Bundles on non-proper schemes: representability

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1 Introduction

Let X_k be a proper irreducible separated scheme of finite type over a field k. We will also assume that X_k satisfies Serre's S_2 condition (for the sake of simplicity the reader may think that X_k is smooth or a locally complete intersection). For a noetherian scheme T over k denote $X_T = (X_k) \times_{Spec(k)} T$ and let $\Phi(T)$ be the collection of all closed subsets $Z \subset X_T$ such that every point $z \in Z$ has codimension ≥ 3 in its fiber over T. Fix a reductive group G over k.

Definition. In the notation above, let $F_G(T)$ be a groupoid category with the objects (E, U) where E is a principal G-bundle defined on an open subscheme $U \subset X_T$, such that the closed complement of U is a subset in $\Phi(T)$. A morphism $(E_1, U_1) \to (E_2, U_2)$ is an isomorphism $E_1|_W \simeq E_2|_W$ on an open subset $W \subset U_1 \cap U_2$ such that the complement of W is again in $\Phi(T)$. The composition of morphisms is defined in an obvious way.

For any morphism $\alpha : T' \to T$ of schemes over T we have pullback functors $\alpha^* : F_G(T) \to F_G(T')$ satisfying the usual compatibility conditions for any pair of morphisms $T'' \xrightarrow{\beta} T' \xrightarrow{\alpha} T$, i.e. F_G is a groupoid over the category of noetherian schemes over k, cf. Section 1 in [Ar]. As usual, we will mostly deal with its restriction to affine noetherian schemes over k, writing X_A and $\Phi(A)$ instead of $X_{Spec(A)}$ and $\Phi(Spec(A))$, respectively. The main goal of this paper is the following result.

Theorem 1 *F_G is an algebraic stack, locally of finite type over k, with separated and quasi-compact diagonal.*

Thus we obtain a partial compactification of the stack of G-bundles on X_k. Our strategy of proof is straightforward, if seldom used: we apply Artin's representability criterion, cf. Theorem 5.3 in [Ar] for a statement and [Li], [Ao] for examples of application.

For the most part of the paper (see Sections 3-7) we consider the case of vector bundles, i.e. work with $G = GL(r)$ for fixed $r \geq 1$; and write F instead of F_G. In Section 8 we show how the proof is extended to the case of general G and also explain why the result fails when the "codimension 3" condition in the definition of $\Phi(T)$ is replaced by "codimension 2".

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2 Depth and local cohomology.

We will freely use definitions and basic properties of local cohomology and depth which can be found e.g. in Section 18 and Appendix 4 to [E] and Sections I-III of [SGA 2]. Observe that the scheme X_T will not satisfy Serre's S_2 condition since no depth assumptions are made for T. However, we can formulate a relative version of this condition.

Definition. For any point $x \in X_T$ let $d(x)$ be the codimension of x in its fiber over T.

Lemma 2 *Let E be a vector bundle on an open subset $U \subset X_T$ and M a coherent sheaf on T. For $t \in T$ let X_t be the fiber of X_T over t and suppose that for every t the structure sheaf of the intersection $U_t = U \cap X_t$ satisfies Serre's condition S_n. Set $E_M = E \otimes_{\mathcal{O}_T} M$. Then for any $x \in U$ one has $depth_{U,x} E_M \geq min(n, d(x))$.*

Definition. We will call the inequality stated in this lemma *the relative S_n condition.* In this paper $n = 2$ or 3.

Proof of the lemma. Since the question is local we can take $E = \mathcal{O}_U$. Thus we can take $T = Spec(A)$, $U = X = Spec(C) \times_k Spec(A)$ and E_M of the form $C \otimes_k M$. Since the fiber over $t \in T$ is given by $X_{k(t)}$, the codimension of x in X_t is equal to the codimension of its image $x' \in X_k$. Setting $r = min(n, d(x))$ we can find a C-regular sequence $f_1, \ldots, f_r$ in the maximal ideal of x' in C. The same f_i viewed as elements of $C \otimes_k A$ will belong to the maximal ideal of x and form a $C \otimes_k M$-regular sequence, which finishes the proof. $\square$

Below, dealing with obstructions, deformations and infinitesimal automorphisms we need the following construction. For for any coherent sheaf E on X_T set

$$H^i_{T,\Phi}(E) = \varinjlim_{Z \in \Phi(T)} H^i(X_T \setminus Z, E)$$

where the filtered direct limit is taken with respect to the inclusion of closed subsets $Z \subset Z'$ in $\Phi(T)$. If $T = Spec(A)$ is affine, we write $H^i_{A,\Phi}(E)$ instead of $H^i_{Spec(A),\Phi}(E)$. If E is defined only on an open subset $U = X_T \setminus Z_0$ with $Z_0 \in \Phi(T)$ we can use the same definition but take the limit over those Z which contain Z_0. Observe that for $i = 0, 1$ the cohomology groups in the limit in fact stabilize under certain restrictions on E:

Lemma 3 *With the notation just introduced, assume that $Z \subset Z'$ are in $\Phi(T)$. If E satisfies the relative S_2 condition on $X_T \setminus Z$ then the natural restriction morphism*

$$\rho_i : H^i(X_T \setminus Z, E) \to H^i(X_T \setminus Z', E)$$

is an isomorphism for $i = 0$ and injective for $i = 1$. If in addition E satisfies the relative S_3 condition on $X_T \setminus Z$ then ρ_i is an isomorphism for $i = 0, 1$ and injective for $i = 2$.

Proof. Denote $U = X_T \setminus Z$, $W = U \cap Z'$ and consider the spectral sequence of local cohomology $H^p(U, \mathcal{H}^q_W(E)) \Rightarrow H^{p+q}_W(U, E)$. By the relative S_2 condition the local cohomology sheaves $\mathcal{H}^i_W(E)$ vanish for $i = 0, 1$ while the relative S_3 condition at the points of W also implies $\mathcal{H}^2_W(E) = 0$. Now the assertion follows from the standard long exact sequence

$$\ldots \to H^i_W(U, E) \to H^i(U, E) \to H^i(U \setminus W, E) \to H^{i+1}_W(U, E) \to \ldots \qquad \square$$

Observe that the relative S_3 condition on E_M also holds if Z contains

$$Z^\circ_T = Z^\circ \times_{Spec(k)} T \in \Phi(T)$$

where $Z^\circ \subset X_k$ is the set of all points in X_k where the S_3 condition fails for the structure sheaf. Observe that Z° is closed by [EGA IV$_2$], Proposition 6.11.2; since its complement contains all points of codimension ≥ 2 we indeed have $Z^\circ_T \in \Phi(T)$.

