Norm functors and effective zero cycles

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Abstract

We compare two known definitions for a relative family of effective zero cycles, based on traces and norms of functions, respectively. In characteristic zero we show that both definitions agree. In the general setting, we show that the norm map on functions can be expanded to a norm functor between certain categories of line bundles, therefore giving a third approach to families of zero cycles.

1 Introduction

Let us start with a simple situation of a quasi-projective scheme X over a perfect field k. We want to understand the notion of a family of degree d effective zero cycles parameterized by a k-scheme T. When $T = Spec(k)$ these are just finite formal linear combinations

$$\xi = \sum d_i x_i \tag{1}$$

of closed points $x_i \in X$ with $d_i \in \mathbb{Z}_{\geq 0}$. Such a cycle has degree $d = \sum_i d_i e_i$ where e_i is the degree of the field extension $k \subset k(x_i)$. We remark here that if $k \subset k(x_i)$ were not separable one would have to work with rational d_i having powers of $p = char\ k$ in denominators, cf. Section 8 of [R2].

To obtain a description which works over an arbitrary base T let f be a regular function defined on an open subset $U \subset X$ containing all x_i. Define the following elements in k

$$\theta(f) = \sum_i d_i \theta_i(f(x_i)); \qquad n(f) = \prod_i n_i(f(x_i))^{d_i}$$

where θ_i and n_i stands for the trace and the norm of the field extension $k \subset k(x_i)$, respectively.

When ξ varies with $t \in T$, denote $X_T = X \times_k T$ and let $\pi_T : X_T \to T$ be the canonical projection. The above construction gives *trace* and *norm* maps

$$\theta : (\pi_T)_* \mathcal{O}_{\widehat{X}_T} \to \mathcal{O}_T; \qquad n : (\pi_T)_* \mathcal{O}_{\widehat{X}_T} \to \mathcal{O}_T.$$

where $\widehat{X}_T$ is the completion of X_T along the closed subset Z swept out by the points $x_i(t)$. Observe that θ is a morphism of $\mathcal{O}_T$-modules, while n is just a multiplicative map. The values on functions pulled back from T are given by $\theta(f) = d \cdot f$ and $n(f) = f^d$. Both trace and norm should be continuous, i.e. factor through $(\pi_T)_*\mathcal{O}_Y$ for some closed subscheme $Y \hookrightarrow X$ with support in X. Both constructions should commute with base change $T' \to T$.

This suggests the idea that a family of zero cycles with base T can be defined by specifying an appropriate closed subset $Z \subset X_T$ and either a trace map θ or a norm map n, as above (the version with norm maps is apparently originally due to Grothendieck, who also applied it to non-effective cycles by restricting to multiplicative groups of $\mathcal{O}_{\widehat{X}_T}$ and $\mathcal{O}_T$). Observe that for a base change $T' \to T$ the trace map automatically pulls back to T', being a morphism of $\mathcal{O}_T$-modules, while with the norm map n we have to *specify* the pullback of n. In other words, we should have a system of maps

$$n_{T'} : (\pi_{T'})_*\mathcal{O}_{\widehat{X}_{T'}} \to \mathcal{O}_{T'}$$

for $T' \to T$, which agree with each other in a natural sense. This seems like an inconvenient detail. However, it turns out that one needs to impose further conditions on θ to get a trace map which comes from a geometric family of cycles (cf. definition after Lemma 2 in Section 3.1 below) while for n the existence of its extensions $n_{T'}$ plays the role of such a condition. In addition, the trace construction only works well when k has characteristic zero (or finite characteristic $p > d$).

The approach using traces was used in characteristic zero by B. Angeniol, cf. [An], and also by Buchstaber and Rees, cf. [BR]. Angeniol extends his definition to cycles of higher dimensions, which leads to a construction of the Chow scheme of cycles. In the affine case the norm approach was used by N. Roby in [Ro], but the global version of his construction was carried out only recently (more than 40 years later!) by D. Rydh in [R1]-[R3]. The latter author deals with a general situation of a separated morphism of algebraic spaces $\pi : X \to S$. He also considers higher dimensional cycles, using an old idea of Barlet that a family of n dimensional cycles over an l-dimensional base can be represented locally as a family of zero cycles over an $n + l$ dimensional base.

In the above construction one should take into account that θ and n could factor through a completion along a smaller closed subset $Z' \subset Z$. In the additive case, B. Angeniol formulates a non-degeneracy condition ensuring that such Z' does not exist. In the multiplicative case, D. Rydh simply considers pairs (Y, n) consisting of a closed subscheme Y and a norm map $n : (\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$, and then uses an equivalence relation which identifies (Y', n') and (Y'', n'') if n' and n'' factor through a norm map defined on a closed subscheme $Y \subset Y' \cap Y''$. In this paper we adopt the second approach, modifying and generalizing the trace definition.

In characterisic zero both functors of families of zero cycles are represented by the symmetric power $Sym^d(X/S)$, i.e. the quotient of the d-fold cartesian product of X over S by the action of the symmetric group Σ_d. In arbitrary characteristic the approach based on traces breaks down: e.g. we are not able to distinguish $\xi = x$ from $\xi = (p+1)x$, while the norms approach leads to the *space of divided powers* $\Gamma^d(X/S)$. There is a natural morphism $Sym^d(X/S) \to \Gamma^d(X/S)$ which is an isomorphism in characteristic zero, but in general only a universal homeomorphism, cf. [R1]. Even for general schemes over a field k (not necessarily quasi-projective) both $Sym^d(X/S)$ and $\Gamma^d(X/S)$ may not be schemes but rather algebraic spaces. Therefore, it is more natural to work in the category of algebraic spaces from the beginning.

The purpose of this paper is to explore a third approach to families of zero cycles, which admits a reasonably straightforward generalization to higher dimensional cycles. In the original setup of a scheme X over k, choose a line bundle L defined on an open subset $U \subset X$ containing all x_i in (1) and assume for simplicity that all $k(x_i)$ are equal to k. Define a one-dimensional vector space

$$N(L) = \prod L_{x_i}^{\otimes d_i}$$

over k. When ξ varies over a base T, this gives a line bundle $N(L)$ on T. Obviously, an isomorphism of line bundles on X induces an isomorphism of bundles on T. However, non-negativity of the coefficients d_i is reflected in the fact that any *any* morphism of invertible $\mathcal{O}_{\widehat{X}_T}$-modules $\psi : L \to M$ gives a morphism of $\mathcal{O}_T$-modules $N(\psi) : N(L) \to N(M)$. We can further consider the line bundles defined only on a neighborhood of Z. Thus, for a closed subscheme $Y \hookrightarrow X_T$ supported at Z we should have a *norm functor*

$$N : PIC(Y) \to PIC(T)$$

where PIC is the category of line bundles and morphisms as $\mathcal{O}$-modules. Again, this functor should come with functorial pullbacks with respect to morphisms of schemes (or algebraic spaces) $T' \to T$. In practice it suffices to restrict to those T' which are affine over T (or even to the full subcategory generated by affine spaces over T).

