author) served as a principal motivation for this work. Many thanks are also due to V. Drinfeld who conjectured the statement of Theorem 6(i), brought the author’s attention to the references [SGA2], [Ha1], and also suggested Example 3 in Section 2.2 below.

This work was supported by the Sloan Research Fellowship.

## 2 Algebraization for vector bundles.

### 2.1 Setup

We refer the reader to Expose III in [SGA2] regarding basic properties of depth and its relation to local cohomology. Let $S$ be an irreducible noetherian scheme of finite type over a field $k$. We will assume that $S$ is proper and satisfies Serre’s $S_2$ condition: for any $s \in S$, depth$_s \mathcal{O}_S \geq \min(\dim \mathcal{O}_S, 2)$.

Let $V \subset S$ be an open subset with closed complement of codimension $\geq 2$ in $S$ and $A$ a complete noetherian local $k$-algebra with residue field $K = A/\mathfrak{m}$ and associated graded $K$-algebra $gr(A) = \oplus_{p \geq 0} gr_p(A) = \oplus_{p \geq 0} \mathfrak{m}^p/\mathfrak{m}^{p+1}$. Define $X = S \times_k \text{Spec}(A)$ and

$$X_p = S \times_k \text{Spec}(A/\mathfrak{m}^{p+1}); \quad U_p = V \times_k \text{Spec}(A/\mathfrak{m}^{p+1}); \quad p \geq 0$$

Let $i_p : U_p \to X_p$ be the natural open embeddings. The completion $\hat{X}$ of $X$ along $X_0$ may be viewed at the limit of $\{X_p\}_{p \geq 0}$, cf. Section 10.6 in [EGAII]. The limit of $i_p$ gives an open formal subscheme $\hat{i} : \hat{U} \to \hat{X}$. The ideal sheaf of $X_0$ in $X$ will be denoted by by $\mathcal{J}_X$ and the closed subset $X_0 \setminus U_0$ by $Z_0$. Finally, $f : X \to \text{Spec}(A)$ is the natural proper projection and, for any $s \in \text{Spec}(A)$, $X_s$ stands for the fiber $f^{-1}(s)$.

Observe that $X$ may no longer satisfy the $S_2$ condition (since we made no depth assumptions on $A$). However, for $f(x) = s$ one can lift a regular sequence from $\mathcal{O}_{X_s,x}$ to $\mathcal{O}_{X,x}$ which gives

**Lemma 1** For any $x \in X$ with $f(x) = s$, depth $\mathcal{O}_{X,x} \geq \min(\dim \mathcal{O}_{X_s,x}, 2)$.

Consider a vector bundle $\mathcal{F}$ on $\hat{U}$, i.e. a sequence of vector bundles $F_p$ on $U_p$ with isomorphisms

$$F_p|_{U_{p-1}} \simeq F_{p-1}; \quad p \geq 1.$$  

(1)
**Definition.** We will say that a vector bundle $F$ on $\hat{U}$ admits an algebraization $(U, F)$ if there exists an open subset $U \subset X$ with $U \cap X_0 = U_0$ and a vector bundle $F$ on $U$ such that $F$ is isomorphic to the completion of $F$, i.e., for $\mathcal{J}_U = \mathcal{J}_X|_U$ there exist isomorphisms $F_p \simeq F/\mathcal{J}_U^{p+1}F$ compatible with (1). In Section 3 we apply similar terminology to principal bundles.

Let $Z$ be the closed subset $X \setminus U$ and $i : U \hookrightarrow X$ the open embedding.

**Lemma 2** Assume that $\text{codim}_{X_0}Z_0 \geq 2$ and an open subset $U \subset X$ satisfies $U \cap X_0 = U_0$. For any $s \in \text{Spec}(\Lambda)$, define $Z_s = Z \cap X_s$. Then $\text{codim}_{X_s}Z_s \geq 2$ for all $s \in \text{Spec} \Lambda$ and $\text{codim}_XZ \geq 2$.

**Proof.** Since $f$ is proper, the image $f(Z_s)$ contains the unique closed point $s_0 \in \text{Spec}(\Lambda)$. Therefore $Z_s \cap X_0 \subset Z_0$ is not empty. By semicontinuity of dimensions in the fibers we have $\text{codim}_{X_s}Z_s \geq \text{codim}_{X_0}(Z \cap X_0) \geq \text{codim}_{X_0}Z_0 = 2$. The second assertion of the lemma follows from the first. $\square$

In our discussion, we repeatedly use the following results

**Proposition 3** In the notation introduced above

(i). Completion along $X_0$ induces an equivalence between the category of coherent sheaves on $X$ and the category of coherent sheaves on the formal scheme $\hat{X}$.

(ii). For any locally free sheaf $F$ (resp. $F_0$) on $U$ (resp. $U_0$) its direct image $i_*F$ (resp. $(i_0)_*F_0$) is coherent. If $\text{codim}_{X_0}Z_0 \geq 3$ then $R^1(i_0)_*F_0$ is also coherent.

(iii). Let $E$ be a coherent sheaf on $X$ and $\psi : E \to i_*i^*E$ the canonical morphism. Then $\psi$ is an isomorphism if and only if $\text{depth}_xE \geq 2$ for any point $x \in Z = X \setminus U$.

**Proof.** Part (i) follows from Corollary 5.1.6 in [EGAIII]. To check the coherence of $i_*F$, by Corollary VIII.2.3 in [SGA2] it suffices to check that $\text{depth}_xF \geq 1$ for any point $x \in U$ such that $\{x\} \cap Z$ has codimension 1 in $\{x\}$. But Lemma 1 and local freeness of $F$ imply that any $x$ with $\text{depth}_xF = 0$ must be generic in its fiber, and the Lemma 2 implies that $\{x\} \cap Z$ would in fact have codimension 2 in $\{x\}$. The same proof applies to $(i_0)_*F_0$. If $\text{codim}_{X_0}Z_0 \geq 3$ then the above argument can also by applied to $R^1(i_0)_*F_0$ once we show that $\text{depth}_xF \geq 2$ for any $x \in U_0$ such that $\{x\} \cap Z_0$ has codimension 1 in $\{x\}$. But by $S_2$ condition $\text{depth}_xF \geq 1$ can only hold for points $x$ of codimension $\leq 1$ in $U_0$, which would imply that $\{x\} \cap Z_0$ has codimension $\geq 2$ in $\{x\}$. This proves (ii). Part (iii) is a particular case of Corollary II.3.5 in loc.cit. $\square$

### 2.2 Examples.

The first example with $\text{codim}_{X_0}Z_0 = 3$ and $K = k$ shows that one may not be able to take $U = U_0 \times_k \text{Spec}(\Lambda)$.

