GERSTENHABER-BATALIN-VILKOVISKI STRUCTURES ON COISOTROPIC INTERSECTIONS

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Abstract. Let Y, Z be a pair of smooth coisotropic subvarieties in a smooth algebraic Poisson variety X. We show that any data of first order deformation of the structure sheaf $\mathcal{O}_X$ to a sheaf of noncommutative algebras and of the sheaves $\mathcal{O}_Y$ and $\mathcal{O}_Z$ to sheaves of right and left modules over the deformed algebra, respectively, gives rise to a Batalin-Vilkoviski algebra structure on the Tor-sheaf $\mathscr{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$. The induced Gerstenhaber bracket on the Tor-sheaf turns out to be canonically defined; it is independent of the choices of deformations involved. There are similar results for Ext-sheaves as well.

Our construction is motivated by, and is closely related to, a result of Behrend-Fantechi [BF], who considered the case of Lagrangian submanifolds in a symplectic manifold.

1. Introduction

1.1. **Main result.** Let $\mathbb{C}$ be a field of characteristic zero. We denote by $\mathbb{C}_\varepsilon := \mathbb{C}[\varepsilon]/(\varepsilon^2)$ the ring of dual numbers and assume that all unlabeled tensor products stand for $\otimes_{\mathbb{C}}$.

We fix X, a smooth complex algebraic Poisson variety. Write $\mathcal{O}_X$ for the structure sheaf, resp. T_X for the tangent sheaf, and $P \in H^0(X, \wedge^2 T_X)$ for the Poisson bivector. Let $\mathcal{A}$ be a sheaf of (not necessarily commutative) $\mathbb{C}_\varepsilon$-algebras equipped with an algebra isomorphism $\mathcal{A}/\varepsilon\mathcal{A} \xrightarrow{\sim} \mathcal{O}_X$ so that $\mathcal{A}$ gives a flat deformation of the structure sheaf $\mathcal{O}_X$. We require, in addition, that the resulting Poisson bracket $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ induced by the commutator in $\mathcal{A}$ be given by the formula $\{f, g\} = \langle P, df \wedge dg\rangle$. A particular example of such a deformation is the sheaf $\mathcal{A} := \mathbb{C}_\varepsilon \otimes \mathcal{O}_X = \mathcal{O}_X \oplus \varepsilon\mathcal{O}_X$, equipped with multiplication given by the well-known formula $f \times g \mapsto f * g = fg + \frac{\varepsilon}{2}\{f, g\}$, for any $f, g \in \mathcal{O}_X$.

Let $Z \subset X$ be a smooth subvariety and $\mathcal{O}_Z$ the structure sheaf of Z. In this paper, we are interested in flat deformations of $\mathcal{O}_Z$, viewed as an $\mathcal{O}_X$-module supported on Z, to either left or right $\mathcal{A}$-module $\mathcal{C}$ set theoretically supported on Z. The flatness assumptions imply that multiplication by ε induces an isomorphism $\mathcal{O}_Y = \mathcal{C}/\varepsilon\mathcal{C} \xrightarrow{\sim} \varepsilon\mathcal{C}$, and similar isomorphisms $\mathcal{O}_X \xrightarrow{\sim} \varepsilon\mathcal{A}$.

We will assume that such a deformation $\mathcal{C}$ admits local k-linear splittings (in the Zariski topology) $\mathcal{C} \simeq (\mathcal{C}/\varepsilon\mathcal{C}) \oplus \varepsilon\mathcal{C}$ such that $(a \oplus 0) * (c \oplus 0) = ac \oplus \varepsilon\alpha(a, c)$ where $\alpha(a, c)$ is an algebraic differential operator in each of its arguments (which satisfies an associativity condition recalled in Section 2). If X is covered by affine open subsets U_i and on each of them a splitting is chosen as above, then on a double intersection $U_i \cap U_j$ the two splittings differ by an automorphism

$$c_1 \oplus \varepsilon c_2 \mapsto c_1 \oplus \varepsilon(\psi_{ij}(c_1) + c_2)$$

where each $\psi_{ij} : \mathcal{O}_Y \to \mathcal{O}_Y$ may be shown to be an algebraic differential operator. The gluing condition for the locally defined $\alpha(a, c)$ reads as follows:

$$\alpha_i(a, c) - \alpha_j(a, c) = a\psi_{ij}(c) - \psi_{ij}(ac).$$

Conversely, a covering of X, a set of operators ψ_{ij} subject to $\psi_{ij} + \psi_{jk} = \psi_{ik}$, and a collection of α_i describing a deformation of $\mathcal{O}_Y|_{U_i}$ to a left module over $\mathcal{A}|_{U_i}$, define a left $\mathcal{A}$-module $\mathcal{C}$ if the above gluing conditions are satisfied.

It follows that for every such $\mathcal{C}$ there is a *transposed* deformation $\mathcal{C}^t$, which is a right $\mathcal{A}$-module, if we set $\psi^t_{ij} = -\psi_{ij}$ and $\alpha^t(c,a) = -\alpha(a,c)$, i.e.

$$(c \oplus 0) * (a \oplus 0) = ac \oplus -\varepsilon\alpha(a,c).$$

The minus signs appear since the opposite algebra $\mathcal{A}^{op}$ may be viewed as a deformation coming from the bivector $-P$.

Next, let $Y, Z \subset X$ be a pair of smooth subvarieties. Then, $Y \cap Z$, a *scheme theoretic* intersection, is a closed subscheme of X with structure sheaf $\mathcal{O}_{Y\cap Z} := \mathcal{O}_Y \bigotimes_{\mathcal{O}_X} \mathcal{O}_Z$. More generally, we have $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$, a coherent sheaf of supercommutative graded $\mathcal{O}_{Y\cap Z}$-algebras. Similarly one has a sheaf $\mathscr{E}xt^{\bullet}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ that comes equipped with the natural structure of a graded $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$-module (the module structure is recalled in Section 3.2).

Recall that a graded commutative algebra $D = \bigoplus_{k\geq 0} D_k$ equipped with an operator $\delta : D_{\bullet} \to D_{\bullet -1}$ is called a Batalin-Vilkovisky (BV) algebra if δ is a differential operator of order ≤ 2 (with respect to multiplication in D) and one has $\delta^2 = 0$. In this case, the formula

$$[x, y] := \delta(x\cdot y) - \delta(x)\cdot y - (-1)^{\deg x} x\cdot\delta(y), \tag{1.1.1}$$

provides $D_{\bullet}$ with a structure of Gerstenhaber algebra (i.e., odd Poisson algebra). See e.g. [BF] for more details on these definitions.