However, on morphisms the correspondence $\psi \mapsto N(\psi)$ is no longer $\mathcal{O}_T$-linear, but rather satisfies $N(\psi)(\pi_T^*(f)s) = f^d N(\psi)(s)$ where s is a local section of $N(L)$ and f is a local section of $\mathcal{O}_T$. Morphisms of modules with this property where also considered by N. Roby, cf. [Ro] where they are called *homogeneous polynomial laws of degree d*. As with norm maps, we should also specify a functorial extension of $\psi \mapsto N(\psi)$ with respect to base changes $T' \to T$. The fact that N is a functor means that $\psi \mapsto N(\psi)$ is multiplicative since compositions should go to compositions. In addition, N should agree with tensor products of line bundles and, similarly to identities:

$$\theta(\pi_T^*(f)) = df; \qquad n(\pi_T^*(f)) = f^d;$$

we should have an isomorphism of functors

$$\eta : N \circ \pi_T^* \simeq \{L \mapsto L^{\otimes d}\}$$

agreeing with base change. This rigidification also ensures that N does not have any non-trivial functor automorphisms.

Besides the generalization to higher dimensional cycles based on the work of F. Ducrot, R. Elkik and E. Muñoz-Garcia, cf. [Du], [El], [MG] this approach to zero cycles also can be used to define the Uhlenbeck compactification of moduli stack of vector bundles on a surface. The standard constructions like Hilbert-to-Chow morphism, sums of cycles and Chow forms are also rather simple in the language of norm functors.

Norms of line bundles were earlier considered in [$\mathrm{EGA_{II}}$] and [De]. More general norms of quasi-coherent sheaves were studied in [Fe] and [R2].

This work is organized as follows. In Section 2 we recall the basic results on polynomial laws, divided powers and norms for finite flat morphisms. In Section 3 we define the functor of families in terms of norms and traces and prove that the two definitions are equivalent when $d!$ is invertible. The norm definition is essentially the one given by D. Rydh in [R1] - in particular the corresponding functor is represented by the space of divided powers - while the trace definition is a generalization of the one given in [An]. We also obtain a formula for the tangent space to a point in the symmetric power, which appears to be new. In Section 4 we prove that divided powers of a line bundle give a line bundle, define norm functors and use them to formulate a third definition for families of zero cycles. We prove that it is equivalent to the definition via norm maps. Finally, in Section 5 we interpret in terms of norm functors such standard constructions as Hilbert-Chow morphism, sums and direct images of cycles, and Chow forms. Quite naturally, our descriptions are closely related to those of [R2].

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2 Preliminaries

2.1 Polynomial laws and divided powers.

We recall some definitions from [Ro], cf. also [La]. In this subsection all rings and algebras will be assumed commutative and with unity, although the theory can be developed in greater generality, cf. *loc. cit.* Let M, N be two modules over a ring A. Denote by $\mathcal{F}_M$ the functor

$$\mathcal{F}_M : A\text{-alg} \to Sets, \qquad A' \to A' \otimes_A M$$

where A-alg is the category of (commutative) A-algebras.

Definition A *polynomial law* from M to N is a natural transformation $F : \mathcal{F}_M \to \mathcal{F}_N$, i.e. for every A-algebra A' it defines a map $F_{A'} : A' \otimes_A M \to A' \otimes_A N$ and for any morphism $A' \to A''$ of A-algebras the natural agreement condition is satisfied. The polynomial law F is *homogeneous of degree* d is $F_{A'}(ax) = a^d F_{A'}(x)$ for any $a \in A'$ and $x \in A' \otimes_A M$. If B and C are A-algebras, then $F : \mathcal{F}_B \to \mathcal{F}_C$ is *multiplicative* if $F_{A'}(1) = 1$ and $F_{A'}(xy) = F_{A'}(x)F_{A'}(y)$ for $x, y \in B \otimes_A A'$.

Denote by $Pol^d(M, N)$ the set of homogeneous polynomial laws of degree d. By Thm IV.1 on p. 266 in [Ro], the functor $Pol^d(M, ?)$ is representable: there exists an A-module $\Gamma^d_A(M)$, called *module of degree d divided powers*, and an isomorphism of functors in N:

$$Pol^d(M, N) \simeq Hom_A(\Gamma^d_A(M), N) \tag{2}$$

Moreover, if B, C are A-algebras then each $\Gamma^d_A(B)$ is also an A-algebra and multiplicative laws in $Pol^d(B, C)$ correspond precisely to A-algebra morphisms $\Gamma^d_A(B) \to C$, cf. Theorem 7.11 in [La] or Proposition 2.5.1 in [Fe].

Explicitly, the direct sum $\Gamma_A(M) = \bigoplus_{d\geq 0} \Gamma^d_A(M)$ may be defined as a unital graded commutative A-algebra with product $\times$, degree d generators $\gamma^d(x)$, $x \in M, d \geq 0$ and relations

$$\gamma^0(x) = 1; \qquad \gamma^d(xa) = \gamma^d(x)a^d; \qquad \gamma^d(x) \times \gamma^e(x) = \binom{d+e}{e} \gamma^{d+e}(x);$$

$$\gamma^d(x+y) = \sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y)$$

In particular, $\Gamma^0_A(M) \simeq A$ and $\Gamma^1_A(M) \simeq M$ with $\gamma^1(x)$ given by x. We briefly summarize the properties of this construction

1. $\Gamma_A(\cdot)$ is a covariant functor from the category of A-modules to the category of graded A-algebras which commutes with base change $A \to A'$.

2. If B is an A-algebra, then the A-algebra $\Gamma^d_A(B)$ satisfies $\gamma^d(xy) = \gamma^d(x)\gamma^d(y)$ for any $x, y \in B$. Below we will also use a formula for arbitrary products which can be found in 2.4.2 of [Fe].

3. The map $\gamma^d : M \to \Gamma^d_A(M)$ is a homogeneous polynomial law of degree d. The isomorphism of (2) is obtained by composing an A-module homomorphism $\Psi_n : \Gamma^d_A(M) \to N$ with γ^d to obtain a polynomial law $n : M \to N$:
$$n = \Psi_n \circ \gamma^d$$

4. When M is flat over A or $d!$ is invertible in A, $\Gamma_A(M)$ is isomorphic to the algebra of symmetric tensors $TS_A(M)$, i.e. the subalgebra $\oplus_{d\geq 0}\left[T^d_A(B)\right]^{\Sigma_d}$ in the tensor algebra $T_A(B)$ equipped with the commutative shuffle product. In the second case we can further identify both algebras with the symmetric algebra $S_A(M)$ (i.e. the *quotient* of the tensor algebra $T_A(M)$ by the obvious relations).

5. If $F \in Pol^d(M, N)$ and we evaluate F at the A-algebra $A' = A[t_1, \dots t_k]$ then $F(t_1 m_1 + \dots t_k m_d) \in N[t_1, \dots, t_k]$ is a sum of degree d monomials in $t_1, \dots, t_k$ and the coefficient of $t_1^{\alpha_1} \dots t_k^{\alpha_k}$ is the value of the corresponding A-module homomorphism $\Psi_F : \Gamma^d_A(M) \to N$ at $\gamma^{\alpha_1}(x_1) \times \dots \times \gamma^{\alpha_k}(x_k)$. This explains the term "degree d homogeneous polynomial law".