**Example 1.** Take $S = X_0 = \mathbb{P}^3$ with homogeneous coordinates $[x : y : z : w]$ and set $V = U_0 = S \setminus [0 : 0 : 0 : 1]$, $A = k[[t]]$ (formal power series in $t$). Define vector bundles $F_p$ as kernels of

$$\varphi_p : \mathcal{O}_{U_p}^{\mathbb{P}^3} \to \mathcal{O}(1)_{U_p} : (s_1 + s_2 + s_3) \mapsto s_1x + s_2y + s_3(z-tw).$$

Observe that $\varphi_p$ is surjective since $t$ is nilpotent on $U_p$ and $[0 : 0 : 1] \notin U_p$.

**Lemma 4** The bundle $F$ admits no algebraization $(U, F)$ with $U = U_0 \times_k \text{Spec}(\Lambda)$.
Proof. Set $F$ to be the kernel of morphism $\varphi : \mathcal{O}^{\oplus 3} \to \mathcal{O}(1)$ of vector bundles on $U$, given by the same formula as for $\varphi_p$. By definition, $\varphi$ is not surjective only at $P = [0 : 0 : t : 1] \in U$ which projects to the generic point $\xi = \text{Spec}(k[t^{-1}, t]) \in \text{Spec}(A)$. The specialization at $t = 0$ is not in $U_0$, hence $P$ is closed in $U$ and $U \setminus P$ is an open subset containing $U_0$. Since on $U \setminus P$ we have the short exact sequence of locally free sheaves
\[ 0 \to F \to \mathcal{O}^{\oplus 3} \to \mathcal{O}(1) \to 0, \]
the restriction of $F$ to each $U_p$ is given by $F_p$, i.e. $F$ is indeed the completion of $F$. On the other hand, $F$ is not locally free at $P$: from $0 \to F \to \mathcal{O}_U^{\oplus 3} \to \mathcal{O}_U \to k_p \to 0$ we immediately get $\mathcal{E}xt^1(F, \mathcal{O}_U) \cong \mathcal{E}xt^3(k_p, \mathcal{O}) \cong k_p$ since the middle two terms are projective.

Suppose that $E$ is a locally free sheaf on $U$ with completion isomorphic to $\mathcal{F}$. We will see later in Proposition 7(ii) that in such situation we must have: $i_*E \cong \widehat{i_*F} \cong \widehat{i_*E}$ hence by Proposition 3(i), $i_*F \cong i_*E$ which contradicts $\mathcal{E}xt^1(F, \mathcal{O}_U) \neq 0$. \(\square\)

The second example illustrates that for $\text{codim}_{X_0} Z_0 = 2$, a pair $(U, F)$ may not exist at all.

Example 2. Consider $A = k[[t]]$ and $S = X_0 = \mathbb{P}^2$ with homogeneous coordinates $(x : y : z)$. Let $V = U_0 = X_0 \setminus P$ where $P = (0 : 0 : 1)$ and define a rank 2 bundle $F_p$ on $U_p = U_0 \times_k \text{Spec}(k[t]/t^{p+1})$ as follows. The affine open subsets $U_p^{(x)}$, $U_p^{(y)}$ given by non-vanishing of $x$, resp. $y$, form a covering of $U_p$ and we can glue trivial rank 2 bundles on these open sets, using the transition function
\[
\left( \begin{array}{c}
1 \\
0 \\
1 \sum_{m=0}^{p} \left( \frac{t^2}{xy} \right)^m \\
1
\end{array} \right)
\]
on $U_p^{(x)} \cap U_p^{(y)}$. Clearly $F_p|_{U_{p-1}} \cong F_{p-1}$ in a natural way, and we obtain a vector bundle $\mathcal{F}$ on $\widehat{U}$.

Lemma 5 There exists no vector bundle $F$ on $U = X \setminus Z$ with $\widehat{F} \cong \mathcal{F}|_U$ for any closed subset $Z \subset X$ such that $Z_0 \subset (Z \setminus X_0)$ and $\text{codim}_{X_0}(Z \setminus X_0) \geq 2$.

Proof. Suppose otherwise and take the direct image of $F$ with respect to the open embedding $i : U \to X$. By Proposition 3, $i_*F$ is coherent and has depth $\geq 2$ at all codimension 2 points of $X$. Since modules of depth 2 over two-dimensional regular local rings are free by Auslander-Buchsbaum formula, $i_*F$ will be locally free in codimension two. Therefore shrinking $Z$ we can assume that $Z$ has codimension 3 in $X$ which in our case means that $Z$ is a finite set of points in $X_0$. Then the short exact sequence of sheaves on $X \setminus Z$
\[ 0 \to F' \to F \to F_p \to 0, \]
(we identify $F_p$ with its direct image on $X \setminus Z$ abusing notation), gives a long exact sequence on $X$:
\[ 0 \to i_*F' \overset{i_*F_p}{\longrightarrow} i_*F \to i_*F_p \to R^1i_*F \overset{i_*F_p}{\longrightarrow} R^1i_*F \]
where $R^1i_*F$ is coherent for the same reason as in Proposition 3(ii). Since $R^1i_*F$ is supported at the finite set $Z$ of closed points, it has finite length at each of them and the last arrow is zero for $p \geq p_0$. For such $p$ we can write $i_*F' \to i_*F_p \to R^1i_*F \to 0$ which gives
\[ i_*F \otimes_{\mathcal{O}_X} k(P) \to i_*F_p \otimes_{\mathcal{O}_X} k(P) \to R^1i_*F \otimes_{\mathcal{O}_X} k(P) \to 0 \]

4
To prove the lemma it suffices to show that \( \dim_k i^*_p F_p \otimes_{\mathcal{O}_X} k(P) \) is unbounded as \( p \to \infty \).