Similarly, given a graded D-module $M = \bigoplus_{k\geq 0} M_k$, a BV-module structure on M is the data of a linear operator $\delta' : M_{\bullet} \to M_{\bullet -1}$ such that $(\delta')^2 = 0$ and such that δ' has order ≤ 2 in the sense that for any homogeneous $x, y \in D$ and $m \in M$, the following equation holds

$$\delta(xy)m - (-1)^{\deg x} x\delta(y)m - \delta(x)ym$$
$$= \delta'(xym) - (-1)^{\deg x} x\delta'(ym) - (-1)^{\deg x \deg y + \deg y} y\delta'(xm) + (-1)^{\deg x + \deg y} xy\delta'(m).$$

In such a case, an analogue of formula (1.1.1) (for δ' instead of δ) gives a pairing $\{-,-\} : D \otimes M \to M$ that makes M a Gerstenhaber module over D.

The main result of this paper, which is our attempt to understand a construction of Behrend-Fantechi [BF], reads

Theorem 1.1.2. *Let Y, Z be a pair of smooth coisotropic subvarieties in a smooth Poisson variety X. Then,*

(i) *Associated with the data of a flat $\mathbb{C}_\varepsilon$-deformation of the sheaf $\mathcal{O}_Y$ to a right $\mathcal{A}$-module $\mathcal{B}$ and of the sheaf $\mathcal{O}_Z$ to a left $\mathcal{A}$-module $\mathcal{C}$, there is a second order differential operator $\delta : \mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathscr{T}or^{\mathcal{O}_X}_{\bullet -1}(\mathcal{O}_Y, \mathcal{O}_Z)$ that provides the algebra $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$ with a structure of BV algebra, i.e., we have $\delta^2 = 0$.*

(ii) *The induced bracket* (1.1.1) *on $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$ is independent of the choice of deformations $\mathcal{B}$ and $\mathcal{C}$. Moreover, flat deformations exist locally which provides $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$ with a* canonical *structure of Gerstenhaber algebra depending only on the Poisson bivector on X.*

(iii) *Similarly, given an additional flat $\mathbb{C}_\varepsilon$-deformation of $\mathcal{O}_Z$ to a* right *$\mathcal{A}$-module $\mathcal{C}'$, there is an associated second order differential operator $\delta' : \mathscr{E}xt^{\bullet}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathscr{E}xt^{\bullet -1}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$, such that $(\delta')^2 = 0$. If $\mathcal{C}' = \mathcal{C}^t$ the corresponding operator δ' provides the sheaf $\mathscr{E}xt^{\bullet}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ with a structure of BV-module over the BV algebra $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$. Moreover, the resulting pairing*

$$\{-,-\} : \mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z) \times \mathscr{E}xt^{\bullet}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathscr{E}xt^{\bullet}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$$

is independent of the choice of deformations $\mathcal{B}$ and $\mathcal{C}$. Thus, it provides $\mathscr{E}xt^{\bullet}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ with a canonical *structure of Gerstenhaber module over the Gerstenhaber algebra $\mathscr{T}or^{\mathcal{O}_X}_{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$.*

Several examples of such BV and Gerstenhaber structures are discussed in §5 below.

1.2. **Construction of the BV differential.** Let $\mathcal{A}$ be any flat $\mathbb{C}_\varepsilon$-deformation of the sheaf $\mathcal{O}_X$ to a sheaf of associative $\mathbb{C}_\varepsilon$-algebras equipped with an algebra isomorphism $\mathcal{A}/\varepsilon\mathcal{A} \simeq \varepsilon\mathcal{A}$. Similarly assume that $\mathcal{O}_Y$ admits a flat deformation $\mathcal{B}$ to a right $\mathcal{A}$-module and $\mathcal{O}_Z$ has a flat deformation $\mathcal{C}$ to a left $\mathcal{A}$-module.

The short exact sequence $0 \to \varepsilon\mathcal{C} \to \mathcal{C} \to \mathcal{C}/\varepsilon\mathcal{C} \to 0$ induces a long exact sequence

$$\ldots \to \mathscr{T}or_{i+1}^{\mathcal{A}}(\mathcal{B}, \mathcal{C}/\varepsilon\mathcal{C}) \to \mathscr{T}or_i^{\mathcal{A}}(\mathcal{B}, \varepsilon\mathcal{C}) \to \mathscr{T}or_i^{\mathcal{A}}(\mathcal{B}, \mathcal{C}) \to \mathscr{T}or_i^{\mathcal{A}}(\mathcal{B}, \mathcal{C}/\varepsilon\mathcal{C}) \to \ldots.$$

Locally, we can choose a projective resolution $P^\bullet$ of $\mathcal{B}$ with $\mathcal{A}$-modules, such that $P^\bullet/\varepsilon P^\bullet$ is a resolution of $\mathcal{O}_Y$ with projective $\mathcal{O}_X$-modules. Further, we have an isomorphism of functors $(\cdot) \otimes_{\mathcal{A}} \varepsilon\mathcal{C} \simeq (\cdot) \otimes_{\mathcal{A}} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ and similarly for $\mathcal{C}/\varepsilon\mathcal{C}$. We deduce canonical isomorphisms

$$\mathscr{T}or_i^{\mathcal{A}}(\mathcal{B}, \varepsilon\mathcal{C}) \simeq \mathscr{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \simeq \mathscr{T}or_i^{\mathcal{A}}(\mathcal{B}, \mathcal{C}/\varepsilon\mathcal{C})$$

Denote by

$$\delta : \mathscr{T}or_{i+1}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathscr{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$$

the resulting connecting morphism. Similarly, suppose that we have a deformation $\mathcal{C}'$ of $\mathcal{O}_Z$ to a *right* $\mathcal{A}$-module. Then there is a long exact sequence

$$\ldots \to \mathscr{E}xt_{\mathcal{A}}^{i-1}(\mathcal{B}, \mathcal{C}'/\varepsilon\mathcal{C}') \to \mathscr{E}xt_{\mathcal{A}}^i(\mathcal{B}, \varepsilon\mathcal{C}') \to \mathscr{E}xt_{\mathcal{A}}^i(\mathcal{B}, \mathcal{C}') \to \mathscr{E}xt_{\mathcal{A}}^i(\mathcal{B}, \mathcal{C}'/\varepsilon\mathcal{C}') \to \ldots$$

In particular, one has a morphism

$$\delta' : \mathscr{E}xt_{\mathcal{A}}^i(\mathcal{B}, \varepsilon\mathcal{C}') \simeq \mathscr{E}xt_{\mathcal{O}_X}^i(\mathcal{O}_Y, \mathcal{O}_Z) \longrightarrow \mathscr{E}xt_{\mathcal{O}_X}^{i+1}(\mathcal{O}_Y, \mathcal{O}_Z) \simeq \mathscr{E}xt_{\mathcal{A}}^{i+1}(\mathcal{B}, \mathcal{C}'/\varepsilon\mathcal{C}')$$

When both $\mathcal{C}$ and $\mathcal{C}'$ are given we will assume that $\mathcal{C}' = \mathcal{C}^t$ or $\mathcal{C}'$ is *transposed to* $\mathcal{C}$. See Section 3.2 regarding the canonical product on Tor and its action on Ext.