2.2 Norms and traces for finite flat morphisms.

Let $\pi : Y \to S$ be a finite flat morphism of schemes or algebraic spaces and assume that $\pi_*\mathcal{O}_Y$ is locally free of constant rank d. We have a natural morphism of $\mathcal{O}_S$-modules

$$\pi_*\mathcal{O}_Y \to End_{\mathcal{O}_S}(\pi_*\mathcal{O}_Y)$$

and taking the composition with trace and determinant we obtain two maps

$$\theta : \pi_*\mathcal{O}_Y \to \mathcal{O}_S; \qquad n : \pi_*\mathcal{O}_Y \to \mathcal{O}_S$$

It is easy to see that θ is a morphism of $\mathcal{O}_S$-modules and that n extends to a homogeneous polynomial law of degree d and therefore defines a section $\sigma : S \to Spec(\Gamma^d_{\mathcal{O}_S}(\mathcal{O}_Y)) =: \Gamma^d(Y/S)$. For any line bundle L on Y we also define its *norm*

$$N(L) = Hom_{\mathcal{O}_S}(\Lambda^d(\pi_*\mathcal{O}_Y), \Lambda^d(\pi_*L)).$$

On the other hand, if $S = Spec(A)$, $Y = Spec(B)$ are affine and L is given by an invertible B-module M then $\Gamma^d_A(M)$ is naturally a $\Gamma^d_A(B)$-module, cf. [Ro], [La]. Using this construction locally on S in the general case, we obtain a quasi-coherent sheaf on $\Gamma^d(Y/S)$ which we denote by $\Gamma^d(L)$. One can show that $\Gamma^d(L)$ is an invertible $\mathcal{O}$-module whenever L is, see Section 4.1 below. We have the following important result

Lemma 1 *In the notation introduced above, $N(L) \simeq \sigma^*\Gamma^d(L)$. We also have canonical isomorphisms*

$$N(L \otimes_{\mathcal{O}_Y} F) \simeq N(L) \otimes_{\mathcal{O}_S} N(F); \qquad N(\pi^*(L)) \simeq L^{\otimes d}$$

where F is an invertible $\mathcal{O}_Y$-module.

Proof The first two assertions are proved in Proposition 3.3 in [Fe] (in fact, F can be a coherent sheaf on Y if the norm is understood as in loc. cit), and the third follows from the above definition of $N(L)$ and the projection formula. □

3 Families of zero cycles via norm and trace maps

3.1 Relation between traces and norms.

First assume that $S = Spec(A)$, $X = Spec(B)$ and $d!$ is invertible in A and fix a homogeneous polynomial law $n : B \to A$ of degree d. Then $\Gamma^d_A(B)$ is isomorphic to the algebra of symmetric tensors $TS^d_A(B)$, cf. [La]. On one hand, A-algebra homomorphisms $\Gamma^d_A(B) \to A$ correspond to homogeneous multiplicative polynomial laws $B \to A$ of degree d. On the other hand, homomorphisms $TS^d_A(B)$ are described by certain "trace morphisms" $\theta : B \to A$ of A-modules, cf. [An] and [BR] (in the latter paper they are called Frobenius n-homomorphisms). We briefly outline the relation between polynomial laws and trace morphisms. It does not seem to appear in the literature as explicitly as below, although many ingredients can be found in [An], [BR] and in Iversen's formalism of linear determinants, cf. [Iv].

The polynomial law n gives, in particular, a map $n_{A[t]} : B[t] \to A[t]$. Imitating the relationship between determinant and trace of a linear operator (compare also with Section 2.2) we define $\theta : B \to A$ as the coefficient of t in $n_{A[t]}(1 + bt)$. More generally, for $k \geq 1$ define a map

$$\Theta_k : B^{\times k} \to A$$

by sending $(b_1, \ldots, b_k)$ to the coefficient of $t_1 \ldots t_k$ in $n_{A[t_1,\ldots,t_k]}(1 + t_1 b_1 + \ldots + t_k b_k)$. In terms of the morphism of A-algebras $\Psi_n : \Gamma^d_A(B) \to A$ corresponding to n, by property (5) in Section 2.1 we have

$$\Theta_k(b_1, \ldots, b_k) = \Psi_n(\gamma^{d-k}(1) \times \gamma^1(b_1) \times \ldots \times \gamma^1(b_k))$$

Lemma 2 *For any degree d polynomial law n the maps Θ_k, $k \geq 1$ satisfy the following properties*

1. *$\Theta_k = 0$ for $k > d$;*
2. *$\Theta_1 = \theta$ and $\Theta_d(x, \ldots, x) = d!n(x)$;*
3. *Θ_k is symmetric in its arguments and A-linear in each of them, i.e. descends to an A-module morphism from the k-th symmetric power $S^k_A(B) \to A$;*

4. if, in addition, n is multiplicative then the following formula holds for all $k \geq 1$

$$\Theta_{k+1}(b_1, \ldots, b_{k+1}) :=$$

$$\theta(b_1)\Theta_k(b_2, \ldots, b_{k+1}) - \Theta_k(b_1 b_2, b_3, \ldots, b_{k+1}) - \Theta_k(b_2, b_1 b_3, \ldots, b_{k+1}) - \ldots - \Theta_k(b_2, b_3, \ldots, b_1 b_{k+1})$$

Proof. The properties (1), (2) immediately follow from the definitions and the identity $[\gamma^1(x)]^{\times d} = d!\gamma^d(x)$. In (3), symmetry also follows immediately from the definitions. Part (4) follows from multiplicativity of Ψ_n and the product formula, cf. 2.4.2 in [Fe]:

$$[\gamma^{d-1}(1) \times \gamma^1(b_1)][\gamma^{d-k}(1) \times \gamma^1(b_2) \times \ldots \times \gamma^1(b_{k+1})] =$$

$$\gamma^{d-k-1}(1) \times \gamma^1(b_1) \times \ldots \times \gamma^1(b_{k+1}) + \sum_{i=2}^{k+1} \gamma^{d-k}(1) \times \gamma^1(b_2) \times \ldots \times \gamma^1(b_1 b_i) \times \ldots \times \gamma^1(b_{k+1}).$$

Finally, A-multilinearity in (3) follows from the linearity of γ^1 and the linearity of Ψ_n. □

Definition. Let B be an A-algebra. A morphism of A-modules $\theta : B \to A$ is a *degree d trace* if

$$\theta(1) = d; \qquad \Theta_{d+1} \equiv 0.$$

where Θ_k are constructed from $\theta =: \Theta_1$ using the formula (4) in Lemma 2.