To that end, replace \( X_0 \) with the affine open subset \( \tilde{X}_0 \cong \mathbb{A}^2 \) given by non-vanishing of \( z \), with affine coordinates \( u = \frac{x}{z}, v = \frac{y}{z} \). Set \( W_0 = U_0 \cap \tilde{X}_0 \) and similarly for \( \tilde{X}_p, W_p, W_p^{(x)} \) and \( W_p^{(y)} \). Then \( i^*_p F_p|_{\tilde{X}_p} \) is the sheaf associated to \( H^0(W_p, F_p|_{W_p}) \) viewed as a module over \( A(\tilde{X}_p) = k[u, v, t]/t^{p+1} \).

By its definition, \( F_p \) is an extension of \( \mathcal{O}_{U_p} \) with \( \mathcal{O}_{U_p} \) which leads to long exact sequence

\[
0 \to H^0(W_p, \mathcal{O}_{W_p}) \to H^0(W_p, F_p|_{W_p}) \to H^0(W_p, \mathcal{O}_{W_p}) \to H^1(W_p, \mathcal{O}_{W_p}).
\]

where the last arrow sends the constant function 1 to the class of the extension. Let \( M_p \) be the kernel of the last arrow. It suffices to show that \( \dim_k (M_p/\langle u, v, t \rangle M_p) \) is unbounded. Computing \( M_p \) via the affine covering \( \{W_p^{(x)}, W_p^{(y)}\} \) we identify it with the kernel of

\[
k[u, v, t]/t^{p+1} \to \frac{1}{uv} k[u^{-1}, v^{-1}, t]/t^{p+1}
\]

where \( \psi_p \) is multiplication by \( \sum_{i=0}^{p} \left( \frac{1}{uv} \right)^i \) (i.e. the upper right corner of the transition matrix in the definition of \( F_p \)), and \( \pi_p \) is the natural projection

\[
k[u, u^{-1}, v, v^{-1}, t]/t^{p+1} \to \frac{1}{uv} k[u^{-1}, v^{-1}, t]/t^{p+1}
\]

It follows that \( M_p \) is generated by the monomials \( t^p, t^i u^{p-i}, t^i v^{p-i} \) for \( i = 0, \ldots, p-1 \), thus

\[
\dim_k (M_p/\langle u, v, t \rangle M_p) = 2p + 1 \to \infty \quad \text{as} \quad p \to \infty. \quad \Box
\]

**Example 3.** (Suggested to the author by V. Drinfeld.) The bundle in the previous example has trivial determinant, but if we don’t insist on that, there is a rank one example: glue two trivial line bundles on \( U_0^{(x)}, U_0^{(y)} \) using the transition function \( \sum_{m=0}^{p} \left( \frac{1}{uv} \right)^m \). The resulting line bundle admits no algebraization since again \( \dim_k (i_p)_* F_p \otimes_{\mathcal{O}_X} k(P) \) is not bounded as \( p \to \infty \).

### 2.3 Algebraization of vector bundles.

**Theorem 6** In the notation of section 2.1,

(i) If \( \text{codim}_{X_0} Z_0 \geq 3 \) then \( F \) admits an algebraization.

(ii) If \( \text{codim}_{X_0} Z_0 \geq 2 \) and the cokernel of the natural morphism \( (i_p)_* F_p|_{X_{p-1}} \to (i_{p-1})_* F_{p-1} \) is supported in codimension \( \geq 3 \) for all \( p \) large enough, then \( F \) admits an algebraization.

(iii) In either of the two situations (codimension \( \geq 3 \) or codimension \( \geq 2 \) with the additional support assumption) the projective system \( \{(i_p)_* F_p\}_{p \geq 0} \) satisfies the Mittag-Leffler condition, the direct image \( \hat{i}_* F \) is coherent and isomorphic to \( \varinjlim (i_p)_* F_p \).

**Proof.** We split the proof of (i) and (ii) in a number of steps. Part (iii) will follow from Step 2.

**Step 1.**

Suppose that \( \hat{i}_* F \) is coherent. By Proposition 3(i) there exists a unique coherent sheaf \( E \) on \( X \) such that \( \hat{E} \cong \hat{i}_* F \). The subset \( U \subset X \) of points where \( E \) is locally free is open and contains \( U_0 \) (e.g. by Nakayama’s Lemma). Shrinking \( U \) if necessary we can achieve \( U \cap X_0 = U_0 \). Now set \( F = E|_U \).
Step 2.
Therefore (i) and (ii) are reduced to showing that, under the conditions stated, \( \hat{i}_* \mathcal{F} \) is coherent. To that end we modify the argument of 0.13.7.7 in [EGAIII] which will also prove (iii). First, as in 0.13.7.2 of loc. cit., we choose injective resolutions \( F_k \to L_k^\bullet \) such that \( L_k^\bullet / J_k^1 L_{k+1}^\bullet \simeq L_k^\bullet \) and the natural filtrations by \( J_k^1 \) agree with those on \( F_k \). Each \( \hat{i}_*(L_k^\bullet) \) is a filtered complex and has a spectral sequence with \( E_1 \) term given by

\[
E_1^{pq} = R^{p+q} \hat{i}_*(J_k^{p} F_k / J_k^{p+1} F_k)
\]

As in 0.13.7.3 of loc.cit. we pass to the limit as \( k \to \infty \) and get a spectral sequence with

\[
E_1^{pq} = R^{p+q} \hat{i}_*(F_p / F_{p+1}) \simeq R^{p+q} \hat{i}_*(F_0) \otimes_K (m^p / m^{p+1}) = R^{p+q} \hat{i}_*(F_0) \otimes_K gr_p(A)
\]

We are interested in the components

\[
E_1^0 = \bigoplus_{p+q=0} E_1^{pq} = \hat{i}_*(F_0) \otimes_K gr(A); \quad E_1^1 = \bigoplus_{p+q=1} E_1^{pq} = R^1 \hat{i}_*(F_0) \otimes_K gr(A).
\]

We would like to show that the spectral sequence converges at the \( E^0 = \oplus E^{p,-p} \) terms. Note that each \( E_1^{k+1} = \oplus E_{k+1}^{p,-p} \) is a quotient of a subsheaf in \( E_1 \) while each \( E_1^{k+1} \) is a subsheaf \( E_0^k \) (since \( E^{p,-p} \) terms are zero). Taking successive preimages of the boundaries in \( E_1, E_2, \ldots, E_1 \) we get a sequence of boundary subsheaves \( B_1 \subset B_2 \subset B_3 \subset \ldots \subset E_1 \), and taking preimages of cycles in \( E_1 \) we get a sequence of cycle subsheaves \( E_1^0 \supset Z_1 \supset Z_2 \supset Z_3 \supset \ldots \). By 0.13.7.6 in loc.cit. these are actually \( \mathcal{O}_{X_0} \otimes_K gr(A) \)-submodules.