1.3. **A conjecture by physicists.** Let X be a smooth algebraic symplectic manifold. Kapustin and Rozansky propose the following

Conjecture 1.3.1. To each pair $Y, Z \subset X$, of smooth Lagrangian submanifolds, one can associate a triangulated category $\mathscr{C}at_X(Y, Z)$ with the following properties:

- The Hochschild homology, resp. cohomology, of the category $\mathscr{C}at_X(Y, Z)$ are given by

$$HH^\bullet(\mathscr{C}at_X(Y,Z)) \cong \mathrm{Tor}_\bullet^X(\mathcal{O}_Y, \mathcal{O}_Z), \quad HH_\bullet(\mathscr{C}at_X(Y,Z)) \cong \mathrm{Ext}_X^\bullet(\mathcal{O}_Y, \mathcal{O}_Z).$$

- The standard Connes differential on $HH_\bullet(\mathscr{C}at_X(Y,Z))$ induces a BV differential on $\mathrm{Ext}^\bullet_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$, resp. the standard Gerstenhaber bracket on $HH^\bullet(\mathscr{C}at_X(Y,Z))$ induces a Gerstenhaber bracket on $\mathrm{Tor}_\bullet^X(\mathcal{O}_Y, \mathcal{O}_Z)$.
- If $X = T^\vee Y$ is the cotangent bundle of Y and $Z = Y$ is the zero section, then $\mathscr{C}at_X(Y,Z) = D^b(\mathrm{Coh}\, Y)$. In this case, we have

$$HH^\bullet(D^b(\mathrm{Coh}\, Y)) = H^\bullet(Y, \wedge^\bullet T_Y) = \mathrm{Tor}_\bullet^{T^\vee Y}(\mathcal{O}_Y, \mathcal{O}_Y).$$

  so that the Gerstenhaber bracket goes to the Schouten bracket on $\wedge^\bullet T_Y$; similarly for $HH_\bullet(D^b(\mathrm{Coh}\, Y)) = H^\bullet(Y, \wedge^\bullet T_Y^\vee)$.
- More generally, let $X = T^\vee Y$ and Y be the zero section as above, and let $Z = \mathrm{Graph}(df)$ where $f \in k[Y]$. Then $\mathscr{C}at_X(Y,Z)$ is the category of matrix factorizations ($F \underset{d'}{\overset{d}{\leftrightarrows}} F'$, $d \circ d' = f \cdot \mathrm{Id} = d' \circ d$), associated with the function f.

Observe that the sheaves $\mathscr{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ are related to the hyper-Tor groups $\mathrm{Tor}_{\bullet}^{X}(\mathcal{O}_Y, \mathcal{O}_Z)$ via the local-to-global spectral sequence $H^{\bullet}(X, \mathscr{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)) \Rightarrow \mathrm{Tor}_{\bullet}^{X}(\mathcal{O}_Y, \mathcal{O}_Z)$. At the same time, the global hyper-Tor may also be calculated by applying $R\Gamma$ to the sheaf of DG algebras $T_{\bullet}$ described in Section 3 below. Thus, we expect that there exists a refined version of our results, in which Gerstenhaber or Batalin-Vilkovisky structures on the cohomology sheaves of $T_{\bullet}$, are replaced by their "strong homotopy" versions on $T_{\bullet}$ itself. In fact, the lemmas of Section 3.2 point towards such a refinement. Similar remarks apply to the Ext groups (local and global), and the resolution $E^{\bullet}$ of Section 3.

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Given a vector bundle (a locally free sheaf) E, we write $E^{\vee}$ for the dual vector bundle. Let Ω_X, resp. $T_X = \Omega_X^{\vee}$, denote the cotangent, resp. tangent, sheaf on a manifold X. Let $N_{X/Y}$ denote the normal sheaf for a submanifold $Y \subset X$.

2. Existence of first and second order deformations.

2.1. **Algebraic setup.** Following Gerstenhaber, a $\mathbb{C}_\varepsilon$-flat deformation of an associative $\mathbb{C}$-algebra A is given by an associative product structure on $A_\varepsilon = A \oplus \varepsilon A$ is defined by

$$(a_1 \oplus 0) * (a_2 \oplus 0) = a_1 a_2 \oplus \varepsilon \alpha_A(a_1, a_2), \qquad a_1, a_2 \in A \subset A_\varepsilon$$

with $\alpha_A : A \otimes A \to A$ a $\mathbb{C}$-linear map. The associativity of the $*$-product is equivalent to the equation

$$\alpha_A(a_1 a_2, a_3) - \alpha_A(a_1, a_2 a_3) + \alpha_A(a_1, a_2) a_3 - a_1 \alpha_A(a_2, a_3) = 0. \tag{2.1.1}$$

Fix a $\mathbb{C}_\varepsilon$-flat deformation of A as above. Let B be a right A-module. Given a right A-module B, one may consider $\mathbb{C}_\varepsilon$-flat extensions of the A-module structure to a right A_ε-module on $B_\varepsilon = B \oplus \varepsilon B$. Explicitly, such an A_ε-module structure on B_ε is determined by a bilinear map $\alpha_B : B \otimes A \to B$. The corresponding right A_ε-action is given by the formula

$$(b \oplus 0) * (a \oplus 0) = ba \oplus \varepsilon \alpha_B(b, a), \quad a \in A, b \in B.$$

The map $\alpha_B : B \otimes A \to B$ must satisfy the associativity equation

$$\alpha_B(b a_2, a_3) - \alpha_B(b, a_2 a_3) + \alpha_B(b, a_2) a_3 - b \alpha_A(a_2, a_3) = 0. \tag{2.1.2}$$

Further, any pair of automorphisms $A_\varepsilon \to A_\varepsilon$, $B_\varepsilon \to B_\varepsilon$ given by

$$a \mapsto a \oplus \varepsilon \beta_A(a); \qquad b \mapsto b \oplus \varepsilon \beta_B(b)$$

with $\mathbb{C}$-linear $\beta_A : A \to A$, $\beta_B : B \to B$, induces equivalent deformations corresponding to

$$\alpha_A(a_1, a_2) \mapsto \alpha_A(a_1, a_2) + \beta_A(a_1 a_2) - a_1 \beta_A(a_2) - \beta_A(a_1) a_2 \tag{2.1.3}$$

$$\alpha_B(b, a_2) \mapsto \alpha_B(b, a_2) + \beta_B(b a_2) - b \beta_A(a_2) - \beta_B(b) a_2 \tag{2.1.4}$$

The deformation extends to $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ if and only if

$$\alpha_A(\alpha_A(a_1, a_2), a_3) - \alpha_A(a_1, \alpha_A(a_2, a_3)) = d\gamma_A(a_1, a_2, a_3) \tag{2.1.5}$$

$$\alpha_B(\alpha_B(b, a_1), a_2) - \alpha_B(b, \alpha_A(a_1, a_2)) = d\gamma_B(b, a_1, a_2) \tag{2.1.6}$$

where $\gamma_A : A \otimes A \to A$, $\gamma_B : B \otimes A \to B$ and $d\gamma_A$, $d\gamma_B$ are defined similarly to the LHS of (2.1.1) and (2.1.2), respectively.