Remark. Since $\theta : B \to A$ admits an obvious A'-linear extension to $\theta_{A'} : A' \otimes_A B \to A'$ for any A'-algebra A, applying part (2) of the lemma, we see that the polynomial law n can be recovered from θ completely. Conversely, given a degree d trace $\theta : B \to A$, one can check that $n(x) = \frac{1}{d!}\Theta_d(x, \ldots, x)$ defines a polynomial law n. In fact, $n(x)$ is homogeneous of degree d by part (3) of Lemma 2 and multiplicative by Theorem 1.5.3 in [An] or Theorem 2.8 in [BR]. Since θ has canonical pullbacks $\theta \otimes 1 : B \otimes A' \to A'$ for all A-algebras A', so does n, i.e. we obtain a polynomial law.

Observe that we can also define θ as a map "tangent" to n, i.e. by considering the A-algebra $A' = A[\varepsilon]/\varepsilon^2$ and then using the identity

$$n_{A'}(1 + \varepsilon b) = 1 + \varepsilon\theta(b).$$

Lemma 3 *The operations $\theta(b) \mapsto n(b) = \frac{1}{d!}\Theta_d(b, \ldots, b)$ and $n(b) \mapsto \theta(b) = \Psi_n(\gamma^{d-1}(1) \times \gamma^1(b))$ define mutually inverse bijections between the set of degree d traces and the set of degree d norm maps.*

Proof. We have seen before that traces are sent to norm maps and the other way around. Let us show that the two constructions are mutually inverse to each other.

First assume that $n(x) = \frac{1}{d!}\Theta_d(x, \ldots, x)$ and let us show that the trace constructed from n coincides with the original $\theta = \Theta_1$. If Θ_k are defined from θ using formula (4) in Lemma 2 one can show that Θ_k are symmetric and multilinear (see e.g. Definition 1.3.1 in [An] where Θ_k are denoted by P^k_θ). The polynomial law $n(x)$ gives an A-algebra homomorphism $\Psi_n : \Gamma^d_A(B) \to A$ and

$$\Theta_d(b, \ldots, b) = \Psi_n(d!\gamma^d(b)) = \Psi_n(\gamma^1(b) \times \ldots \gamma^1(b))$$

Since $\gamma^1(b)$ is A-linear in b and $d!$ is invertible in A, by an easy polarization argument we conclude that $\Theta_d(b_1, \ldots, b_d) = \Psi_n(\gamma^1(b_1) \times \ldots \times \gamma^1(b_d))$. Using the recursive definition of Θ_k we get

$$\Theta_{k+1}(1, b_2, \ldots, b_{k+1}) = (d-k)\Theta_k(b_2, \ldots, b_{k+1})$$

and by descending induction on k we conclude that $\Theta_k(b_1, \ldots, b_k) = \Psi_n(\gamma^{d-1}(1) \times \gamma^1(b_1) \times \ldots \times \gamma^1(b_k))$. In particular,

$$\theta(b) = \Psi_{\frac{1}{d!}\Theta_d(b,\ldots,b)}(\gamma^{d-1}(1) \times \gamma^1(b)).$$

On the other hand, if we start with a polynomial law $n(x)$ and set $\theta(b) = \Psi_n(\gamma^{d-1}(1) \times \gamma^1(b))$ then Lemma 2 tells us that $n(b) = \frac{1}{d!}\Theta_d(b, \ldots, b)$, as required. □

Examples.

(1) Let $f_i : B \to A$ be A-algebra homomorphisms for $i = 1, \ldots, d$. Then the product $n = f_1 \ldots f_d$ is a degree d homogeneous polynomial law and $\theta = f_1 + \ldots + f_d$ is the degree d trace corresponding to it, while Θ_k is given by the k-th elementary symmetric function in the f_i (up to a scalar).

(2) Let $A = k$ be a field of characteristic p with $p = 0$ or $p > d$ and Q a k-point of $Spec(B)$ with $k(Q) = k$, corresponding to the evaluation homomorphism $B \to k$, $b \mapsto b(Q)$. Consider the polynomial law $b \mapsto b(Q)^d$ corresponding to the effective cycle $[dQ] \in Spec(\Gamma^d_k(B)) \simeq Sym^d(Spec(B))$. We have the following formula for the dual of the tangent space at $[dQ]$:

$$T^\vee_{[dQ]} \simeq \mathfrak{m}_Q/\mathfrak{m}_Q^{d+1}$$

In fact, by assumption on k an element of $T_{[dQ]}$ corresponds to a degree d trace

$$\theta = \theta' + \varepsilon\theta'' : B \to k[\varepsilon]/\varepsilon^2 = k \oplus \varepsilon k$$

with $\theta'(f) = d \cdot f(Q)$. Since $\theta(1) = d$, θ'' vanishes on the subspace of constants $k \subset B$ and therefore we can identify it with a linear function of $\mathfrak{m}_Q$. Let us show that the condition $\Theta_{d+1}(b_1, \ldots, b_{d+1}) = 0$ for all $b_i \in B$, is equivalent to $\theta''(\mathfrak{m}_Q^{d+1}) = 0$. In fact, since Θ_{d+1} is multilinear, we can assume that each b_i is either 1 or in $\mathfrak{m}_Q$. If at least one of the b_i is 1, by symmetry we can assume that $b_1 = 1$ and then $\theta(1) = d$ together with the formula (4) in Lemma 2 immediately give the vanishing of Θ_{d+1}. If all the arguments $b_1, \ldots, b_{d+1}$ are in $\mathfrak{m}_Q$, then it is easy to show by induction using the same formula, that $\Theta_{l+1}(b_1, \ldots, b_{l+1}) = (-1)^l l!\ \varepsilon\theta''(b_1 \ldots b_{l+1})$ with $l \geq 0$ and the usual convention $0! = 1$. In particular, $\Theta_{d+1}(b_1, \ldots, b_{d+1}) = (-1)^d d!\ \varepsilon\theta''(b_1 \ldots b_{d+1})$ and using our assumption on k again, we see that $\Theta_{d+1} \equiv 0$ if and only if θ'' descends to a linear function on $\mathfrak{m}_Q/\mathfrak{m}_Q^{d+1}$, which proves the assertion.

3.2 Functors of zero cycles.

Definitions.

(1) Let $\pi : X \to S$ be a separated morphism of algebraic spaces, cf. [Kn]. Let $Chow^n_{\pi,d}$ be a functor on the category of algebraic spaces over S, sending $T \to S$ to a set $Chow^n_{\pi,d}(T)$ of equivalence classes of pairs (Y, n), where $Y \hookrightarrow X_T$ is a closed algebraic subspace which is integral over T (i.e.

affine and such that locally over T every regular function on Y satisfies a monic polynomial with coefficients in $\mathcal{O}_T$), and $n : (\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$ is a multiplicative polynomial law of degree d. Two pairs (Y_1, n_1), (Y_2, n_2) are called equivalent if there is a third pair (Y, n), such that Y is an algebraic subspace in $Y_1 \cap Y_2$ which is integral over T, and n_i is equal to the composition of the natural morphism $(\pi_T)_*\mathcal{O}_{Y_i} \to (\pi_T)_*\mathcal{O}_Y$ with n, for $i = 1, 2$. The inverse image of (Y, n) with respect to an S-morphism $\phi : T' \to T$ is given by (Y', n') where $Y' = Y \times_T T'$ and n' is described on elements of an affine covering $T' = \cup Spec(A_i)$ by restricting n to those $\mathcal{O}_T$-algebras which factor through A_i.