Suppose that sequence of cycles stabilizes, i.e. for some \( r_0 \) one has \( Z_r = Z_{r_0} \) whenever \( r \geq r_0 \), then by 0.13.7.4 in [EGAIII], the projective system \( \{ \hat{i}_*(F_k) \}_{k \geq 0} \) satisfies the Mittag-Leffler condition and the associated graded of \( \hat{i}_*(\mathcal{F}) \) is precisely \( Z_{r_0} \subset \hat{i}_*(F_0) \otimes_K gr(A) \). But \( \hat{i}_*(F_0) \) is a coherent by Proposition 3(ii), hence the subsheaf \( gr(\hat{i}_*\mathcal{F}) \subset \hat{i}_*(F_0) \otimes_K gr(A) \) is a coherent \( \mathcal{O}_{X_0} \otimes_K gr(A) \)-module, by the noetherian property of \( X_0 \) and \( A \). By loc.cit. 13.7.7.2, \( \hat{i}_* \mathcal{F} \) is itself coherent on \( \hat{X} \). Also, \( \hat{i}_* \mathcal{F} \simeq \varprojlim (i_p)_* \mathcal{F}_p \) by 0.13.7.5.1 in loc.cit..

Step 3.
Now the assertion of the theorem is reduced to showing that the sequence of cycles \( Z_1 \supset Z_2 \supset \ldots \) stabilizes. By definition of \( Z_r \) this is equivalent to saying that the higher differentials of the spectral sequence \( d_r : E_1^0 \to E_1^1 \) become zero for \( r \geq r_0 \). That in turn is equivalent to saying that the sequence of boundaries \( B_1 \subset B_2 \subset B_3 \subset \ldots \), also stabilizes.

If \( \text{codim}_{X_0} Z_0 \geq 3 \) by Proposition 3(ii), \( R^1(i_0)_* F_0 \) is also coherent and \( \{ B_r \}_{r \geq 1} \) stabilizes by the noetherian property of \( R^1(i_0)_* F_0 \otimes_K gr(A) \), which proves (i). If \( \text{codim}_{X_0} Z_0 \geq 2 \) we need to find a coherent subsheaf of \( R^1(i_0)_* F_0 \otimes_K gr(A) \) containing \( B_r \) for all \( r \geq 1 \).

Step 4.
At this point we reduced (ii) to showing that, under the assumptions stated, there exists a coherent subsheaf \( G \subset R^1(i_0)_* F_0 \) such that \( B_r \subset G \otimes_K gr(A) \) for all \( r \). By 0.11.2.2 in [EGAIII] for \( r \geq p \) the term \( B_r^{p,-p} \) is the image of the connecting homomorphism

\[
\hat{i}_* F_p \to \hat{i}_* F_{p-1} \overset{\rho_p}{\to} R^1 \hat{i}_* F_0 \otimes_K (m^p / m^{p+1})
\]

in the long exact sequence obtained by applying \( \hat{R} \) to the short exact sequence on \( \hat{U} \):

\[
0 \to F_0 \otimes_K (m^p / m^{p+1}) \to F_p \to F_{p-1} \to 0.
\]
Observe that by our assumptions each $\text{Im} (\rho_p)$ is coherent, and supported in codimension $\geq 3$ for $p \gg 0$. Therefore we are done once we show that the subsheaf of $R^1 (\rho_p)_x F_0$ formed by all sections with support in codimension $\geq 3$, is coherent whenever $\text{codim} X_0 Z_0 \geq 2$ and $F_0$ is locally free on $U_0$.

**Step 5.**

Set $Q = (i_0)_x F_0$, a coherent sheaf on $X_0$ by Step 2. By the standard exact sequence we have $\mathcal{H}^2_{Z_0} Q = R^1 (i_0)_x Q | U_0 = R^1 (i_0)_x F_0$, so it suffices to show that $\mathcal{H}^0_{\geq 3} \mathcal{H}^2_{Z_0} Q$ is coherent where $\mathcal{H}^0_{\geq 3}$ is the functor of sections supported in codimension $\geq 3$. Let $\mathcal{H}^0_{\geq 3}$ be the higher derived functors.

First, the standard spectral sequence for the composition of functors $R \mathcal{H}^0_{\geq 3}, R \mathcal{H}^0_{Z_0}$ has $E^{p,q}_2 = \mathcal{H}^p_{\geq 3} \mathcal{H}^q_{Z_0} Q$. But $\mathcal{H}^i_{Z_0} Q = 0$ for $i = 0, 1$ by Proposition 3(iii), so

$$\mathcal{H}^i_{\geq 3} \mathcal{H}^i_{Z_0} Q \simeq \mathcal{H}^i Q$$

where the local cohomology $\mathcal{H}^i_q$ has family of supports $\Phi = \{\text{all codim} \geq 3 \text{ closed subsets in } Z_0\}$.

**Step 6.**

To show that $\mathcal{H}^i_q Q$ is coherent note that by [Ha2] the scheme $X_0$ has a dualizing complex $\omega$ of the form

$$0 \rightarrow \mathcal{K}^0 \rightarrow \ldots \rightarrow \mathcal{K}^{\dim X_0} X_0 \rightarrow 0$$

with $\mathcal{K}^i = \bigoplus_{\dim \mathcal{O}_{X_0, x} = i} J(x)$ and each $J(x)$ is the direct image of the injective envelope of the residue field $k(x)$ with respect to the natural morphism $i^x : \text{Spec} (\mathcal{O}_{X_0, x}) \rightarrow X_0$. By definition of a dualizing complex, the double complex $K^{p,q} = \text{Hom} (\text{Hom} (Q, \mathcal{K}^q), \mathcal{K}^p)$ has total complex quasi-isomorphic to $Q$. Moreover, by Proposition IV.2.1 and the remark on page 123 in [Ha2], the total complex is a flasque resolution of $Q$ and hence can be used to compute $\mathcal{H}^i_q (Q)$. This leads to a spectral sequence:

$$E^{p,q}_2 = \mathcal{E}xt^p_{\Phi} (\mathcal{E}xt^{-q} (Q, \omega), \omega) \Rightarrow \mathcal{H}^{p+q} \Phi (Q)$$

where $\mathcal{E}xt^p_{\Phi} = R^p (\Gamma_\Phi \circ \text{Hom})$ and the $\mathcal{E}xt$ sheaves are understood in the sense of hypercohomology.