2.2. **Deformation complex.** The identities of the previous subsection can be reformulated as follows. The A-module structure on B defines a homomorphism $g : A \to \mathrm{End}_{\mathbb{C}}(B)$ of algebras over $\mathbb{C}$, and deforming the algebra/module structure amounts to deforming g to an algebra homomorphism $A_\varepsilon \to \mathrm{End}_{\mathbb{C}_\varepsilon}(B_\varepsilon)$. Observes that $\mathrm{End}_{\mathbb{C}_\varepsilon}(B_\varepsilon)$ is the trivial deformation of $\mathrm{End}_{\mathbb{C}}(B)$. Thus, adjusting the definitions of [GS], [FMY] (i.e. removing the term responsible for the deformation of $\mathrm{End}_{\mathbb{C}}(B)$) we introduce the deformation complex

$$C^n_g = C^n(A, A) \oplus C^{n-1}(A, \mathrm{End}_{\mathbb{C}}(B)) = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A) \oplus \mathrm{Hom}_{\mathbb{C}}(A^{\otimes(n-1)}, \mathrm{End}_{\mathbb{C}}(B))$$

where $C^n(A, X)$ is the standard complex of Hochschild cochains of a A-bimodule X. The differential of C^n_g is given by

$$d_g(\alpha_A \oplus \alpha_B) = d_H\alpha_A \oplus (g\alpha_A - d_H\alpha_B)$$

where d_H is the standard Hochschild differential, cf. *loc. cit.* Denote $H^n_g := H^n(C^\bullet_g)$.

Equations (2.1.1), (2.1.2) say that $\alpha = \alpha_A \oplus \alpha_B$ is a cocycle in C^2_g. Equations (2.1.3) and (2.1.4) say that the equivalence class of the deformation depends only on the image of α in H^2_g.

To reinterpret integrability conditions recall that by *loc. cit.* $C^{\bullet-1}_g$ has a structure of a DG Lie algebra such that $C^{\bullet-1}(A, A)$ with its Gerstenhaber bracket, is a quotient DG Lie algebra of $C^{\bullet-1}_g$. Explicitly, up to a choice of signs for $\alpha_A \oplus \alpha_B \in C^n_g$, $\alpha'_A \oplus \alpha'_B \in C^m_g$ one has

$$[\alpha_A \oplus \alpha_B, \alpha'_A \oplus \alpha'_B] =$$

$$\big(\alpha_A \circ \alpha'_A - (-1)^{(n-1)(m-1)}\alpha'_A \circ \alpha_A\big) \oplus \big(\alpha_B \circ \alpha'_A + \alpha_B \cup \alpha'_B - (-1)^{(n-1)(m-1)}(\alpha'_B \circ \alpha_A + \alpha'_B \cup \alpha_B)\big)$$

where

$$\alpha_B \circ \alpha'_A = \sum_{s=1}^{n} (-1)^{(s-1)(m-1)}\alpha_B(1_A^{\otimes(s-1)} \otimes \alpha'_A \otimes 1_A^{\otimes(n-s)})$$

and similarly for the other terms. The cup product $\alpha_B \cup \alpha'_A : B \otimes A^{\otimes^{m+n-1}} \to B$ is the composition of $\alpha_B \otimes \alpha'_A : B \otimes A^{\otimes^{m+n-1}} \to B \otimes A$ and the action map $B \otimes A \to B$.

Then (2.1.5), (2.1.6) say that $\frac{1}{2}[\alpha, \alpha] = d_g(\gamma)$, i.e. that $[\alpha, \alpha]$ represents the zero class in H^3_g.

Observe that $C^{\bullet-1}(A, \mathrm{End}_{\mathbb{C}}(B))$ is a subcomplex of $C^\bullet_g$, and $C^\bullet(A, A)$ is a quotient complex of $C^\bullet_g$. The corresponding long exact sequence of cohomology reads

$$\ldots \to \mathrm{Ext}^{n-1}_A(B, B) \to H^n_g \to H^n(A, A) \xrightarrow{g} \mathrm{Ext}^n_A(B, B) \to \ldots \qquad (2.2.1)$$

We see that an n-cocycle $\alpha_A \in C^2(A, A)$ may be lifted to a class in H^2_g if and only if the map $g \circ \alpha_A : A \times A \to \mathrm{End}_{\mathbb{C}} B$, that represents the image of the class of α_A under the connecting homomorphism, gives the zero class in $\mathrm{Ext}^2_A(B, B)$.

Similarly, given an algebra homomorphism $h : A \to (\mathrm{End}_{\mathbb{C}} C)^{op}$, one can introduce a deformation complex with

$$C^n_{g,h} = C^n(A, A) \oplus C^{n-1}(A, \mathrm{End}_{\mathbb{C}}(B)) \oplus C^{n-1}(A, \mathrm{End}_{\mathbb{C}}(C)),$$

the differential

$$d_g(\alpha_A \oplus \alpha_B \oplus \alpha_C) = d_H\alpha_A \oplus (g\alpha_A - d_H\alpha_B) \oplus (h\alpha_A - d_H\alpha_C)$$

and cohomology groups $H^\bullet_{g,h}$. There is a long exact sequence

$$\ldots \to \mathrm{Ext}^{n-1}_A(B, B) \oplus \mathrm{Ext}^{n-1}_A(C, C) \to H^n_{g,h} \to H^n(A, A) \xrightarrow{g} \mathrm{Ext}^n_A(B, B) \oplus \mathrm{Ext}^n_A(C, C) \to \ldots$$

2.3. **Local deformations.** Let now X be a smooth affine variety and $Y \subset X$ a smooth closed subvariety. Write $A := \mathbb{C}[X]$, resp. $B := \mathbb{C}[Y]$, for the corresponding coordinate rings.