Corollary 4 *If $Z^\circ_T \subset Z$ then*

$$H^i_{T,\Phi}(E_M) = H^i(X_T \setminus Z, E_M)$$

for $i = 0, 1$ and $E_M = E \otimes_T M$ as above. Moreover, if j stands for the open embedding $X_T \setminus Z \hookrightarrow X_T$ then the sheaves $j_* E_M$ and $R^1 j_* E_M$ are coherent on X_T and if $T = Spec(A)$ is affine the two stable cohomology groups are finitely generated A-modules.

Proof. Stabilization follows immediately from the previous lemma. Coherence of the two direct images is due to [SGA 2], VII.2.3 while the finite generation is proved by combining the spectral sequence $H^p(X_T, R^q j_*(E_M)) \Rightarrow H^{p+q}(X_T \setminus Z, E_M)$ with the fact that X_T is proper over T. $\square$

Remark. Of course, for $i = 2$ even the cohomology group $H^2(\mathbb{P}^3_k \setminus P, \mathcal{O})$ is infinite dimensional over k for any closed point P.

3 Locally finite presentation.

In this section we do not use the S_2 assumption on X_k. Let $R = \varinjlim R_\alpha$ be a filtered direct limit of k-algebras.

Proposition 5

$$\varinjlim F(R_\alpha) \to F(R)$$

is an equivalence of categories.

Proof. The assertion means that any object $(E, U) \in F(R)$ is an image of some $(E_\alpha, U_\alpha) \in F(R_\alpha)$ and that, whenever (E_α, U_α) and (E_β, U_β) give isomorphic objects in $F(R)$, there exists γ such that $\gamma \geq \alpha$, $\gamma \geq \beta$ and the corresponding objects in $F(R_\gamma)$ are isomorphic. In addition, a similar condition should hold for morphisms.

To prove the assertion for objects, consider a vector bundle E on $U \subset X_R$ and take a finite affine covering $\{U_i\}$ of U such that $E|_{U_i}$ is trivial.

Using the results of Sections 8.2-8.5 of [EGA IV$_3$] we see that there exists α and open subsets U_i^α such that $U_i = \pi_\alpha^{-1}(U_i^\alpha)$ where $\pi_\alpha : X_R \to X_{R_\alpha}$ is the natural projection. Since in general a scheme W is affine iff the canonical morphism $W \to Spec(\Gamma(W, \mathcal{O}_W))$ is an isomorphism, by increasing α if necessary we can assume that all U_i^α are affine.

The transition functions for E given by $\phi_{ij} : U_i \cap U_j \to GL_r(k)$ can be viewed as automorphisms of the trivial bundle. Increasing α we can assume that ϕ_{ij} arise from regular maps $\phi_{ij}^\alpha : U_i^\alpha \cap U_j^\alpha \to GL(r)$. Increasing α again we can assume that ϕ_{ij}^α satisfy the cocycle condition and thus define a vector bundle E^α on $U^\alpha = \bigcup_i U_i^\alpha$. By construction, $E \simeq \phi_\alpha^* E^\alpha$. To show that the closed complement Z^α of U^α is in $\Phi(Spec(R_\alpha))$ (again, after a possible increase of α) note that $U = \pi_\alpha^{-1}(U^\alpha)$ and the fibers of $X_R \to Spec(R)$ are obtained from the fibers $X_{R_\alpha} \to Spec(R_\alpha)$ by extension of scalars. Therefore the closed subset W of points $s \in Spec(R_\alpha)$ for which codimension of $Z^\alpha \cap X_s$ is ≤ 2 has empty preimage in $Spec(R)$. Therefore, for some $\alpha' \geq \alpha$ the preimage of W in $Spec(R_{\alpha'})$ is empty and we can replace α by α'.

To prove surjectivity on morphisms, let (E_1, U_1) and (E_2, U_2) be two objects in $F(R)$ and suppose we are given an isomorphism $\phi : E_1|_U \simeq E_2|_U$ where $U \subset U_1 \cap U_2$ is open with its closed complement in $\Phi(R)$.

By the previous argument, we can assume that E_i is isomorphic to the pullback of some vector bundle E_i^α on an open subset $U_i^\alpha \subset X \times Spec(R_\alpha)$. Increasing α we can assume that U is the preimage of an open subset $U^\alpha \subset U_1^\alpha \cap U_2^\alpha$. Then $E_1^\alpha|_{U^\alpha}$ and $E_2^\alpha|_{U^\alpha}$ become isomorphic after pullback to U hence by *loc. cit.* by increasing α we can find an isomorphism $\phi^\alpha : E_1^\alpha|_{U^\alpha} \simeq E_2^\alpha|_{U^\alpha}$ which induces ϕ on U. As before, we may have to increase α one more time to ensure that the complement of U^α is in $\Phi(R_\alpha)$.

Injectivity on morphisms is an immediate consequence of Theorem 8.5.2 in *loc. cit.* $\Box$

4 Small affine pushouts.

Let A_0 be a noetherian k-algebra, and $A' \to A$ a surjection of two infinitesimal extensions of A_0 such that $M = ker(A' \to A)$ is a finite A_0 module. Let B be a noetherian ring and $B \to A$ a morphism, such that the composition $B \to A \to A_0$ is surjective.

Denote by B' the pushout $A' \times_A B$, i.e. the subset of pairs $(a, b) \in A' \times B$ which have the same image in A. Then $B' \to B$ is surjective and its kernel may be identified with M viewed as a B-module. Observe that $Spec(B')$ is homeomorphic to $Spec(B)$, while $Spec(A'), Spec(A)$ and $Spec(A_0)$ are homeomorphic to each other, and $Spec(A_0) \to Spec(B)$ is naturally a closed subscheme by the assumption.