Only finitely many terms $E^{p,q}_2$ with $p + q = 2$ will be non-trivial: since $\mathcal{K}^q$ are injective, the non-vanishing implies $0 \leq (-q) \leq \dim \mathcal{K} X_0$. Thus it suffices to show that $E^{2,2-p}_2 = \mathcal{E}xt^p_{\Phi} (\mathcal{E}xt^{2-p} (Q, \omega), \omega)$ is coherent for $p \geq 2$.

An important observation which we use below is that $\mathcal{K}^p$ has no sections supported in codimension $\geq p + 1$.

**Step 7.**

First observe that $\mathcal{E}xt^2_{\Phi} (G, \omega) = 0$ for any quasi-coherent sheaf $G$ since $\mathcal{K}^2$ has no sections supported in codimension $\geq 3$ and hence no sections with support in $\Phi$. Hence we can assume that $p \geq 3$.

We first claim that $\dim \mathcal{O}_{X_0, x} \text{Supp} (\mathcal{E}xt^{2-p} (Q, \omega)) = d \geq p \geq 3$. In fact, let $x \in \text{Supp} (\mathcal{E}xt^{2-p} (Q, \omega))$ be a point with $\dim \mathcal{O}_{X_0, x} = d$. By local duality, cf. V.6 in [Ha2], the nonvanishing of the stalk $\mathcal{E}xt^{2-p} (Q, \omega)_x$ is equivalent to the non-vanishing of local cohomology $H^{d+2-p} (x)$ which implies $d + 2 - p \geq 0$ and $d \geq p - 2 \geq 1$. If $d = 1$ then $p = 3$ and also $x \notin Z_0$ hence the stalk $Q_x$ is free. Thus $H^0 (\mathcal{O}) \neq 0$, contradicting the $S_2$ assumption. If $d \geq 2$ then applying the $S_2$ condition when $x \notin Z_0$ and Proposition 3(iii) when $x \in Z_0$ we actually have $d + 2 - p \geq 2$ so $d \geq p$ as required.

By primary decomposition, the coherent sheaf $\mathcal{E}xt^{2-p} (Q, \omega)$ admits a finite filtration by coherent subsheaves such that all successive quotients have irreducible supports of codimension $\geq p$. By the standard long exact sequence for $\mathcal{E}xt^p_{\Phi} (\cdot, \omega)$ is suffices to show that $\mathcal{E}xt^p_{\Phi} (G, \omega)$ is coherent whenever $p \geq 3$ and $G$ is a coherent sheaf with irreducible support $Y$ of codimension $\geq p$.
If \( Y \not\subseteq Z_0 \) for any \( W \) in the family \( \Phi \), the intersection \( Y \cap W \) is not equal to \( Y \) and therefore has codimension \( \geq p + 1 \). But then \( \mathcal{E}xt^p_\mathfrak{q}(G, \omega) = 0 \) because any section \( p \) of \( \mathcal{H}om(G, \mathcal{K}_p) \) representing a class in \( \mathcal{E}xt^p_\mathfrak{q}(G, \omega) \) has zero values since \( \mathcal{K}_p \) has no sections supported in codimension \( \geq p + 1 \). If \( Y \subseteq Z_0 \) then \( Y \) is an element of \( \Phi \) and \( \mathcal{E}xt^p_\mathfrak{q}(G, \omega) \cong \mathcal{E}xt^p(G, \omega) \) since all sections of \( \mathcal{H}om(G, \mathcal{K}_p) \) have support in \( \Phi \). But \( \mathcal{E}xt^p(G, \omega) \) is coherent which finishes the proof. \( \square \)

The converse to Theorem 6 can be formulated as follows.

**Proposition 7** In the setting of Section 2.1, assume that \( \mathcal{F} \) admits an algebraization \((U, F)\) and view each \( F_p \) as a sheaf on \( U \). Then

(i). The cokernel of \( i_* F_p \to i_* F_{p-1} \) is supported in codimension \( \geq 3 \) for \( p \gg 0 \). (ii). The isomorphism \( \hat{\mathcal{F}} \cong \mathcal{F} \) extends to direct images: \( i_* \mathcal{F} \cong \hat{i}_* \mathcal{F} \). In particular, \( \hat{i}_* \mathcal{F} \) is coherent.

Proof. To prove (i) observe that the cokernel of \( i_* F_p \to i_* F_{p-1} \) is annihilated by \( \mathcal{J}_X \), being a subsheaf of \( R^1 i_* F_0 \otimes_K \text{gr}_p(A) \), and is therefore isomorphic to the cokernel of \( i_* F_p|_{X_0} \to i_* F_{p-1}|_{X_0} \).

We will first show that the natural map \( i_* F_p|_{X_0} \to i_* F_0 \) is an embedding of sheaves for all \( p \). Considering the exact sequence

\[
0 \to \mathcal{J}_X(i_* F_p) \to i_* F_p \to i_* F_p|_{X_0} \to 0
\]

and its map to the first terms of the sequence

\[
0 \to i_*(\mathcal{J}_U F_p) \to i_* F_p \to i_* F_0 \to R^1 i_*(\mathcal{J}_U F_p) \to \ldots
\]

we see that \( i_* F_p|_{X_0} \to i_* F_0 \) is an embedding precisely when the natural map \( \mathcal{J}_X(i_* F_p) \to i_*(\mathcal{J}_U F_p) \) is an isomorphism. Observe that \( i_* \mathcal{O}_U = \mathcal{O}_X \) hence \( i_* \mathcal{J}_U \) is a sheaf of ideals in \( \mathcal{O}_X \).