A bivector $P \in H^0(X, \wedge^2 T_X)$ with a vanishing Schouten bracket gives a Poisson structure on A. We will say that Y is *coisotropic* with respect to P if P projects to zero in $H^0(Y, \wedge^2 N_{X/Y})$.

Proposition 2.3.1. *Let $\alpha_A \in C^2(A, A)$ be a 2-cocycle. Then, we have*

(i) *Y is a* coisotropic *subvariety in X if and only if there exists $\alpha_B : B \otimes A \to B$ giving a first order deformation of B.*

(ii) *If $[\alpha_A, \alpha_A] = 0$, then any such α_B automatically extends to a second order deformation, i.e. satisfies* (2.1.6).

(iii) *If, moreover, $\alpha_A(a_1, a_2) = \frac{1}{2}\langle P, da_1 \wedge da_2\rangle$ is given by a bivector field $P \in H^0(X, \wedge^2 T_X)$ such that $[P, P] = 0$, then one may choose $\alpha_B : B \otimes A \to B$ to be a sum of a bidifferential operator of bidegree $(1, 1)$ and a bidifferential operator of bidegree $(0, 2)$.*

Proof. For (i) just note that $g\alpha_A = d_H \alpha_B$ in $C^2(A, \mathrm{End}_{\mathbb{C}}(B))$ means that (2.1.2) holds by definition of the maps involved. For (ii) observe that, in the smooth case, the long exact sequence (2.2.1) takes the form

$$\ldots \to H^0(X, \wedge^{n-1} T_X) \to H^0(Y, \wedge^{n-1} N_{X/Y}) \to H^n_g \to H^0(X, \wedge^n T_X) \to \ldots$$

By definition of C^n_g the connecting homomorphism $H^0(X, \wedge^n T_X) \to H^0(Y, \wedge^n N_{X/Y})$ is given by composition of restriction to Y and projection onto $N_{X/Y}$. Observe that this morphism is onto since Y is affine. Thus the long exact sequence splits into short sequences

$$0 \to H^n_g \to H^0(X, \wedge^n T_X) \to H^0(Y, \wedge^n N_{X/Y}) \to 0$$

The map $H^{\bullet-1}_g \to H^0(X, \wedge^{\bullet-1})$ is a Lie algebra morphism by definitions of the brackets involved. Then $[\alpha, \alpha] \in H^3_g$ has zero image in $H^0(X, \wedge^3 T_X)$ by the assumption on α_A. Hence $[\alpha, \alpha] = 0$ due to the short exact sequence above.

Part (iii) requires a finer argument. Assume α_B exists and denote $\phi(a) := \alpha_B(1_B, a) : A \to B$. Then

$$\phi(xa) - \phi(x)a + 1_B \alpha_A(x, a) = 0; \qquad x \in I, a \in A \tag{2.3.2}$$

(since the LHS is $-\alpha_B(1_B x, a)$). On the other hand, given such a ϕ we can define

$$\alpha_B(b, a) = \phi(\tilde{b}a) - \phi(\tilde{b})a + 1_B \alpha_A(\tilde{b}, a).$$

where $\tilde{b}$ is any lift of $b \in B$ to A. The condition (2.3.2) ensures that the RHS is independent on the choice of the lift. The same equation ensures that also the cocycle condition holds for $\alpha_B(b, a)$.

Thus, it suffices to find a second order operator $\phi : A \to B$ satisfying (2.3.2). We will provide one which is a composition of two first order operators $\psi : A \to I/I^2$ and $\phi_I : I/I^2 \to B$ such that ϕ_I satisfies (2.3.2) and ϕ restricts to the canonical (A-linear) projection $I \to I/I^2$.

To define ψ we choose a splitting $\pi : \Omega^1_X|_Y \to N^\vee_{X/Y}$ of the short exact sequence

$$0 \to N^\vee_{X/Y} \to \Omega^1_X|_Y \to \Omega^1_Y \to 0$$

(which exists since Y is affine) and set $\psi(a) = \pi\big[(da)|_Y\big]$. By definition of π, the restriction of ψ to I is just the canonical projection $I \to I/I^2 \simeq \Gamma(Y, N^\vee_{X/Y})$.

To find $\phi_I : I/I^2 \to B$ as above we first observe that it can be considered as a differential operator over B, rather than A, since I acts trivially on both source and target. There is a standard exact sequence

$$0 \to Hom_{\mathcal{O}_Y}(N^\vee_{X/Y}, \mathcal{O}_Y) \to Diff^{\leq 1}(N^\vee_{X/Y}, \mathcal{O}_Y) \to Hom_{\mathcal{O}_Y}(N^\vee_{X/Y} \otimes_{\mathcal{O}_Y} \Omega^1_Y, \mathcal{O}_Y)$$

where the last arrow is given by the symbol map. Since by assumption P projects to zero in $\wedge^2 T_Y$, it gives a well-defined element in the next associated graded quotient of the filtration on $\wedge^2 T_X|_Y$ induced by the normal exact sequence, and this quotient is $N_{X/Y} \otimes_{\mathcal{O}_Y} T_Y$. Then (2.3.2) simply says that the symbol of ϕ_I is given by the image of $-\frac{1}{2}P$ in $Hom_{\mathcal{O}_Y}(N^\vee_{X/Y} \otimes_{\mathcal{O}_Y} \Omega^1_Y, \mathcal{O}_Y)$. Since Y is affine, the symbol map is onto: in general its cokernel is embedded into $Ext^1_Y(N^\vee_{X/Y}, \mathcal{O}_Y)$ as can be seen by considering the bundle $\mathcal{P}^1(N^\vee)$ of principal parts, cf. [EGA], which satisfies

$$Diff^{\leq 1}(N^\vee_{X/Y}, \mathcal{O}_Y) \simeq Hom_{\mathcal{O}_Y}(\mathcal{P}^1(N^\vee), \mathcal{O}_Y)$$

and fits into a short exact sequence

$$0 \to N^\vee_{X/Y} \to \mathcal{P}^1_Y(N^\vee_{X/Y}) \to N^\vee_{X/Y} \otimes_{\mathcal{O}_Y} \Omega^1_Y \to 0.$$

We also observe there that for general Y the obstruction to existence of ϕ_I is the cup product of the Atiyah class of $N^\vee_{X/Y}$ in $Ext^1_Y(N^\vee_{X/Y}, N^\vee_{X/Y} \otimes_{\mathcal{O}_Y} \Omega^1_Y)$ and (up to sign) the image of the Poisson bivector in $Hom_{\mathcal{O}_Y}(N^\vee_{X/Y} \otimes_{\mathcal{O}_Y} \Omega^1_Y, \mathcal{O}_Y)$.