(2) Let $\pi : X \to S$ be as before and assume that $d!$ defines an invertible regular function on S. Let $Chow^{\theta}_{\pi,d}$ be a functor on the category of algebraic spaces over S, given by equivalence classes of pairs (Y, θ), where $Y \hookrightarrow X_T$ is a closed algebraic subspace of X_T which is integral over T and $\theta : (\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$ is a degree d trace. Equivalence of such pairs is defined in a similar way. Note that for a pullback (Y', θ') with respect to $\phi : T' \to T$ we can define θ' simply as $\theta \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$.

Proposition 4 *Assume that $d!$ is an invertible regular function on S. There exists an isomorphism of functors $Chow^{n}_{\pi,d} \simeq Chow^{\theta}_{\pi,d}$.*

Proof. Follows immediately from Lemma 3 □

Remarks.

(1) The definition of $Chow^{n}_{\pi,d}$ is a restatement of Definition 3.1.1 in [R1]. Therefore, this functor is represented by the space of divided powers $\Gamma^d(X/S)$. The definition of $Chow^{\theta}_{\pi,d}$ is a version of Definition on page 7 of [An] applied to zero cycles, but stated here in greater generality.

(2) Let $n : B \to A$ be a degree d norm map. Then, following the Definition 2.1.5 in [R1] we define the characteristic polynomial of $b \in B$ by the formula

$$\chi_{n,b}(t) := n_{A[t]}(b - t) = \sum_{k=0}^{d} (-1)^k \Psi_n(\gamma^k(1) \times \gamma^{d-k}(b)) t^k \in A[t]$$

In the notation of Lemma 2 we have $\chi_{n,b}(t) = \sum_{k=0}^{d} (-1)^k \frac{\Theta_{d-k}(b,\ldots,b)}{(d-k)!} t^k$. Now let $J_n \subset B$ be the ideal generated by $\chi_{n,b}(b)$ for all $b \in B$, called the *Cayley-Hamilton ideal* of n. Then by Proposition 2.1.6 in [R1] the norm map n factors through the quotient B/J_n. Similarly, for a degree d trace $\theta : B \to A$ the Section 1.6 of [An] defines an ideal $J_\theta \subset B$ such that θ factors through B/J_θ.

We observe here that $J_n = J_\theta$ if n and θ are related by the bijection of Lemma 2. In fact, by Definition 1.6.2.2 of *loc. cit.* J_θ is generated by values of the polarized version of $\chi_{n,b}$ only Θ_{d-k} are defined based in θ, as in part (4) of Lemma 2 while in the case of J_n they are defined through n. Since $d!$ is assumed invertible in A the values of the polarized version generates the same ideal as values of $\chi_{n,b}$ itself.

(3) In [An] traces were defined on the completion $\widehat{X}_T$ at a closed subset $Z \subset X_T$ which is proper and of pure relative dimension zero over T. But by Corollary 1.6.3 in *loc. cit.*, a degree d trace $\theta : (\pi_T)_*\mathcal{O}_{\widehat{X}_T} \to \mathcal{O}_T$ descends to the subscheme Y given by $\mathcal{O}_{Y'}/\mathcal{J}_\theta$ which is integral over T. Therefore we can restrict to integral subschemes in the definition.

4 Families of zero cycles via norm functors

4.1 Divided powers of line bundles.

Let $\pi : X \to S$ be an affine morphism of algebraic spaces. For any quasi-coherent sheaf L on X, the sheaf $\Gamma^d_{\mathcal{O}_S}(\pi_* L)$ is a module over the $\mathcal{O}_S$-algebra $\Gamma^d_{\mathcal{O}_S}(\pi_*\mathcal{O}_X)$, cf. [La]. This gives a quasi-coherent sheaf $\Gamma^d(L)$ on $\Gamma^d(X/S)$.

We now recall a construction from Section 3 of [Fe] presenting it here in a sheafified version. Let L' and L'' be two quasi-coherent sheaves on X. There is a unique functorial morphism of $\mathcal{O}_S$-modules

$$\Gamma^d_{\mathcal{O}_S}(\pi_* L') \otimes_{\mathcal{O}_S} \Gamma^d_{\mathcal{O}_S}(\pi_* L'') \to \Gamma^d_{\mathcal{O}_S}(\pi_* L' \otimes_{\mathcal{O}_S} \pi_* L'')$$

which sends $\gamma^d(x)\otimes\gamma^d(y)$ to $\gamma^d(x\otimes y)$, cf. [Ro], [La], [Fe]. The image on a general element is given by formula 2.4.2 in [Fe]. The composition of this map with $\Gamma^d_{\mathcal{O}_S}(\pi_* L' \otimes_{\mathcal{O}_S} \pi_* L'') \to \Gamma^d_{\mathcal{O}_S}(\pi_*(L' \otimes_{\mathcal{O}_X} L''))$ descends to a morphism of $\Gamma^d_{\mathcal{O}_S}(\pi_*\mathcal{O}_X)$-modules

$$\Gamma^d_{\mathcal{O}_S}(\pi_* L') \otimes_{\Gamma^d_{\mathcal{O}_S}(\pi_*\mathcal{O}_X)} \Gamma^d_{\mathcal{O}_S}(\pi_* L'') \to \Gamma^d_{\mathcal{O}_S}(\pi_*(L' \otimes_{\mathcal{O}_X} L'')) \tag{3}$$

The following result does not seem to appear in the literature:

Lemma 5 *The morphism* (3) *is an isomorphism if at least one of the sheaves L', L'' is an invertible $\mathcal{O}_X$-module. In particular $\Gamma^d(L)$ is an invertible module on $\Gamma^d(X/S)$ if L is an invertible module on X. The map $L \mapsto \Gamma^d(L)$ extends to a functor $N : PIC(X) \to PIC(\Gamma^d(X/S))$ between the categories PIC of invertible modules (and morphisms as $\mathcal{O}$-modules). The functor N is equipped with isomorphisms $\Gamma^d(L') \otimes_{\mathcal{O}_{\Gamma^d(X/S)}} \Gamma^d(L'') \simeq \Gamma^d(L' \otimes_{\mathcal{O}_X} L'')$ which agree with commutativity and associativity isomorphisms for tensor product of line bundles. If $\pi^d : \Gamma^d(X/S) \to S$ is the canonical morphism then the induced map*

$$\pi_*\mathcal{H}om_{\mathcal{O}_X}(L', L'') \to \pi^d_*\mathcal{H}om_{\mathcal{O}_{\Gamma^d(X/S)}}(\Gamma^d(L'), \Gamma^d(L''))$$

extends canonically to a homogeneous polynomial law of degree d.

Proof. We here prove the first two assertions, since the statements about the PIC functor follow from a routine check based on the definitions involved.