Using Lemma 1 and the Cohen-Macaulay assumption on \( X_0 \) we see that \( \mathcal{H}^t_\mathfrak{p} \mathcal{O}_X = \mathcal{H}^t_{\mathfrak{p} X_0} \mathcal{O}_{X_0} = 0 \) for \( t = 0, 1 \). By the short exact sequence \( 0 \to \mathcal{J}_X \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0 \) we derive \( \mathcal{H}^t_{\mathfrak{p}} \mathcal{J}_X = 0 \) for \( t = 0, 1 \) and hence \( \mathcal{J}_X = i_* \mathcal{J}_U \) by Proposition 3 (iii). Then

\[
i_* (\mathcal{J}_U F_p) = (i_* \mathcal{J}_U)(i_* F_p) = \mathcal{J}_X i_* F_p
\]

as required. Similarly, \( i_* F|_{X_0} \to i_* F_0 \) is an embedding. So for any \( p \geq 1 \) we have embeddings

\[
i_* F|_{X_0} \to i_* F_p|_{X_0} \to i_* F_{p-1}|_{X_0} \to i_* F_0
\]

Consequently, the coherent sheaf \( \mathcal{K} = \text{Coker}(i_* F)|_{X_0} \to i_* F_0 \) has a decreasing filtration by images of \( i_* F_p|_{X_0} \) and each \( \text{Coker}(i_* F_p|_{X_0} \to i_* F_{p-1}|_{X_0}) \) is its successive quotient. But \( \mathcal{K} \) is a coherent sheaf with \( \text{Supp}(\mathcal{K}) \subseteq Z_0 \) and \( Z_0 \) has at most finitely many points of codimension 2. Since for each point \( x \in X_0 \) of codimension 2, the localization \( \mathcal{K}_x \) is a module of finite length, only finitely many successive quotients of the filtration of \( \mathcal{K} \) can be non-trivial in codimension 2, which proves (i).

To prove (ii) first observe that \( \hat{i}_* \mathcal{F} \) and \( E = i_* \mathcal{F} \) are coherent by Theorem 6 (iii) and Proposition 3 (ii), respectively. By Proposition 3 (i) we can find a sheaf \( E' \) such that \( \hat{E}' \cong \hat{i}_* \mathcal{F} \). The isomorphism \( \hat{E}|_{U_0} \cong \mathcal{F} \) extends uniquely to a morphism of sheaves \( \hat{\phi} : \hat{E} \to \hat{i}_* \mathcal{F} = \hat{E}' \). By Proposition 3 (i), \( \hat{\phi} \) is the completion of a unique morphism \( \phi : E \to E' \) which by Corollary 10.8.14 in [EGAI] should be an isomorphism on an open subset \( W \) containing \( U_0 \). Shrinking \( W \) if necessary we can assume \( W \subseteq U \). By Lemma 2, each point \( x \in U \setminus W \) has codimension \( \geq 2 \) in its fiber, hence
depth₁E ≥ 2 by Lemma 1. For x ∈ X \ U we still have depth₂E ≥ 2 by Proposition 3(iii). Applying the same result to j : W ⊆ X instead of U we see that E = j∗j∗E. By adjunction of j∗ and j the isomorphism \( \phi|W \)⁻¹ : j∗E' → j∗E extends uniquely to a morphism \( \psi : E' \to j_{*}j_{*}E = E. \)

By construction, the composition \( \psi \phi : E \to E \) restricts to identity on W hence \( \psi \phi = Id_{E} \), by the same adjunction. Similarly, the composition \( \hat{\phi} \psi = E' \to \hat{E}' \) restricts to identity on \( \hat{U} \) and since \( \hat{E}' \cong \hat{i}_{*}F \), we must have \( \hat{\phi} \psi = Id_{\hat{E}'} \) by Proposition 3(i). We have proved that \( E = i_{*}F \cong E' \). Since \( \hat{E}' = \hat{i}_{*}F \) we conclude that \( \hat{i}_{*}F = \hat{i}_{*}F \).

Corollary 8 The following conditions are equivalent:

1. The cokernel of \( (i_{p})_{*}F_{p}|_{X_{p-1}} \to (i_{p-1})_{*}F_{p-1} \) is supported in codimension ≥ 3 for \( p \gg 0 \).
2. The projective system \( \{i_{*}F_{p}\}_{p \geq 1} \) satisfies the Mittag-Leffler condition.
3. The direct image \( \hat{i}_{*}F \) is coherent.
4. The bundle \( F \) admits an algebraization.

Proof. The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are established in the proof of Theorem 7. The implication (iv) ⇒ (i) is proved in Proposition 7. If the projective system \( \{i_{*}F_{p}\}_{p \geq 1} \) satisfies the Mittag-Leffler condition, by 0.13.3.1 in [EGAIII] the natural map \( \hat{i}_{*}F \to \lim \hat{i}_{*}F_{p} \) is an isomorphism. By the Mittag-Leffler condition we can replace \( \hat{i}_{*}F_{p} \) by a system of subsheaves \( G_{p} \subset \hat{i}_{*}F_{p} \) so that the property \( \hat{i}_{*}F \cong \lim G_{p} \) still holds and \( G_{p}|_{X_{p-1}} \to G_{p-1} \) is surjective. Since each \( G_{p} \) is coherent by the noetherian property of \( X_{p} \), Proposition 10.11.3 in [EGAI] tells that \( \lim G_{p} \) is also coherent. Therefore, (ii) ⇒ (iii).

Remark. Suppose that \( X_{0} \) is a smooth projective surface over \( K, \xi = k_{1}P_{1} + \ldots + k_{l}P_{l} \) an effective zero cycle and \( F_{0} \) a rank \( n \) vector bundle on \( U_{0} = X_{0} \setminus \{ P_{1}, \ldots, P_{l} \} \). The pair \( (F_{0}, \xi_{0}) \) should define a point \( Spec(K) \to Uhl_{n} \) of the Uhlenbeck functor. Assume that \( (F, \xi) : Spec(A) \to Uhl_{n} \) extends \( (F_{0}, \xi_{0}) \). Then it is expected that \( Coker(i_{*}F \to i_{*}F_{0}) \) can be supported only at the points \( P_{1}, \ldots, P_{l} \), with multiplicities bounded by \( k_{1}, \ldots, k_{l} \), respectively (in the differential geometry picture, cf. [DK], \( \xi_{0} \) represents the singular part of a connection which may be smoothed out by \( F \) but may not acquire any negative coefficients; since the multiplicities of \( Coker(i_{*}F \to i_{*}F_{0}) \) measure the local change of \( c_{2} \), one obtains the bound mentioned). But the proof of Proposition 7 shows that the multiplicities of \( Coker(i_{*}F \to i_{*}F_{0}) \) give an upper bound for the total sum, over all \( p \), of similar multiplicities for \( Coker((i_{p})_{*}F_{p}|_{X_{p-1}} \to (i_{p-1})_{*}F_{p-1}) \). Hence the condition of Corollary 8(i) is rather natural from the point of view of Uhlenbeck spaces.