Thus, we found a degree 2 operator $\phi := \psi \circ \phi_I : A \to B$ which satisfies (2.3.2) and thus

$$\alpha_B(b, a) = \phi(\tilde{b}a) - \phi(\tilde{b})a + 1_B \alpha_A(\tilde{b}, a) : B \otimes A \to B$$

makes sense and satisfies the cocycle condition. It remains to show that it has required order with respect to its arguments. It has order ≤ 2 is the argument a since this holds for $\phi(-)$ and $\alpha_A(\tilde{b}, -)$. Moreover, using the definitions we easily derive

$$\alpha_B(b_1, b_2) - b_1 \alpha_B(b_2, a) = b_2(\alpha_B(b_1, a) - b_1 \phi(a))$$

thus α_B has order ≤ 1 in the b argument. In addition, one can show that the expression

$$\alpha'_B(b, a) = \alpha_B(b, a) - b\phi(a)$$

is a derivation in the second argument, which means that α_B has no term of bidegree $(1, 2)$. Hence the total order of α_B is ≤ 2 and the assertion follows. □

Remark 2.3.3. Since (iii) is rather important in our setting, we sketch two more proofs.

Second proof. Equation (2) says that $d\alpha_B = \alpha_A \cdot Id_B$ holds in $C^2(A, \mathrm{End}_{\mathbb{C}}(B))$. Let $D^\bullet(A, \mathrm{End}_{\mathbb{C}}(B))$ be a subcomplex of the Hochschild complex formed by cochains given by multidifferential operators. The arguments from [Kon], pp. 16-17 can be used to show that the cohomology groups of the complex $D^\bullet(A, \mathrm{End}_{\mathbb{C}}(B))$ are $\wedge^\bullet N_{X/Y}$. Therefore, it follows from a version of the Hochschild-Kostant-Rosenberg result that the imbedding $\jmath : D^\bullet(A, \mathrm{End}_{\mathbb{C}}(B)) \hookrightarrow C^\bullet(A, \mathrm{End}_{\mathbb{C}}(B))$ induces an isomorphism $H^\bullet(\jmath)$, on cohomology.

Now, the Poisson bivector on X gives a class $\alpha_A \cdot Id_B \in D^2(A, \mathrm{End}_{\mathbb{C}}(B))$. Since Y is coisotropic, $\jmath(\alpha_A \cdot Id_B) \in C^2(A, \mathrm{End}_{\mathbb{C}}(B))$ is a coboundary. By injectivity of $H^2(\jmath)$ the class $\alpha_A \cdot Id_B \in D^2(A, \mathrm{End}_{\mathbb{C}}(B))$ is itself a coboundary, i.e. $d\alpha_B = \alpha_A \cdot Id_B$ for some $\alpha_B \in D^2(A, \mathrm{End}_{\mathbb{C}}(B))$. A separate easy calculation shows that the symbol of α_B is a linear combination of maps $Sym^2\Omega_{A/\mathbb{C}} \to B$ and $\Omega_{A/\mathbb{C}} \otimes \Omega_{B/\mathbb{C}} \to B$, i.e. its component $Sym^2\Omega_{B/\mathbb{C}} \to B$ is actually zero.

Third proof. We will look for α_B of the form

$$\alpha_B(b,a) = \rho(db,da) + b\eta(a)$$

where $\rho : \Omega_{B/\mathbb{C}} \otimes_A \Omega_{A/\mathbb{C}} \to B$ is a A-linear map and $\eta : A \to B$ a second order operator satisfying $\eta(1) = 0$. If $\sigma\eta : Sym^2_A \Omega_{A/\mathbb{C}} \to B$ is the A-linear map given by the principal symbol of η, then equation (2.1.2) becomes equivalent to

$$\rho(d(a_2|_Y), da_3) = \sigma\eta(da_2, da_3) + \alpha_A(a_2,a_3)|_Y$$

Choose a splitting $\Omega^1_X|_Y \simeq N^\vee_{X/Y} \oplus \Omega^1_Y$ as in the first proof, and let π, p be the first and the second projection, respectively. Since

$$P|_Y : \Omega^1_X|_Y \otimes_{\mathcal{O}_Y} \Omega^1_X|_Y \to \mathcal{O}_Y$$

vanishes on $N^\vee_{X/Y} \otimes_{\mathcal{O}_Y} N^\vee_{X/Y}$ we can set

$$\sigma\eta(da_2, da_3) = \frac{1}{2}P|_Y(pda_2, \pi da_3) - \frac{1}{2}P|_Y(\pi da_2, pda_3); \quad \rho(d(a_2|_Y), da_3) = \frac{1}{2}P|_Y(pda_2, da_3 + pda_3).$$

Observe that ρ indeed depends only on $d(a_2|_Y)$. Now we can lift the principal symbol $\sigma\eta$ to the required $\eta : A \to B$ by an argument similar to the one in the first proof.

Remark 2.3.4. It can be shown using the arguments of the proof above and those of Section 4 below that the existence of a (not necessarily split) deformation for $\mathcal{O}_Y$ with Y coisotropic, is equivalent to the vanishing of a certain class in $H^1(Y, N_{X/Y})$. This class is the cup product of the Atiyah class in $H^1(Y, \Omega^1_Y \otimes_{\mathcal{O}_Y} End(N_{X/Y}))$ with the image of P in $H^0(Y, T_Y \otimes N_{X/Y})$.

3. An algebraic construction of BV operators

3.1. **Complexes computing** $\mathrm{Tor}^A_\bullet(B,C)$ **and** $\mathrm{Ext}^\bullet_A(B,C)$**.** In this subsection we fix a commutative algebra A and a pair of A-modules B, C. We have associated algebra homomorphisms $g : A \to \mathrm{End}_\mathbb{C} B$, resp. $h : A \to \mathrm{End}_\mathbb{C} C$.

Recall that the A-module B admits a free bar resolution $B \otimes T(A) \otimes A \to B$, cf. [We]. Therefore $\mathrm{Tor}^A_\bullet(B,C)$ and $\mathrm{Ext}^\bullet_A(B,C)$ can be computed as cohomology of complexes $T_\bullet$ and $E^\bullet$, respectively, with

$$T_i = B \otimes A^{\otimes i} \otimes C, \quad E^i = \mathrm{Hom}_k(B \otimes A^{\otimes i}, C)$$

The differentials are given by

$$d_T(b \otimes a_1 \otimes \ldots \otimes a_i \otimes c) = ba_1 \otimes \ldots \otimes a_i \otimes c + (-1)^i b \otimes a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i c + \\ + \sum_{s=1}^{i-1} (-1)^s b \otimes a_1 \otimes \ldots \otimes a_s a_{s+1} \otimes \ldots \otimes a_i \otimes c$$

$$d_E\phi(b \otimes a_1 \otimes \ldots \otimes a_{i+1}) = -\phi(ba_1 \otimes \ldots \otimes a_{i+1}) + (-1)^{i-1}\phi(b \otimes a_1 \otimes \ldots \otimes a_i)a_{i+1} \\ + \sum_{s=1}^{i-1} (-1)^{s-1}\phi(b \otimes a_1 \otimes \ldots \otimes a_s a_{s+1} \otimes \ldots \otimes a_{i+1})$$

We consider deformations of (A,B,C). Such a deformation is determined by an element of the deformation complex $C^2_{g,h}$ given by the cocycle $(\alpha_A, \alpha_B, \alpha_C)$. Working with $T_\bullet$ we always assume that α_B gives a deformation of B to a right module and α_C a deformation of C to a left module.