Fix an object $a = (E_A, U_A) \in F(A)$. Let $F_a(B)$ the groupoid of extensions of a over $Spec(B)$, and similarly for A', B'.

Proposition 6 *The natural functor*

$$F_a(B') \to F_a(A') \times F_a(B)$$

is an equivalence of groupoids.

Proof. Suppose that $(E_{A'}, U_{A'})$, (E_B, U_B) are two extensions of (E, U) to $X_{A'}$ and X_B, respectively. Since $Spec(A)$ and $Spec(A')$ are homeomorphic, shrinking $U_{A'}, U_B$ and U_A, if necessary, we can assume $U_{A'} \simeq U_A \simeq U_B \cap X_{A_0}$ (homeomorphisms induced by the natural embeddings). Denote by $U_{B'}$ the subset U_B viewed as an open subset of $X_{B'}$. We have a commutative diagram

$$\begin{array}{ccc} U_A & \xrightarrow{p} & U_{A'} \\ q\big\downarrow & & i\big\downarrow \\ U_B & \xrightarrow{j} & U_{B'} \end{array}$$

where the horizontal arrows are homeomorphisms. Since $q^*E_B \simeq E_A$, $p^*U_{A'} \simeq E_A$ and $ip = jq$ there will be an exact sequence on $U_{B'}$

$$i_*E_{A'} \oplus j_*E_B \to (ip)_*E_A \to 0$$

where the first arrow is given by the difference of the obvious canonical maps. One can check that the kernel $E_{B'}$ of the first arrow is a locally free sheaf of rank r on $U_{B'} \subset X_{B'}$ such that $i^*E_{B'} \simeq E_{A'}$, $j^*E_{B'} \simeq E_B$, $(ip)^*E_{B'} \simeq E_A$ in a compatible way. A straightforward check shows that the correspondence $(E_{A'}, E_B) \mapsto E_{B'}$ induces which is an equivalence of categories. $\square$

5 Automorphisms, deformations, obstructions.

We keep the notation of Section 4 and recall that for a finitely generated A_0-module M and a vector bundle E we denote $E_M = E \otimes_{A_0} M$.

5.1 Aut, D, O

Infinitesimal automorphisms.

Let $A = A_0$, $A' = A_0 \oplus M$. Any $a_0 \in F(A_0)$ given by a pair (E, U) admits a trivial extension to $A_0 \oplus M$ defined by $E' = E \oplus E_M$. We are interested in the group of automorphisms $Aut_{a_0}(A_0 + M)$ of the bundle E', which restrict to identity over A_0. Every such automorphism is defined uniquely by a morphism $E \to E_M$ defined, perhaps, on a smaller open subset $V \subset U$. In other words

$$Aut_{a_0}(A_0 + M) = H^0_{A_0,\Phi}(End(E)_M)$$

By Corollary 4 this is a finitely generated module over A_0.

Deformations

Now consider $D_{a_0}(M)$, the set of isomorphism classes of extensions of $a_0 = (E, U)$ to A'. A standard argument, cf. e.g. Chapter IV of [I], identifies $D_{a_0}(M)$ with $H^1_{A_0,\Phi}(End(E)_M)$ which is also finitely generated over A_0, as established in Corollary 4.

Obstructions

Our goal is to define a finitely generated submodule of $H^2_{A_0,\Phi}(End(E)_M)$ which will serve as obstruction module for our problem. First consider a square zero extension $0 \to M \to A' \to A \to 0$ and $a = (E_A, U_A) \in F(A)$. Let $U_{A'} \subset X_{A'}$ be the open subset homeomorphic to U_A, with its natural structure of an open subscheme of $X_{A'}$. By [I] there is an obstruction to deforming E_A over $U_{A'}$, given by a class

$$\omega(E_A) \in Ext^2_{U_A}(End(E_A), M)$$

which is a Yoneda product of two classes

$$a(E_A) \in Ext^1_{U_A}(End(E_A), L_{U_A}), \qquad \kappa(U/U') \in Ext^1_{U_A}(L_{U_A}, M).$$

Here L_{U_A} is the cotangent complex of U over k, $a(E)$ is the Atiyah class of E and $\kappa(U_A/U_{A'}) \in Ext^1_{U_A}(L_{U_A}, M)$ is the Kodaira-Spencer class of $U_{A'}$ viewed as a deformation of U_A, cf. *loc.cit.* Moreover, since X_A is a direct product of X_k and $Spec(A)$, its cotangent complex over k splits into a direct sum $L_{X_k} \oplus L_A$ of the pullbacks of cotangent complexes from X_k and $Spec(A)$, respectively. Since $X_{A'}$ and $U_{A'}$ viewed as deformations of X_A and U_A, respectively, are induced by $Spec(A) \to Spec(A')$, the Kodaira-Spencer class $\kappa(U_A/U_{A'})$ belongs to the direct factor $Ext^1_{U_A}(L_A, M) \subset Ext^1_U(L_{U_A}, M)$. Therefore, $\omega(E_A)$ may be viewed as the product of the Kodaira-Spencer class with the image of the Atiyah class $a'(E) \in Ext^1_{U_A}(End(E_A), L_A)$. If we represent $a'(E)$ by an extension

$$0 \to L_A \to Q \to End(E_A) \to 0,$$

then $\omega(E_A)$ will become the image of $\kappa(U_A/U_{A'})$ under the connecting homomorphism:

$$\ldots \to Ext^1_{U_A}(Q, M) \to Ext^1_{U_A}(L_A, M) \to Ext^2_{U_A}(End(E_A), M) \to \ldots$$

Therefore, the obstruction $\omega(E_A)$ can be viewed as a element of the following cokernel $O_a(U_A, M)$:

$$Ext^1_{U_A}(Q, M) \to Ext^1_{U_A}(L_A, M) \to O_a(U_A, M) \to 0.$$

Since the cotangent complex of $Spec(A)$ is concentrated in non-positive degrees and has finitely generated cohomology, we can find a free resolution $\ldots \to L_2 \to L_1 \to L_0 \to L_A \to 0$ and therefore a locally free resolution $\ldots \to L_2 \to L_1 \to Q_0 \to Q \to 0$. The standard spectral sequence $E_1^{p,q} = Ext^p_{U_A}(L_q, M) \Rightarrow Ext^{p+q}_{U_A}(L_A, M)$ shows that $Ext^1_{U_A}(L_A, M)$ has a two-step filtration with associated graded depending only on $E_1^{p,q}$ for $p \leq 1$ and $q \leq 2$. Similar argument applies to $Ext^1_{U_A}(Q, M)$ which allows us to conclude that $O_a(U_A, M)$ is finitely generated over A and independent of U_A as long as $U_A \cap Z_A^\circ = \emptyset$ (where Z_A° is defined before Corollary 4). For such U_A, denote the stabilized module $O_a(U_A, M)$ simply by $O_a(M)$.