3 Algebraization of principal bundles.

Let \( G \) be an affine algebraic group over \( k \). We keep the notation of Section 2.1. and consider left principal \( G \)-bundles which are locally trivial in fppf topology. For such a \( G \)-bundle \( P \) (over \( \hat{U} \) or an open subset \( U \subset X \) and any scheme \( Y \) over \( k \) with left \( G \)-action, denote by \( P_{Y} = G \setminus (Y \times_{k} P) \) the associated fiber bundle, i.e. the quotient by the left diagonal action of \( G \). For instance, when \( \rho : G \to H \) is a homomorphism of linear algebraic groups over \( k \), we can consider a left \( G \)-action on \( H \) given by \( g \cdot h = h \rho(g)^{-1} \) and then \( P_{H} \) is simply the principal \( H \)-bundle induced via \( \rho \).

9
Theorem 9 Assume that the identity component \( G^0 \) is reductive. Then a principal \( G \)-bundle \( \mathcal{P} \) over the formal scheme \( \hat{U} \) admits an algebraization if and only if for a fixed exact representation \( G \to GL(V) \) the associated vector bundle \( \mathcal{P}_V \) admits an algebraization, i.e. satisfies the conditions of Corollary 8.

The “only if” part is obvious. Since by a result of Haboush, cf. Theorem 3.3 in [Ha1], the quotient \( GL(V)/G \) is affine, the “if” part follows from the following general statement.

Proposition 10 Let \( H \) an affine algebraic group over \( k \) and \( G \) its closed subgroup such that \( H/G \) is affine. Suppose that \( \mathcal{P} \) is a principal \( G \)-bundle over \( \hat{U} \) such that the associated principal \( H \)-bundle \( Q = \mathcal{P}_H \) admits an algebraization. Then \( \mathcal{P} \) admits an algebraization.

First we establish a preparatory result. As before, \( U \subset X \) is an open subset satisfying \( U \cap X_0 = U_0 \).

Lemma 11 Let \( H \) be a linear algebraic group over \( k \), \( Q \) be a principal \( H \)-bundle on \( U \) and \( \hat{Q} \) its completion. Let also \( Y \) be an affine \( H \)-variety. Then for any section \( \hat{s} : \hat{U} \to \hat{Q}_Y \) there exists a section \( s : W \to Q_Y \) on an open subset \( W \subset U \) containing \( U_0 \), with completion equal to \( \hat{s} \). If \( (W, s) \) and \( (W', s') \) are two such algebraizations, then \( s \) and \( s' \) are equal on \( W \cap W' \).

Proof. One can find a \( H \)-invariant linear subspace \( V^\vee \subset k[Y] \) containing a set of generators of \( k[Y] \) as a \( k \)-algebra. Then the surjection \( Sym^*_k(V^\vee) \to k[Y] \) gives an \( H \)-equivariant closed embedding \( Y \hookrightarrow V \) into the dual space \( V \). This induces closed embeddings \( Q_Y \hookrightarrow Q_V \) and \( \hat{Q}_Y \hookrightarrow \hat{Q}_V \).

Therefore \( \hat{s} \) becomes a section of the vector bundle \( \hat{Q}_V \). By Proposition 7(ii) the completion of the coherent sheaf \( i_*Q_V \) is isomorphic to \( \hat{i}_*\hat{Q}_V \) and therefore by Proposition 3(i) there exists a unique section \( \hat{s} \) of \( i_*Q_V \) with completion given by \( \hat{i}_*\hat{s} \). Set \( s = \hat{s}|_{U} \).

It remains to show that \( s(W) \subset Q_V \) on some \( W \) as above. Let \( A = Sym^*(Q_Y^\vee) \) be the sheaf of symmetric algebras on \( U \) corresponding to \( Q_Y \) and \( \mathcal{I} \subset A \) the ideal sheaf of \( Q_Y \). The section \( s \) gives the evaluation morphism \( \rho : A \to \mathcal{O}_U \). The sheaf \( G = \rho(\mathcal{I}) \) is coherent, being a sub sheaf of \( \mathcal{O}_U \). Since \( \hat{s} \) takes values in \( \hat{Q}_V \), the completion \( \hat{G} \) is zero. By Corollary 10.8.12 in [EGAI] this implies \( \text{Supp}(G) \cap U_0 = \emptyset \) hence \( W = U \setminus \text{Supp}(G) \) satisfies the conditions of the lemma. The uniqueness of \( s \) follows from the uniqueness of \( \hat{s} \). \( \square \)

Proof of Proposition 10. Let \( (U, Q) \) be an algebraization of \( \mathcal{Q} \). In general, giving a principal \( G \)-bundle is equivalent to giving a principal \( H \)-bundle \( \mathcal{R} \) together with a reduction to \( G \), i.e. a section of the associated bundle \( \mathcal{R}_{H/G} \) with the fiber \( H/G \). Since \( Q \) is induced from \( \mathcal{P} \), we get a section \( \hat{s} : \hat{U} \to Q_{H/G} \) and by the above lemma there exists \( s : W \to Q_{H/G} \) such that \( \hat{s} \) is equal to its completion. Then \( \mathcal{P} \) admits an algebraization \( (W, P) \) where \( P \) is the pullback of the principal \( G \)-bundle \( Q \to Q_{H/G} \) via \( s : W \to Q_{H/G} \). \( \square \)

4 Categorical formulations.

Proposition 12 The functor \( F \mapsto \hat{F}|_G \) induces an equivalence between the full subcategory of all coherent sheaves \( E \) on \( X \) which are locally free at the points of \( U_0 \subset X \) and have \( \text{depth}_x E \geq 2 \) at the points where \( E \) is not locally free, and the full subcategory of locally free sheaves on \( \hat{U} \) admitting algebraization.