The triple $(\alpha_A, \alpha_B, \alpha_C)$ induces an operation $\delta_\alpha : T_i \to T_{i-1}$ given essentially by the same formula as d_T but ba_1 is replaced by $\alpha_B(b, a_1)$, $a_s a_{s+1}$ by $\alpha_A(a_s, a_{s+1})$ and $a_i c$ by $\alpha_C(a_i, c)$. If, in addition $\alpha'_C : C \otimes A \to A$ gives a deformation to a right module, then the triple $(\alpha_A, \alpha_B, \alpha'_C)$ induces an operation $\delta'_\alpha : E^i \to E^{i+1}$ given by a formula similar to d_E (this time $\phi(X)a_i$ is replaced by $\alpha'_C(\phi(X), a_i)$. The following result is proved by direct computation

Lemma 3.1.1. *Let δ_α be the operator on $T_\bullet$ constructed from a triple $(\alpha_A, \alpha_B, \alpha_C)$.*

(1) *If (α_A, α_B) is a cocycle in C^2_g and (α_A, α_C) is a cocycle in C^2_h then*

$$\delta_\alpha d_T + d_T\delta_\alpha = 0$$

(2) *If $(\tilde\alpha_A, \tilde\alpha_B) - (\alpha_A, \alpha_B) = d(\beta_A, \beta_B)$ and $(\tilde\alpha_A, \tilde\alpha_C) - (\alpha_A, \alpha_C) = d(\beta_A, \beta_C)$ then*

$$\delta_{\tilde\alpha} - \delta_\alpha = d_T\delta_\beta + \delta_\beta d_T$$

where

$$\delta_\beta(b \otimes a_1 \otimes \ldots \otimes a_n \otimes c) = \beta_B(b) \otimes a_1 \otimes \ldots \otimes a_n \otimes c +$$
$$+ \sum_i (-1)^i b \otimes a_1 \otimes \ldots \otimes \beta_A(a_i) \otimes \ldots \otimes a_n \otimes c + (-1)^{n+1} b \otimes a_1 \otimes \ldots \otimes a_n \otimes \beta_C(c)$$

(3) *If* (2.1.5), (2.1.6) *hold (with similar equation and notation assumed for α_C), then*

$$\delta^2_\alpha = d_T\delta_\gamma + \delta_\gamma d_T$$

Similar identities hold for the map δ'_α constructed from $(\alpha_A, \alpha_B, \alpha'_C)$, with d_T replaced by d_E.

We now interpret δ_α in the context of the long exact sequence of Section 1.2. Since B_ε is flat over $\mathbb{C}_\varepsilon$ we can construct a bar resolution using tensor products over $\mathbb{C}_\varepsilon$:

$$\ldots \to B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \to B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \to B_\varepsilon \to 0$$

where the bar differential is defined using the deformed product $A_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \to A_\varepsilon$ and the deformed action $B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \to B_\varepsilon$. In particular, $\mathrm{Tor}_i^{A_\varepsilon}(B_\varepsilon, C_\varepsilon)$ is the homology of the complex with the i-th term

$$T_i^\varepsilon = B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon^{\otimes_{\mathbb{C}_\varepsilon} i} \otimes_{\mathbb{C}_\varepsilon} C_\varepsilon \simeq \left[B \otimes A^{\otimes i} \otimes C\right] \oplus \varepsilon\left[B \otimes A^{\otimes i} \otimes C\right] = T_i \oplus \varepsilon T_i$$

It is easy to see that the differential of this complex is $d_\varepsilon = d + \varepsilon\delta_\alpha$. The spectral sequence of the filtered complex $\varepsilon T_\bullet \subset T^\varepsilon_\bullet$ boils down to the long exact sequence

$$\ldots \to H_i(T_\bullet, d) \to H_i(T^\varepsilon_\bullet, d_\varepsilon) \to H_i(T_\bullet, d) \to H_{i-1}(T_\bullet, d) \to \ldots$$

By definition $H_i(T_\bullet, d) = \mathrm{Tor}_i^A(B, C)$. The connecting differential $\delta : H_i(T_\bullet, d) \to H_{i-1}(T_\bullet, d)$ is computed as usual: we take a representative $x \in T_i \subset T_i^\varepsilon$, assume that $dx = 0$ then $d_\varepsilon x \in \varepsilon T_i \in T_i^\varepsilon$ and we set $\delta(x)$ to be represented by $\frac{1}{\varepsilon} d_\varepsilon x$. Due to the definition of d_ε this is precisely δ_α.

In the case of $\mathrm{Ext}^\bullet_A(B, C)$ assume that both B, C are deformed as right modules. Again, if $(\alpha_A, \alpha_B, \alpha'_C)$ satisfies the cocycle condition determining the first order deformation, the operation δ'_α descends to δ' on $\mathrm{Ext}^\bullet_A(B, C)$ defined in Section 1.2. In fact, $\mathrm{Ext}^i_{A_\varepsilon}(B_\varepsilon, C_\varepsilon)$ may be computed as the cohomology of the complex

$$\begin{aligned} E^i_\varepsilon = \mathrm{Hom}_{A_\varepsilon}(B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon^{\otimes_{\mathbb{C}_\varepsilon} i} \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon, C_\varepsilon) &= \mathrm{Hom}_{\mathbb{C}_\varepsilon}(B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon^{\otimes_{\mathbb{C}_\varepsilon} i}, C_\varepsilon) \\ &= \mathrm{Hom}_{\mathbb{C}_\varepsilon}([B \otimes A^{\otimes i}] \oplus \varepsilon[B \otimes A^{\otimes i}], C \oplus \varepsilon C) \\ &= \mathrm{Hom}_k(B \otimes A^{\otimes i}, C) \oplus \varepsilon\, \mathrm{Hom}_k(B \otimes A^{\otimes i}, C) =: E^i \oplus \varepsilon E^i. \end{aligned}$$

The differential d_ε again splits into $d_E + \varepsilon\delta'_\alpha$ hence the connecting differential $\mathrm{Ext}^{i-1}_A(B, C) \to \mathrm{Ext}^i_A(B, C)$ is induced by δ'_α.