Assume now that A is itself an extension

$$0 \to N \to A \to A_0 \to 0$$

where N is nilpotent and acts on M by zero (so that M is an A_0-module). Both $Ext^1_{U_A}(Q, M)$ and $Ext^1_{U_A}(L_A, M)$ in this case are A_0-modules, hence $O_a(M)$ is also a finitely generated A_0-module. In fact, if $a_0 = (E, U)$ is the restriction of $a = (E_A, U_A)$ and $Q^\bullet$ and $L^\bullet$ are the above locally free resolutions, then

$$Ext^1_{U_A}(Q, M) = Ext^1_U(Q^\bullet \otimes_A A_0, M); \qquad Ext^1_{U_A}(L_A, M) = Ext^1_U(L^\bullet \otimes_A A_0, M)$$

and by virtue of the spectral sequence just mentioned we can assume that $Q^\bullet$, $L^\bullet$ are concentrated in degrees $[-2, 0]$. Denoting $P^\bullet = \mathcal{H}om_U(Q^\bullet \otimes_A A_0, \mathcal{O}_U) = (P^0 \to P^1 \to P^2)$ and similarly for $R^\bullet = \mathcal{H}om_U(L^\bullet \otimes_A A_0, \mathcal{O}_U) = (R^0 \to R^1 \to R^2)$ we see that

$$O_a(M) = Coker\Big[H^1(U, P^\bullet \otimes M) \to H^1(U, R^\bullet \otimes M)\Big]$$

By the standard argument, cf. [I], if the obstruction $\omega(E_A)$ vanishes, all deformations of E_A over $U_{A'}$ form a pseudo-torsor over $D_{a_0}(M)$.

5.2 Etale localization, completions, constructibility.

Let $p : A \to B$ be etale and consider $p_0 : A_0 \to B_0$ defined by $B_0 = B \otimes_A A_0$, $p_0 = p \otimes_A A_0$. Consider $a = (E_A, U_A) \in F(A)$ and let b be its pullback in $F(B)$, and similarly for $a_0 = (E, U) \in F(A_0)$, $b_0 \in F(B_0)$.

Proposition 7 *There exist natural isomorphisms:*

$$O_b(M \otimes_{A_0} B_0) \simeq O_a(M) \otimes_{A_0} B$$

and

$$Aut_{b_0}(B_0 + M \otimes_{A_0} B_0) \simeq Aut_{a_0}(A_0 + M) \otimes_{A_0} B_0; \quad D_{a_0}(M \otimes_{A_0} B_0) \simeq D_{a_0}(M) \otimes_{A_0} B_0$$

Proof. For Aut and D this follows from their identification with $H^i(U, End(E)_M)$ for $i = 0, 1$ and some $U \subset X_{A_0}$, and etale localization for cohomology. For O, one uses the definition

$$Coker\Big[Ext^1_{U_A}(Q, M) \to Ext^1_{U_A}(L_A, M)\Big]$$

with some $U_A \subset X_A$, and then applies etale localization for Ext^1 plus the idenity $L_B = L_A \otimes_A B$ which holds for any etale extension $A \to B$, cf. Chapter II of [I]. $\square$

Proposition 8 *Let $\mathfrak{m} \subset A_0$ be a maximal ideal. Then*

$$D_{a_0}(M) \otimes_{A_0} \widehat{A}_0 \simeq \varprojlim D_{a_0}(M/\mathfrak{m}^n M)$$

and similarly for $Aut_{a_0}(A_0 + M)$.

Proof. Both follow immediately from Proposition 0.13.3.1 in [EGA III_1] applied to the completion of the open subscheme U for which $H^i(U, End(E)_M)$ compute for $i = 0, 1$ the modules Aut_{a_0} and D_{a_0}, respectively. $\square$

Proposition 9 *Assume that the ring A_0 is reduced. Then there exists an open dense subset of points $p \in Spec(A_0)$, so that*

$$D_{a_0}(M) \otimes_{A_0} k(p) \simeq D_{a_0}(M \otimes_{A_0} k(p)),$$

and similarly for $Aut_{a_0}(A_0 + M)$ and $O_a(M)$.

Proof.

Step 1. Without loss of generality we can also assume that $Spec(A_0)$ is irreducible, i.e. A_0 is a domain. It suffices to show that, replacing $Spec(A_0)$ by its affine open subset, one can achieve

$$H^i(U, P^\bullet_M) \simeq H^i(U, P^\bullet) \otimes_{A_0} M; \quad i = 0, 1$$

for arbitrary finitely generated A_0-module M and a fixed complex $P^\bullet = (P^0 \to P^1 \to P^2)$ of vector bundles on U.