To prove the isomorphism we can assume that $X = Spec(B)$ and $S = Spec(A)$ are affine and L' is given by a projective B-module P of rank 1. Observe that for any finite subset of points $x_1, \ldots, x_l$ there is an affine open subset $U \subset X$ containing these points, such that $L|_U$ is trivial. In fact, we can assume that no x_i is in the closure of another x_j (otherwise we can erase x_j from the list, since trivialization of L in a neighborhood of x_i will also give a trivialization for x_j). Since X is affine, we can choose sections $l_1, \ldots, l_d$ in $H^0(X, L)$, generating the stalks $L_{x_1}, \ldots, L_{x_d}$, respectively. Also, we can choose functions $f_1, \ldots, f_d$ in $H^0(X, \mathcal{O}_X)$ such that each (image of) f_i generates the stalk $\mathcal{O}_{x_i}$ and vanishes at x_j for $i \neq j$. Then the section $l = l_1 f_1 + \ldots + l_d f_d$ generates the stalks of L at $x_1, \ldots, x_d$ and hence defines a trivialization of L in an affine neighborhood U of $x_1, \ldots, x_d$.

Choose and fix a point $\beta \in \Gamma^d(X/S)$. It suffices to prove that (3) is an isomorphism in a neighborhood of α. By the results of Section 2 in [R1] one can find finitely many points $x_1, \ldots, x_l$ in X, with $l \leq d$ and a closed affine subscheme $Y \subset X$ supported on the union of x_i, such that β

is a point in the closed subscheme $\Gamma^d(Y/S)$. Choosing an open affine neighborhood U of $x_1, \ldots, x_l$ and a trivialization $L'|_U$ as above, we obtain a open affine neighborhood $\Gamma^d(U/S) \subset \Gamma^d(X/S)$ of β, and the trivialization of L' on U induces an isomorphism of $\Gamma^d(L')|_{\Gamma^d(U/S)}$ with the structure sheaf. Thus, on $\Gamma^d(U/S)$ the map (3) is an isomorphism and $\Gamma^d(L')$ is locally free of rank 1. Note that, once we know the isomorphism, the fact that $\Gamma^d(L')$ is invertible may also be proved by choosing L'' to be the dual of L'. □

Remarks.

(1) By equation (2.4.3.1) of [Fe] for any invertible $\mathcal{O}_S$-module P and any $\mathcal{O}_X$-module F one has

$$\Gamma^d(\pi^* P \otimes_{\mathcal{O}_X} F) \simeq (\pi^d)^*(P^{\otimes d}) \otimes_{\mathcal{O}_{\Gamma^d(X/S)}} \Gamma^d(F)$$

(2) Suppose that S can be covered by affine open subsets V_i such that L is trivial on $\pi^{-1}(V_i)$ (e.g. that we are in the situation of Section 2.2). Then $\Gamma^d(L)$ is trivial on the open subset $(\pi^d)^{-1}(V_i)$. If ϕ_{ij} are the transition functions for L then their norms $\gamma^d(\phi_{ij})$ are the transition functions of $\Gamma^d(L)$. This construction was originally given for the setting of Section 2 by Grothendieck in [$\mathrm{EGA}_{\mathrm{II}}$], 6.5.

4.2 Norm functors

Definitions.

(1) Let $\pi : Y \to S$ be a separable morphism of algebraic spaces. Denote by $\mathcal{PIC}(Y/S)$ the category with objects given by (S', L) where $S' \to S$ is an algebraic space over S and L is a line bundle on $Y_{S'} = Y \times_S S'$. A morphism $(\xi, \rho) : (S_1, L_1) \to (S_2, L_2)$ in the category $\mathcal{PIC}(Y/S)$ is given by a morphism $\xi : S_1 \to S_2$ of algebraic spaces over S, plus a morphism $\rho : L_1 \to \xi^*(L_2)$ of coherent sheaves on Y_{S_1}. There is an obvious forgetful functor $p_Y : \mathcal{PIC}(Y/S) \to \mathrm{Sp}/S$ to the category of algebraic spaces over S, given by $(T, L) \mapsto T$ and $(\xi, \rho) \mapsto \xi$. When $Y = S$ and π is the identity morphism we write $\mathcal{PIC}(S)$ instead of $\mathcal{PIC}(S/S)$. There is a natural functor $\mathcal{PIC}(S) \to \mathcal{PIC}(Y/S)$ given by pullback of L from S' to $Y_{S'}$. We denote this functor simply by π^*.

(2) Let $\pi : Y \to S$ be as above. A *norm functor of degree d over π* is a triple $\mathcal{N} = (N, \mu, \epsilon)$ where N is a functor $\mathcal{PIC}(Y/S) \to \mathcal{PIC}(S)$ such that $p_S \circ N = p_Y$. In other words, a pair $(S', L) \in Ob(\mathcal{PIC}(Y/S))$ is sent to a pair $(S', M) \in Ob(\mathcal{PIC}(S))$ and sometimes we will abuse notation by dropping S' and writing $M = N(L)$. Further, for any pair (S', L_1), (S', L_2) of objects in $\mathcal{PIC}(Y/S)$ with the same S' we require an isomorphism

$$\mu_{S', L_1, L_2} : N(L_1) \otimes_{\mathcal{O}_{S'}} N(L_2) \simeq N(L_1 \otimes_{\mathcal{O}_{Y_{S'}}} L_2)$$

such that the system of isomorphisms $\mu = \mu_{\{S', \cdot, \cdot\}}$ agrees with the the base cange and the standard symmetry and associativity isomorphisms for line bundles on $Y_{S'}$ and S', respectively (see e.g. the last two diagrams on p. 36 of [Du]). Finally ϵ is an isomorphism of functors $\mathcal{PIC}(S) \to \mathcal{PIC}(S)$:

$$\epsilon : N \circ \pi^* \simeq (\,\cdot\,)^{\otimes d}$$

such that $\mu_{\{S', \cdot, \cdot\}} \circ (N\pi^* \otimes N\pi^*)$ is given by the canonical isomorphism $L_1^{\otimes d} \otimes_{\mathcal{O}_{S'}} L_2^{\otimes d} \simeq (L_1 \otimes_{\mathcal{O}_{S'}} L_2)^{\otimes d}$.

(3) Let $\pi : X \to S$ be a separated morphism of algebraic spaces. Let $Chow^N_{\pi,d}$ be the functor from the category Sp/S of algebraic spaces over S to sets, sending $T \to S$ to equivalence classes of data $(Y, \mathcal{N})$ where $Y \hookrightarrow X_T$ is a closed algebraic subspace, integral over T, and $\mathcal{N}$ is a degree d norm functor over $(\pi_T)|_Y$. Two pairs $(Y_1, \mathcal{N}_1)$ and $(Y_2, \mathcal{N}_2)$ are called equivalent if there is a third subspace $Y \subset Y_1 \cap Y_2$ and a degree d norm functor $\mathcal{N}$ over $(\pi_T)|_Y$ together with isomorphisms between N_i and the composition of N with the restriction from Y_i to Y, which are further required to agree with ϵ_i and μ_i in the obvious sense.