Proof. Let \( (U, F) \) be an algebraization of \( \mathcal{F} \). Then the sheaf \( E = i_*F \) satisfies \( E \simeq i_*i^*E \) hence by Proposition 3(iii) \( \text{depth}_x E \geq 2 \) for all \( x \in Z = X \setminus U \). We also observe that \( E \) is uniquely
Theorem 13
results on algebraization of principal bundles

Also denote by $\hat{E} \simeq \hat{i}_*F$. Thus the functor described is essentially surjective on objects. For the morphisms, let $\mathcal{F}_1, \mathcal{F}_2$ be a pair of vector bundles on $\hat{U}$ with algebraization $(U, F_1)$ and $(U, F_2)$, respectively, which we may assume to be defined on the same $U$. Denote by $E_1 = i_*F_1, E_2 = i_*F_2$ the corresponding coherent sheaves on $X$. Then $\text{Hom}_\mathcal{F}(\mathcal{F}_1, \mathcal{F}_2) = \text{Hom}_X(\hat{i}_*\mathcal{F}_1, \hat{i}_*\mathcal{F}_2) = \text{Hom}_X(E_1, E_2)$ where the first equality is by adjunction of $i^*$ and the second by Propositions 3(ii) and 7(ii). $\square$

To formulate a result for principal bundles, let $\mathcal{B}(G, U_0)$ be the groupoid category in which the objects are given by pairs $(U, P)$ where $U \subset X$ is an open subset with $U \cap X_0 = U_0$, and $P$ is a principal $G$-bundle on $U$. Morphisms from $(U, P)$ to $(U', P')$ are given by the set of equivalence classes of pairs $(W, \psi)$ where $W \subset U \cap U'$ is an open subset with $W \cap X_0 = U_0$ and $\psi : P|_W \to P'|_W$ an isomorphism of $G$-bundles. Two such pairs $(W, \psi)$ and $(W, \psi')$ are equivalent if $\psi = \psi'$ on $W \cap W'$. Also denote by $\text{Bun}(G, \hat{U})$ the groupoid category of $G$-bundles on the formal scheme $\hat{U}$. Completion along $U_0$ defines a functor $\Psi : \mathcal{B}(G, U_0) \to \text{Bun}(G, \hat{U})$. The following statement summarizes our results on algebraization of principal bundles.

**Theorem 13**

With the notation of Section 2.1,

(i). For any affine algebraic group $G$ over $k$, $\Psi : \mathcal{B}(G, U_0) \to \text{Bun}(G, \hat{U})$ is full and strict.

(ii). For $G = GL_n(k)$ the essential image of $\Psi$ is the full subcategory of rank $n$ vector bundles $\mathcal{F} = \varprojlim F_p$ on $\hat{U}$ which satisfy the equivalent conditions (i)-(iii) of Corollary 8.

(iii). Let $G \hookrightarrow H$ be a closed embedding of affine algebraic groups over $k$ such that $H/G$ is affine. Then the natural functor from $G$-bundles to $H$-bundles induces an equivalence of categories

$$\mathcal{B}(G, U_0) \simeq \text{Bun}(G, \hat{U}) \times_{\text{Bun}(H, \hat{U})} \mathcal{B}(H, U_0)$$

**Proof.** To prove (i) suppose that $P, P'$ are two principal bundles on $\hat{U}$ admitting algebraizations $P, P'$, respectively, which we may assume to be defined on the same $U \subset X$. Let $\psi : P \to P'$ be an isomorphism. We need to prove that there exists (perhaps after shrinking $U$) a unique isomorphism $\psi : P \to P'$ with completion given by $\hat{\psi}$. Let $\text{Isom}(P, P')$ be the bundle of isomorphisms $P \to P'$. Considering graphs of isomorphisms, we can identify $\text{Isom}(P, P') \simeq G \setminus (P \times_U P')$. On the other hand, $P \times_U P'$ is a principal bundle over $G \times_k G$. Define a left action of $G \times_k G$ on $G$ by $(g, h) \cdot f = gfh^{-1}$, then $G \setminus (P \times_U P') \simeq (P \times_U P')_G$. Since $\psi$ gives a section $\hat{s}$ of $\text{Isom}(P, P')$, applying Lemma 11 to $H = G \times_k G$ and $Y = G$, we get a unique algebraization $s : W \to (P \times_U P')_G \simeq \text{Isom}(P, P')|_W$, which corresponds to the required isomorphism $\psi$. This proves (i).

The statement of (ii) for objects holds by Corollary 8 and for morphisms by (i).

For (iii) first observe that the compositions $\mathcal{B}(G, U_0) \to \mathcal{B}(H, U_0) \to \text{Bun}(H, \hat{U})$ and $\mathcal{B}(G, U_0) \to \text{Bun}(G, \hat{U}) \to \text{Bun}(H, \hat{U})$ are canonically isomorphic, therefore one does get a functor

$$\mathcal{B}(G, U_0) \to \text{Bun}(G, \hat{U}) \times_{\text{Bun}(H, \hat{U})} \mathcal{B}(H, U_0)$$

On objects, this functor is an equivalence if for a $G$-bundle $\mathcal{P}$ on $\hat{U}$, an $H$-bundle $Q$ on $U \subset X$ and an isomorphism $\phi : \mathcal{P}_H \simeq \hat{Q}$, there exists an open subset $W \subset U$ with $W \cap X_0 = U_0$, a $G$-bundle $\mathcal{P}$ on $W$ and isomorphisms $\hat{P} \simeq \mathcal{P}$ and $P_H \simeq Q|_W$ which induce $\phi$ in a natural way. This is equivalent to finding an algebraization of the section $\hat{s} : \hat{U} \to \hat{Q}_{H/G}$ induced by $\phi$, which was done in the proof of Proposition 10. On morphisms, without loss of generality it suffices to consider two $G$-bundles.
$P,P'$ defined on the same open set $U$, and isomorphisms $\psi : P_H \simeq P'_H, \hat{\phi} : \hat{P} \to \hat{P}'$ which have the same image in $Bun(H,\hat{U})$. We need to show that there exists a unique isomorphism $\phi : P \to P'$ inducing $\phi$ and $\psi$ in the natural sense. But by (i) there exists a unique $\phi$ with completion equal to $\hat{\phi}$. Since by assumption the isomorphisms $\psi' = \phi_H$ and $\psi$ are equal after completion, $\psi' = \psi$ by part (i). This finishes the proof. □

References


Address: Department of Mathematics, MSTB 103, UC Irvine, Irvine CA 92697.