We first consider the case when $P^\bullet = P^0 =: P$ is a single vector bundle in degree zero. Denote X_{A_0} by X and the open embedding $U \to X$ by j. Recall that the sheaves $j_*(P_M)$, $R^1 j_*(P_M)$ are coherent on X by Corollary 4. Recall also the standard exact sequence for a quasicoherent sheaf $\mathcal{F}$ on X:

$$\ldots \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, j^*\mathcal{F}) \to H^{i+1}_Z(X, \mathcal{F}) \to \ldots$$

and the spectral sequence $E_2^{p,q} = H^q(X, \mathcal{H}^p_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$. Since $\mathcal{F} = j_* P_M$ satisfies $\mathcal{F} \simeq j_* j^* \mathcal{F}$ we have $\mathcal{H}^0_Z(j_* P_M) = \mathcal{H}^1_Z(j_* P_M) = 0$. This gives an isomorphism $H^0(U, P_M) \simeq H^0(X, j_* P_M)$ and a long exact sequence

$$0 \to H^1(X, j_* P_M) \to H^1(U, P_M) \to H^0(X, R^1 j_* P_M) \to H^2(X, j_* P_M) \tag{1}$$

Step 2. Since X is proper over $Spec(A_0)$, for any coherent sheaf $\mathcal{F}$ on X and any finitely A_0-module M one has

$$H^i(X, \mathcal{F}_M) \simeq H^i(X, \mathcal{F}) \otimes_{A_0} M, \quad i \geq 0 \tag{2}$$

after a localization of A_0 at powers of a nonzero element $f \in A_0$. In fact, only finitely many A_0-modules $H^i(X, F)$ are non-zero and by properness they are finitely generated over A_0. Applying the following Generic Freeness Lemma, cf. Theorem 14.4 in [E], we can ensure that after a localization of A_0 the cohomology modules will be free of finite rank:

Lemma 10 *Let A_0 be a noetherian domain and B a finitely generated A_0-algebra. If N is a finitely generated B-module, there exists a nonzero element $t \in A_0$, such that the localization $N[t^{-1}]$ is free over $A_0[t^{-1}]$.*

By combining Cech complex with projective resolutions we can find a complex $C(F)$ of flat A_0-modules, such that $C(F) \otimes_{A_0} M$ computes the cohomology of F_M for any M. This gives a second quadrant spectral sequence

$$E_2^{p,-q} = Tor_q^{A_0}(H^p(X, F), M) \Rightarrow H^{p-q}(X, F \otimes M)$$

which by flatness of $H^i(X, F)$ reduces to isomorphisms $H^i(X, F \otimes M) = H^i(X, F) \otimes M$, as required.

Step 3. Denote the exact sequence (1) by $K^\bullet(M)$. Localizing A_0 further we can assume that

$$Coker\left[H^0(X, R^1 j_* P) \to H^2(X, j_* P)\right]$$

is also free over A_0. Then $K^\bullet(A_0)$ is a complex of projective A_0-module and therefore $K^\bullet(A_0) \otimes_{A_0} M$ is also exact. Comparing $K^\bullet(A_0) \otimes_{A_0} M$ with $K^\bullet(M)$ and using the isomorpism of Step 2, we reduce the isomorphisms

$$H^i(U, P_M) \simeq H^i(U, P) \otimes_{A_0} M \quad i = 0, 1 \tag{3}$$

to the following

Lemma 11 *In the notation introduced above,*

$$j_*(P_M) \simeq j_*(P)_M, \qquad R^1 j_*(P_M) \simeq R^1 j_*(P)_M \tag{4}$$

Proof of the lemma. Since the statement in local we can assume that $X = Spec(B)$ is affine. Let N be the B-module corresponding to the coherent sheaf $j_*(P)$ and $I \subset B$ an ideal such that $Supp(B/I) = Z$. Then the local cohomology modules $H^0_I(N)$, $H^1_I(N)$ vanish by Step 1. Hence by Proposition 18.4 in [E], $depth_I(N) \geq 2$ and there is an N-regular sequence $(f, g) \in I$. We can also assume that (f, g) is regular on $N_M = N \otimes_{A_0} M$ for any M. Indeed, by regularity on N we have an exact sequence

$$0 \to N \to N \oplus N \to N \to N/(f,g)N \to 0$$

of finitely generated B-modules. Localizing A_0 we may assume that all modules in this sequence are free over A_0 therefore the complex

$$0 \to N_M \to N_M \oplus N_M \to N_M \to \big(N/(f,g)N\big) \otimes_{A_0} M \to 0$$

is also exact. Since its first three terms give the Koszul complex of N_M, (f, g) is N_M-regular. In particular, $H^0_I(N_M) = H^1_I(N_M) = 0$. Since N_M corresponds to the sheaf $j_*(P)_M$, the vanishing of local cohomology implies the first isomorphism of the lemma

$$j_*(P)_M \simeq j_* j^*(j_*(P)_M) \simeq j_*(P_M).$$

The second isomorphism $R^1 j_*(P_M) \simeq R^1 j_*(P)_M$ is equivalent to

$$H^2_I(N_M) \simeq H^2_I(N) \otimes_{A_0} M.$$

We claim that one can replace I by an ideal $I' = (f, g, h) \subset I$, where $h \in I$, such that $H^2_I(N_M) \simeq H^2_{I'}(N_M)$ for all M. In fact, by prime avoidance, cf. Lemma 3.3 in [E], one can find $h \in I$ such that $h \notin P$ whenever P is an associated prime of $B/(f, g)B$ which does not contain I. Since P is locally free away from Z, the sequence (f, g) is $\mathcal{O}_U$-regular. This implies that $Z' = V(f, g, h)$ has codimension 3 in X at any point of of $Z' \setminus Z$. Recall that by our assumption Z contains all points where the fiberwise S_3 conditon is violated therefore by Lemma 2 the sheaf corresponding to N_M satisies the relative S_3 condition on U and $\mathcal{H}^2_{Z'}(P \otimes_{A_0} M)|_U = 0$. Therefore

$$H^2_{I'}(N_M) \simeq H^0_I(H^2_{I'}(N_M)) \simeq H^2_I(N_M)$$

where the last isomorphism uses the spectral sequence $H^p_I(H^q_{I'}(N_M)) \Rightarrow H^{p+q}_I(N_M)$ coming from $R\Gamma_{I_1+I_2} \simeq R\Gamma_{I_1} \circ R\Gamma_{I_2}$; and vanishing of $H^0_{I'}(N_M)$ and $H^1_{I'}(N_M)$ due to N_M-regular sequence $(f, g) \in I'$.