Remark. Since by definition a norm functor is local on S, we obtain a map

$$(\pi_{S'})_*\mathcal{H}om_{\mathcal{O}_{Y_{S'}}}(L_1, L_2) \to \mathcal{H}om_{\mathcal{O}_{S'}}(N(L_1), N(L_2)) \tag{4}$$

This map is not $\mathcal{O}_{S'}$ linear; it is rather a polynomial law of degree d. To show this, it suffices to assume $L_1 = \mathcal{O}_{Y_{S'}}$. In fact, since by definition N preserves tensor products and sends the trivial bundle to the trivial bundle we have $N(L^\vee) = N(L)^\vee$. The left hand side can be rewritten as $(\pi_{S'})_*\mathcal{H}om_{\mathcal{O}_{Y_{S'}}}(\mathcal{O}_{Y_{S'}}, L_1^\vee \otimes L_2)$ while the right hand side becomes

$$\mathcal{H}om_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, N(L_1)^\vee \otimes N(L_2)) \simeq \mathcal{H}om_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, N(L_1^\vee \otimes L_2))$$

A local section f of $\mathcal{O}_{S'}$ acts on $(\pi_{S'})_*\mathcal{H}om_{\mathcal{O}_{Y_{S'}}}(\mathcal{O}_{Y_{S'}}, L_1^\vee \otimes L_2)$ by composition with the "multiplication by f" endomorphism of $\mathcal{O}_{Y_{S'}} \simeq \pi_{S'}^*\mathcal{O}_{S'}$. By definition of ϵ, the norm functor sends it to multiplication by f^d.

Lemma 6 *Let $\pi : Y \to S$ be an integral morphism of algebraic spaces with universally topologically finite fibers and let L be a line bundle on Y. Any point $s \in S$ has an etale neighborhood $U \subset S$ such that the restriction on L on $\pi^{-1}(U)$ is trivial.*

Proof. It suffices to assume that S, and hence also Y, are affine. Since the fiber $\pi^{-1}(s)$ is finite, repeating the argument in Lemma 5 we can find a section l of L on Y which generates the stalks of L at each of the points in $\pi^{-1}(s)$. The subset $W \subset Y$ of points where l *fails* to generate the stalk of L is closed in Y and disoint from the fiber $\pi^{-1}(s)$. Its image $\pi(W)$ is closed in Y since π is integral, and does not contain s. Hence s admits an affine Zariski neighborhood $U \subset (Y \setminus \pi(W))$ such that on $\pi^{-1}(U)$ the line bundle L is trivialized by the section l. $\square$

Lemma 7 *In $\pi : Y \to S$ is integral, then any norm functor has no nontrivial automorphisms.*

Proof. A functor automorphism is given by a family of isomorphisms $\phi_{(T,L)} : N(L) \to N(L)$ for all objects (T, L) of $\mathcal{PIC}(Y)$. If L is pulled back from T this automorphism has to be identity since it has to respect ϵ. By the previous lemma, we can find an etale open cover $\{U_i\}$ of T, such that L is trivial over each U_i. Then the restriction of $\phi_{(T,L)}$ to each U_i is the identity due to the agreement with ϵ hence $\phi_{(T,L)}$ is itself identity. $\square$

Remark. For a general π the previous result fails. One possible example is the situation when Y and S are over a field k, $Y = Y_0 \times_{Speck} S$ and there exists a non-trivial group homomorphism $Pic(Y_0) \to \mathcal{O}_S^*$ (where Pic is the group of isomorphism classes of line bundles on Y_0).

Proposition 8 *The functor* $Chow^N_{\pi,d}$ *is isomorphic to* $Chow^n_{\pi,d}$ *and is therefore represented by the space of divided powers* $\Gamma^d(X/S)$.

Proof. An S-morphism $T \to \Gamma^d(X/S)$ induces a norm functor by Lemma 5 by using the pullback of line bundles $\Gamma^d(L)$ to T. Conversely, taking $L_1 = L_2 = \mathcal{O}_Y$ in (4) we obtain a norm map $(\pi_T)_*\mathcal{O}_{Y_T} \to \mathcal{O}_T$.

It is obvious that $Chow^n_{\pi,d} \to Chow^N_{\pi,d} \to Chow^n_{\pi,d}$ is identity since we are essentially expanding the data involved in the definition of $Chow^n_{\pi,d}$ and then forgetting the extra data constructed.

In the opposite direction, suppose we have a closed subspace $Y \subset X_T$ and a norm functor $\mathcal{N} = (N, \mu, \epsilon)$ over $(\pi_T)|_Y$ which we use to extract the polynomial law $(\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$ and thus obtain an S-morphism $\sigma : T \to \Gamma^d(Y/S)$. We need to construct isomorphisms $N(L) \simeq \sigma^*(\Gamma^d(L))$ for all line bundles L, which commute with pullbacks, agree with multiplicativity isomorphisms and give identity on $L^{\otimes d}$ on bundles pulled back from T to Y. In other words, we need to prove an isomorphism

$$\mathcal{O}_S \otimes_{\Gamma^d_{\mathcal{O}_S}((\pi_T)_*\mathcal{O}_Y)} \Gamma^d_{\mathcal{O}_S}((\pi_T)_*L) \simeq N(L)$$

It suffices to construct a morphism from the left hand side to the right hand side and then apply Lemma 5 to find a Zariski open covering $\{U_i\}$ of T such that L is trivial on the preimage of U_i, in which case the isomorphism becomes a tautology. To that end, observe that (4) gives a polynomial law $(\pi_T)_*L \to N(L)$ hence a morphism of $\mathcal{O}_S$-modules

$$\mu_L : \Gamma^d_{\mathcal{O}_S}((\pi_T)_*L) \to N(L)$$

The fact that μ_L descends to the above tensor product is equivalent to the formula

$$\mu_L(f \cdot s) = \mu_{\mathcal{O}_Y}(f) \cdot \mu_L(s) \tag{5}$$

where f, resp. s, is a local section of $\Gamma^d_{\mathcal{O}_S}((\pi_T)_*\mathcal{O}_Y)$, resp. $\Gamma^d_{\mathcal{O}_S}((\pi_T)_*L)$, and the module structure of the left hand side is given e.g. by formula 7.6.1 in [La]. But (5) follows from the fact that N is a functor, i.e. maps compositions of morphisms to compositions of morphisms, and the fact that after a faithfully flat base change the $\mathcal{O}_T$-module $\Gamma^d_{\mathcal{O}_T}((\pi_T)_*L)$ is generated by $\gamma^d((\pi_T)_*L)$.

Agreement of $\sigma^*(\Gamma^d(L)) \simeq N(L)$ with multiplicativity isomorphisms and normalization on line bundles pulled back from T, also follows from the functor property of N. □

4.3 Non-homonegeous norm functors

One can also give a definition of a non-homogeneous norm functor as a triple (N, μ, ϵ) where N and μ are as before and ϵ is an isomorphism

$$\epsilon : N(\mathcal{O}_Y) \simeq \mathcal{O}_S$$

which sends the identity endomorphism of $\mathcal{O}_Y$ to the identity endomorphism of $\mathcal{O}_S$.