The local cohomology $H^i_{I'}(N_M)$ are computed by the complex $C^\bullet(P) \otimes_{A_0} M$ where $C^\bullet(P)$ is the Cech complex of the vector bundle P on $X \setminus Z'$ with respect to the affine covering $X_f \cup X_g \cup X_h$. Observe that each term of $C^\bullet(P)$ is flat over A_0, and the tensor product $C^\bullet(P) \otimes_{A_0} M$ can be identified with the Cech complex of P_M. Then the second quadrant spectral sequence

$$E_2^{-p,q} = Tor_p^{A_0}(H^q(C^\bullet(P)), M) \Rightarrow H^i(C^\bullet(P) \otimes_{A_0} M)$$

shows that $H^q_{I'}(N_M) \simeq H^q_{I'}(N) \otimes_{A_0} M$ holds for any M, q if all $H^q_{I'}(N)$ are flat over A_0. For $q = 0, 1$ these modules vanish as above while for $q = 2$ we have

$$H^2_{I'}(N) = H^2_I(N) = \Gamma(X, R^1 j_* P)$$

which is finitely generated over B since $R^1 j_* P$ is coherent. By Generic Freeness we can assume that $H^2_{I'}(N)$ is free over A_0 after a localization. The only remaining local cohomology group is

$$H^3_{I'}(N) = \varinjlim N/(f^n, g^n, h^n)N$$

Since a filtered direct limit of projectives is flat, it suffices to show that $N_n = N/(f^n, g^n, h^n)N$ are projective over a dense open subset of $V \subset Spec(A_N)$, *for all* $n \geq 1$. By a standard combinatorial argument N_n admits a filtration with successive quotients of the type

$$N_{p,q,r} = f^p g^q h^r N/(f^{p+1} g^q h^r, f^p g^{q+1} h^r, f^p g^q h^{r+1})N.$$

and it suffices to ensure that these modules are projective over some $V \subset Spec(A_0)$.

Both B and N have three filtrations by powers of f, g, h respectively and taking associated graded quotients $gr(\cdot) = gr_h gr_g gr_f(\cdot)$ we obtain a $\mathbb{Z}_3$-graded ring $gr(B)$ and a finitely generated module $gr(N) = \bigoplus_{p,q,r \geq 0} N_{p,q,r}$. Applying Generic Freeness to $gr(N)$ we can localize A_0 at a single element and assume that $gr(N)$ is free over A_0. Then each direct factor $N_{p,q,r}$ must be projective over A_0, hence N_n is also projective over A_0 and the direct limit of N_n is flat, as required. This proves the lemma. $\square$

Therefore, for a vector bundle P on U we have established isomorphisms

$$H^i(U, P_M) \simeq H^i(U, P) \otimes_{A_0} M; \qquad i = 0, 1.$$

Step 4. Now we return to the general case of a complex $P^\bullet = (P^0 \to P^1 \to P^2)$ of vector bundles on U. Consider the spectral sequence $E_1^{r,q}(M) = H^q(U, P^r_M) \Rightarrow H^{r+q}(U, P^\bullet_M)$. Then $E_2^{r,q}$ is the r-th cohomology of

$$E_1^{\bullet,q}(M) = \left[H^q(U, P^0_M) \to H^q(U, P^1_M) \to H^q(U, P^2_M)\right]$$

By previous step for $q = 0, 1$ we can localize A_0 to achieve $H^q(U, P^r_M) \simeq H^q(U, P^r) \otimes_{A_0} M$. In addition, we can ensure that the cohomology of the complexes $E_1^{\bullet,q}(A_0)$ for $q = 0, 1$, are free finitely generated A_0-modules. The second assumption guarantees that for $E_1^{\bullet,q}(A_0)$ cohomology commutes with $\otimes_{A_0} M$; which in view of the first assumption gives $E_2^{r,q}(M) = E_2^{r,q}(A_0) \otimes_{A_0} M$ for $q = 0, 1$.

Now the proposition follows by the isomorphism $H^0(U, P^\bullet_M) = E_2^{0,0}(M)$ and the exact sequence

$$0 \to E_2^{1,0}(M) \to H^1(U, P^\bullet_M) \to E_2^{0,1}(M) \to E_2^{2,0}(M) \qquad \square$$

6 Effectiveness

Proposition 12 *Let $\widehat{A}$ be a complete local algebra with residue field of finite type over k and maximal ideal $\mathfrak{m}$, then the canonical functor*

$$F(\widehat{A}) \to \varprojlim F(\widehat{A}/\mathfrak{m}^n)$$

is an equivalence of categories.

Proof. Let $\{(E_n, U_n) \in F(\widehat{A}/\mathfrak{m}^n)\}$ be a sequence representing an object on the right hand side. Shrinking U_n as in Section 2 we can assume that each E_n satisfies the relative S_3 condition on U_n. But then by the tangent-obstruction theory and stabilization of cohomology the set of isomorphism classes of extensions of E_n to $F(A/\mathfrak{m}^{n+1})$, i.e. the cohomology $H^1(?, End(E_n) \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1})$, will be the same over U_n as over any open subset $W \subset U_n \cap U_{n+1}$ with closed complement in $\Phi(\widehat{A}/\mathfrak{m}^n)$. Therefore we can assume that E_{n+1} is defined also on U_n and by induction all E_n are defined on the same open subset U'. Then by the main result of [B] there exists a bundle E on an open subset $U \subset Spec(\widehat{A})$ such that (E, U) restricts to (E_n, U') in each $F(A/\mathfrak{m}^n)$. On morphisms the assertion also follows from the main result of *loc.cit.* $\square$

7 Properties of the diagonal

Lemma 13 *Let G be a coherent sheaf on X_k, A a noetherian k-algebra and H a coherent sheaf on X_A. Then there exists a finitely generated A-module Q, unique up to canonical isomorphisms, and a natural isomorphism of covariant functors (with argument M)*

$$Hom_{X_A}(H, (G \otimes_k A) \otimes_A M) \simeq Hom_A(Q, M)$$

from the category of A-modules to itself.

Proof. First assume that X_k is projective. Since $G \otimes_k A$ is flat over A and H is a cokernel of a morphism of locally free sheaves, the assertion is an immediate consequence of Corollary 7.7.8 in [EGA III_2].