Observe that the proof of Lemma 6 still works in this case and hence non-homogeneous norm functors form a set. As in the homogeneous case, any such functor gives a multiplicative polynomial law

$$\pi_*\mathcal{O}_Y \to \mathcal{O}_S$$

and hence by Section 2 of [R2] it defines a section

$$S \to \Gamma^\star(Y/S) = \coprod_{d \geq 0} \Gamma^d(Y/S).$$

Repeating the argument of the previous subsection one shows that the functor of zero cycles $Chow^N_{\pi,\star}$ defined via non-honogeneous norm functors is isomorphic to the functor of zero cycles defined via non-homogeneous norm maps $Chow^n_{\pi,\star}$. Therefore $Chow^N_{\pi,\star}$ is represented by the space of effective zero cycles $\Gamma^\star(X/S)$. Details are left to the motivated reader.

5 Standard constructions

Hilbert-Chow morphism.

If $\pi : Y \to S$ finite and flat and $\pi_* \mathcal{O}_Y$ is locally free of constant rank d on S the construction of section 2.2 gives the norm of a line bundle L on Y. Lemma 1 shows that the norm of line bundles defines a norm functor $\mathcal{P}IC(Y/S) \to \mathcal{P}IC(S)$ inducing a morphism of representing spaces:

$$Hilb^d(X/S) \to Chow^N_{\pi,d}(X/S).$$

Sums of cycles

If $(Y_1, \mathcal{N}_1)$ and $(Y_2, \mathcal{N}_2)$ are two families of zero cycles of degrees d_1 and d_2, respectively, then their sum has degree $d_1 + d_2$ and is given by $(Y_1 \cup Y_2, (\mathcal{N}_1 \circ i_1^*) \otimes (\mathcal{N}_2 \circ i_2^*))$ where $(i_l)_*$ is the functor defined by restriction of line bundles from $Y = Y_1 \cup Y_2$ to Y_l for $l = 1, 2$. This induces the sum morphism

$$\pi_{d_1,d_2} : \Gamma^{d_1}(X/S) \times_S \Gamma^{d_2}(X/S) \to \Gamma^{d_1+d_2}(X/S)$$

Universal family.

For $T = \Gamma^d(X/S)$ consider the base change morphism $\pi_T : X_T \to T$. Set $Y = \Gamma^{d-1}(X/S) \times_S X$ which maps to T via the addition morphism $\pi_{d-1,1}$. By [R2], Y is integral over T and can be identified with a closed subspace of $\Gamma^d(X/S) \times_S X$ via the morphism $(\xi, x) \mapsto (\xi + x, x)$. There does not seem to be a quick definition of the corresponding universal norm functor $N : \mathcal{P}IC(Y/T) \to \mathcal{P}IC(T)$, as is also the case with the universal norm map $(\pi_{d-1,1})_* \mathcal{O}_Y \to \mathcal{O}_T$. However, if $\eta : \Gamma^{d-1}(X/S) \times_S X \to X$ is the canonical projection, it follows easily that the composition

$$\mathcal{P}IC(X) \xrightarrow{\eta^*} \mathcal{P}IC(Y/T) \xrightarrow{N} \mathcal{P}IC(T) = \mathcal{P}IC(\Gamma^d(X/S))$$

is given simply by the functor $L \mapsto \Gamma^d(L)$.

Direct image of cycles.

Let $\pi' : Z \to S$ be another separated morphism of algebraic spaces and $f : X \to Z$ is a morphism over S. Take a faimily of zero cycles on X is represented by a pair $(Y, \mathcal{N})$. By Section 2 of [R1] we can assume that Y is has univerally topologically finite fibers over S and hence by the appendix to *loc. cit.* $f(Y)$ is a well-defined algebraic subspace of Z which is integral over S. One can also give a more direct proof of this result using the approximation results (Theorem D) in [R4]. The direct image cycle is defined by $(f(Y), \mathcal{N} \circ f^*)$, which induces a morphism

$$Chow^N_{\pi,d}(X/S) \to Chow^N_{\pi',d}(Z/S)$$

Chow forms.

Assume that $X = Proj(\mathcal{A})$ where $\mathcal{A} = \bigoplus_{l \geq 0} \mathcal{A}_l$ is a graded $\mathcal{O}_S$-algebra generated over $\mathcal{A}_0 = \mathcal{O}_S$ by its first component $\mathcal{A}_1$. Then the natural sheaf $\mathcal{O}(1)$ on X is invertible.

Let $(Y, \mathcal{N})$ be a pair representing an element in $Chow^N_{\pi,d}(T)$ with $\xi : T \to S$ and denote the inverse image of $\mathcal{O}(1)$ on Y by L. By assumption a local section of $\xi^*\mathcal{A}_l$ on $U \subset T$ gives a section of $L^{\otimes l}$ on $\pi_T^{-1}(U)$ and hence by the Remark in Section 4.2 a section of $N(L^{\otimes l}) \simeq N(L)^{\otimes l}$ on U itself. Therefore we obtain a degree d homogenous polynomial law

$$\xi^*\mathcal{A}_l \to N(L)^{\otimes l}$$

and therefore a morphism of $\mathcal{O}_S$-modules

$$\Omega_l : \Gamma^d_{\mathcal{O}_T}(\xi^*\mathcal{A}_l) \to N(L)^{\otimes l}$$

which we call the l-th *Chow form* of $(Y, \mathcal{N})$. It is easy to see that for any point $t \in T$ a local section ϕ of $\mathcal{A}_l$ which does not vanish at t gives a local section of $N(L)^{\otimes l}$ which does not vanish at t. Therefore Ω_l is a surjective morphism of sheaves. Moreover, by multiplicativity of N for a section ϕ_l of $\mathcal{A}_k$ and a section ϕ_m of $\mathcal{A}_m$ we have equality

$$\Omega_{m+l}(\phi_l\phi_m) = \Omega_l(\phi_l) \otimes \Omega_m(\phi_m)$$

of local sections of $N(L)^{\otimes(m+l)}$. Therefore we obtain a surjective morphism of $\mathcal{O}_S$-algebras

$$\xi^*\Big(\bigoplus_{l\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_l)\Big) \simeq \bigoplus_{l\geq 0} \Gamma^d_{\mathcal{O}_T}(\xi^*\mathcal{A}_l) \to \bigoplus_{l\geq 0} N(L)^{\otimes l}$$

hence an S-morphism

$$T \to Proj(\bigoplus_{l\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_l))$$

Lemma 9 *In the situation described above, $Proj(\bigoplus_{l\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_l)) \simeq \Gamma^d(X/S)$. Moreover, if $l \geq r(d-1)$ then $\Gamma^d(X/S)$ is isomorphic to a closed subscheme of $\mathbb{P}(\Gamma^d(\mathcal{A}_l))$. When S is a scheme over $\mathbb{Q}$, the assertion holds for any $l \geq 1$.*

Proof. See Corollary 3.2.8 and Proposition 3.2.9 in [R3]. □

Corollary 10 *A family of cycles $(Y, \mathcal{N})$ is uniquely determined by its l-th Chow form $\Omega_l : \Gamma^d_{\mathcal{O}_T}(\xi^*A_l) \to N(L)^{\otimes l}$ where l is given by the previous lemma.*

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