For proper X_k we use the Chow lemma and the standard pattern of Section 5 in [EGA III_1]. Fixing H, we will say that G is representable if a module Q as in the statement exists. Consider an exact sequence if coherent sheaves on X_k:

$$0 \to G_1 \to G_2 \to G_3 \to 0.$$

We claim that if G_2, G_3 are representable then G_1 is, and if G_1, G_3 are representable then G_2 is. The first assertion is quite easy as the morphism $G_2 \to G_3$ corresponds to the morphism $Q_3 \to Q_2$ of representing A-modules and we can set $Q_1 = Coker(Q_3 \to Q_2)$. For the second assertion we first show that the functor $M \mapsto R(M) = Hom_{X_A}(H, (G \otimes_k A) \otimes_A M)$ commutes with projective limits. In fact, choose an affine covering $X_k = \cup U_i$ and compute $R(M)$ as the kernel of the first arrow in the corresponding Cech complex

$$\bigoplus_i Hom_{(U_i)_A}(H, (G \otimes_k A) \otimes_A M) \to \bigoplus_{i \neq j} Hom_{(U_i \cap U_j)_A}(H, (G \otimes_k A) \otimes_A M)$$

where we omits the notation for restriction of sheaves to U_i and $U_i \cap U_j$, respectively. Since projective limits are left exact and commute with $Hom(H, \cdot)$ by universal property of projective limits, it suffices to show that $(G \otimes_k A) \otimes_A (\cdot)$ commutes with projective limits, which is obvious since the first tensor factor is free over A. By a theorem of Watts, [W], since $R(M)$ is left exact and commutes with projective limits, it is representable by an A-module Q: $R(M) = Hom_A(Q, M)$

although in general$Q$ is not finitely generated. Now, if G_1, G_3 as above are represented by finitely generated A-modules Q_1, Q_3 and G_2 by an A-module Q_2, the exact sequence of sheaves induces an exact sequence of modules

$$Q_1 \to Q_2 \to Q_3 \to 0$$

and since Q_1, Q_3 are finitely generated, the same holds for Q_2.

Now we prove the assertion for all proper separated X_k by induction on the dimension of $Supp(G)$ (as before, A and H are fixed). Choosing an appropriate closed subscheme in X_k we can assume $Supp(G) = X_k$. Then applying Chow Lemma we find a projective subscheme $\widetilde{X}_k$ over k and a projective morpism $h : \widetilde{X}_k \to X_k$ which is an isomorphism over a dense open subset of X_k. Let $h_A : \widetilde{X}_A \to X_A$ be the morphism obtained by base change $Spec(A) \to Spec(k)$. By adjunction

$$Hom_{\widetilde{X}_A}(h_A^* H, (h^*(G) \otimes_k A) \otimes M) \simeq Hom_{X_A}(H, (h_* h^*(G) \otimes_k A) \otimes M)$$

so the coherent sheaf $h_* h^*(G)$ is representable. Since the kernel and the cokernel of $\phi : G \to h_* h^*(G)$ are zero on a dense open subset of X_k, by induction they are representable. Hence by above argument $Im(\phi) = Ker(h_* h^*(G) \to Coker(\phi))$ is representable and from the exact sequence $0 \to Ker(\phi) \to G \to Im(\phi) \to 0$ the sheaf G is also representable, as required. $\square$

Corollary 14 *Let E be a vector bundle on $U = X_A \setminus Z$ with $Z \in \Phi(A)$ and $Y \to U$ a closed subscheme in the total space of E over U. Then the functor $Sec(Y/U)$ on (Aff/A) which sends an A-algebra B to the set of sections $U_B = U \times_A B \to Y_B = Y \times_A B$ is represented by an affine scheme of finite type over A.*

Proof. First we deal with $Sec(E/U)$. By definition

$$Sec(E/U)(B) = Hom_{\mathcal{O}_U-alg}(Sym^\bullet(E^*), \mathcal{O}_U \otimes_A B) = Hom_{\mathcal{O}_U}(E^*, \mathcal{O}_U \otimes_A B)$$

If $j : U \to X_A$ is the open embedding, then by adjunction $Hom(j^*(\cdot), \cdot) = Hom(\cdot, j_*(\cdot))$ and the S_2 condition we have

$$Hom_{\mathcal{O}_U}(E^*, \mathcal{O}_U \otimes_A B) = Hom_{\mathcal{O}_{X_A}}(j_*(E), \mathcal{O}_{X_A} \otimes_A B) = Hom_A(Q, B) = Hom_{A-alg}(Sym^\bullet_A(Q), B)$$

where the second equality holds for some finitely generated A-module Q by the previous lemma. Thus $Sec(E/U)$ is represented by $Spec(Sym^\bullet_A(Q))$.

For a closed subscheme Y we can find a $\mathcal{O}_U$-coherent subsheaf N of $Sym^\bullet_{\mathcal{O}_U}(E^*)$ which generates the ideal subsheaf of Y as $Sym^\bullet_{\mathcal{O}_U}(E^*)$-module, e.g. by following the pattern of Proposition 9.6.5 in [EGA I]. A section $s : U_B \to E_B$ induces a homomorphism of $\mathcal{O}_U$-algebras $Sym^\bullet_{\mathcal{O}_U}(E^*) \to \mathcal{O}_U \otimes_A B$ and s factors through Y_B precisely when the restriction $\phi : N \to \mathcal{O}_U \otimes_A B$ vanishes. Fixing a coherent sheaf N' on X_A which restricts to N we get an isomorphism

$$Hom_{X_A}(N', \mathcal{O}_{X_A} \otimes B) \simeq Hom_U(N, \mathcal{O}_U \otimes_A B)$$

which follows by adjunction of j^*, j_* and $j_*(\mathcal{O}_U \otimes_A B) = \mathcal{O}_{X_A} \otimes B$. By Lemma ?? there exists a finitely generated A-module Q such that the above Hom groups can be identified with $Hom_A(Q, B)$. Denote the corresponding homomorphism $Q \to B$ by the same letter ϕ. Then for any B-algebra $B \to B'$ the induced section $s_{B'}$ corresponds to the composition of $\phi : Q \to B$ with $B \to B'$. It follows that $s_{B'}$ factors through $Y_{B'}$ presicely when $Ker(B \to B')$ contains the ideal generated by $\phi(Q) \subset B$. Therefore $Sec(Y/U)$ is a closed subfunctor of $Sec(E/U)$ and the assertion follows. $\square$.

Proposition 15 *The diagonal of F is representable, quasi-compact and separated.*

Proof. Although representability of the diagonal follows formally from the previou sresults, it is useful to establish it directly: if S is an algebraic space then a morphism $S \to F \times_k F$ corresponds to a pair of rank r bundles E_1, E_2 which we may assume to be defined on a common open subset $X_S \backslash Z$. Then the fiber product with the diagonal is the functor of isomorphisms $Isom(E_1, E_2)$. Although such isomorphisms correspond to sections with values in an *open* subset of the vector bundle $Hom(E_1, E_2)$, the isomorphism condition is equivalent to the nonvanishing of the determinant, i.e. the induced section of $L = Hom(\Lambda^r E_1, \Lambda^r E_2)$. Therefore the subscheme of isomorphisms is isomorphic to the closed subscheme in the total space of $Hom(E_1, E_2) \oplus L^*$, formed by all sections (ϕ, s) such that $\det(\phi)s = 1$. Now Corollary 14 gives representability in the case when S is affine, and uniqueness of the representing module Q from Lemma 13 in general case.

Quasi-compactness follows immediately from the fact that $Isom(E_1, E_2)$ is affine over S.

By valuative criterion, cf. Theorem 7.3 in [LM-B], separatedness reduces to the following fact: if R is a discrete valuation ring and (E_1, U_1), (E_2, U_2) two objects in $F(R)$ then $H^0(U_1 \cap U_2, \mathcal{H}om(E_1, E_2))$ is torsion free. Denote $U = U_1 \cap U_2$, $E = Hom(E_1, E_2)$ and let $t \in R$ be a local parameter. Then we need to show that $ts = 0$ for $s \in H^0(U, E)$ implies $s = 0$. This question is local on U hence we can assume that E is a trivial bundle and $U = Spec(B)$, for a flat R-algebra B. Then a short exact sequence $0 \to R \xrightarrow{t} R \to K \to 0$ gives $0 \to B \xrightarrow{t} B \to B \otimes_R K \to 0$, which proves the assertion. $\square$

Remark. Observe that by Lemma 13 and the proof of Corollary 14, for any pair of bundles E, F on $U = X_A \setminus Z$ the functor on A-modules, which sends M to $Hom_U(E, F \otimes_A M)$ is represented by a finitely generated A-module Q, unique up to canonical isomorphism: $Hom_U(E, F \otimes_A M) \simeq Hom_A(Q, M)$. Glueing such representing modules and applying generic freeness one immediately obtains an analogue of Proposition 2.2.3(i) in [Li] which eliminates further possible pathologies of the diagonal morphism.

8 Representability of principal bundles.

Proof of Theorem 1. To prove that $F_{GL(r)}$ is an algebraic stack, locally of finite type and separated over k, we just need to collect the results of the previous sections and compare with the conditions of Artin's representability criterion, cf. Theorem 5.3 in [Ar]. The "limit preserving" condition is proved in Section 3, Schlessinger's condition S1 in Section 4, while S2 is establihsed in Section 5.1. Effectiveness (condition (2) of Artin's criterion) is given by Section 3, while part (3) of *loc.cit.* is proved in Section 5.2. Finally local quasi-separation (part (4) of Artin's criterion) is established in a stronger form in Section 7.

For a general reductive group G over k we use a result of Haboush, cf. [Ha], and choose an exact finite dimensional representation $\rho : G \to GL(r)$ with $Y = GL(r)/G$ affine. Moreover, Y is isomorphic to a closed $GL(r)$-orbit of a vector in a finite dimensional rational $GL(r)$-module W.

Then each principal G-bundle P induces a principal $GL(r)$-bundle $E = P_\rho$. Conversely, for any principal $GL(r)$-bundle E over a scheme U its reduction of the structure group to G may be viewed as a regular section $U \to Y_E = E \times_{GL(r)} Y$. Moreover, the scheme Y_E which is affine and flat over

Y_E may be realized as a closed subscheme of a total space of a vector bundle W_E on U, induced from E via the homomorphism $GL(r) \to GL(W)$.

This construction and Corollary 14 shows that the morphism $F_G \to F_{GL(r)}$ is representable in the sense of Definition 3.9 in [LM-B] and since $F_{GL(r)}$ is an algebraic stack, by Proposition 4.5 (ii) of *loc.cit.* $F(G)$ is also an algebraic stack.

Alternatively, we can re-prove the results of Sections 3, 4 and 7 for F_G by using the proved facts for vector bundles and reducing the structure group from $GL(r)$ to G. The Effectiveness property of Section 6 is proved in [B] by a similar strategy. Finally, by Chapter VI of [I] the arguments of Section 5 carry over to G after a minor modification. Let $\Omega(G)$ be the space of G-invariant (from the right) differential forms on G, with the natural left G-action (in characetistic zero this is just the adjoint representation). For any G-bundle P denote by $ad(P)$ the vector bundle induced from P via the action homomorphism $G \to GL(\Omega(G))$. Then all arguments of Section 5 carry over if $Ext^i(End(E), \cdot)$ are replaced by $Ext^i(ad(P), \cdot)$ and $H^i(\cdot, End(E)_M)$ by $H^i(\cdot, ad(P)^\vee_M)$, where $ad(P)^\vee$ stands for the dual bundle. This finishes the proof of Theorem 1. $\square$

Remark. If we replace the "codimension 3" condition in the definition of $\Phi(T)$ (see Section 1), by "codimension 2", the stack F_G will not longer be algebraic. The most obvious reason is that the tangent space $H^1_{A_0,\Phi}(End(E))$ will no longer be finitely generated over A_0. On the other hand, the results of Sections 3, 4 and 7 remain valid while the Effectiveness of Section 6, which definitely fails by itself, may be repaired introducing an additional condition as it is done in [B]. It is conjectured by V. Drinfeld that in this case F_G is an inductive limit of algebraic stacks, locally of finite type over k. We plan to return to this topic, as well as the related Uhlenbeck functor, in future work.

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