Observe that for δ, resp. δ', part (1) of the Lemma 3.1.1 implies that δ_α, resp. δ'_α, does descend to (co)homology. Part (2) says that the operator on cohomology does not change of $(\alpha_A, \alpha_B, \alpha_C)$ is replaced by $(\alpha_A, \alpha_B, \alpha_C) + d(\beta_A, \beta_B, \beta_C)$. Part (3) says that integrability conditions imply $\delta^2 = 0$, resp. $(\delta')^2 = 0$.

3.2. **Multiplicative properties of $T_\bullet$ and $E^\bullet$.** We begin with a rather general result.

Let (D, d) be a differential associative $\mathbb{Z}/2\mathbb{Z}$-graded algebra with an odd differential d and an odd linear map $\delta : D \to D$ vanishing on the identity. We write $|x| \in \mathbb{Z}/2\mathbb{Z}$ for the parity of a homogeneous element $x \in D$, and introduce the notation $[d, \delta]_+ := d\delta + \delta d$. Define a bracket $[-,-] : D \times D \to D$ as follows

$$[x, y] = \delta(xy) - \delta(x)y - (-1)^{|x|}x\delta(y) \qquad x, y \in D.$$

Also, for any $x, y, z \in D$, put

$$\Xi(x, y, z) := \delta(xyz) - (-1)^{|x|}x\delta(yz) - \delta(xy)z - (-1)^{|y|(|x|-1)}y\delta(xz).$$

A straightforward computation yields the following result

Lemma 3.2.1. *The following identities hold:*

(1) $$d[x, y] - [dx, y] - (-1)^{|x|}[x, dy] = [d, \delta]_+(xy) - [d, \delta]_+(x)y - x[d, \delta]_+(y)$$

(2) $$[x, yz] - [x, y]z - (-1)^{|y||x|}y[x, z] = \Xi(x, y, z) + (-1)^{|x|+|y|}xy\delta(z) + (-1)^{|x|}x\delta(y)z + \delta(x)yz.$$

(3) $$\begin{aligned}&[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y]\\ &= \delta^2(xyz) - z\delta^2(xy) - x\delta^2(yz) - y\delta^2(xz) + yz\delta^2(x) + xz\delta^2(y) + zy\delta^2(z)\\ &\quad + \delta(\Xi(x, y, z)) - \Xi(\delta(x), y, z) - (-1)^{|x|}\Xi(x, \delta(y), z) - (-1)^{|x|+|y|}\Xi(x, y, \delta(z)).\end{aligned}$$

By [We], Exercise 8.6.5, Section 8.7.5 and Lemma 8.7.15 and a similar statement for Ext groups, we have

Lemma 3.2.2. *The algebra* $\mathrm{Tor}^A_\bullet(B, C)$ *is isomorphic to the homology of the DG algebra* $T_\bullet$ *with the shuffle product* $\bullet : T_i \otimes T_j \to T_{i+j}$ *given by*

$$(b \otimes a_1 \otimes \ldots \otimes a_i \otimes c) \bullet (b' \otimes a_{i+1} \otimes \ldots \otimes a_{i+j} \otimes c') = \sum_{\sigma \in Sh(i,j)} (-1)^\sigma bb' \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(i+j)} \otimes cc'.$$

The $\mathrm{Tor}^A_\bullet(B, C)$*-module* $\mathrm{Ext}^\bullet_A(B, C)$ *is isomorphic to the cohomology of the* $T_\bullet$*-module* $E^\bullet$ *with the action* $\bullet' : T_i \otimes E^j \to E^{j-i}$ *given by*

$$(b \otimes a_1 \otimes \ldots \otimes a_i \otimes c) \bullet' \phi(b' \otimes a_{i+1} \otimes \ldots \otimes a_j) = \sum_{\sigma \in Sh(i,j)} (-1)^\sigma \phi(bb' \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(i+j)})c$$

Lemma 3.2.3. *Let* T *be the DG algebra computing the* Tor *groups and* $\delta = \delta_\alpha$ *as introduced in section 2.3 One has* $\Xi(x, y, z) = 0$ *for all* x, y, z *precisely when*

$$\alpha_B(b_1 b_2, a) - b_1\alpha_B(b_2, a) - b_2\alpha_B(b_1, a) + b_1 b_2 \alpha_B(1, a) = 0 \tag{3.2.4}$$

for all $b_1, b_2 \in B$, $a \in A$, *and similarly for* α_C. *If* $\alpha'_C = \alpha^t_C$, *the same conditions ensure that*

$$\delta_\alpha(xy)m - x\delta_\alpha(y)m - y\delta_\alpha(x)m = \delta'_\alpha(xym) - x\delta'_\alpha(ym) - y\delta'_\alpha(xm) + xy\delta'_\alpha(m) \tag{3.2.5}$$

Proof. Let $x = b_x \otimes x_1 \dots x_{l_x} \otimes c_x$ and use the similar notation for y, z. If we plug in the formula for δ into the definition of $\Xi(x, y, z)$ we get three kinds of terms: those which involve α_B, α_A and α_C, respectively. For instance, the terms in $\delta(xyz)$ coming from α_A, will involved tensor factors of the type

$$\alpha_A(x_i, x_{i+1}), \alpha_A(y_j, y_{j+1}), \alpha_A(z_s, z_{s+1}),$$

$$\alpha_A(x_i, y_j), \alpha_A(y_j, x_i), \alpha_A(x_i, z_s), \alpha_A(z_s, x_i), \alpha_A(y_j, z_s), \alpha_A(z_s, y_j)$$

For $x\delta(yz)$ we need to include only those terms in which α_A is applied to y_j ans z_s but not to x_i, and so on. Hence the terms in $\Xi(x, y, z)$ which depend on α_A cancel out by inclusion-exclusion formula. Looking at terms which involve α_B we get for $\delta(x, y, z)$:

$$\alpha_B(b_x b_y b_z, x_1), \alpha_B(b_x b_y b_z, y_1), \alpha_B(b_x b_y b_z, z_1)$$

For $x\delta(yz)$ we get

$$b_x \alpha_B(b_y b_z, y_1) + b_x \alpha_B(b_y b_z, z_1)$$

and similarly for other summands in $\Xi(x, y, z)$. Extracting the terms which only contain x_1 we get

$$\alpha_B(b_x b_y b_z, x_1) - b_y \alpha_B(b_x b_z, x_1) - b_z \alpha_N(b_x b_y, x_1) + b_y b_z \alpha_B(b_x, x_1) = 0$$

For $b_x = 1$ this gives (3.2.4). On the other hand, if (3.2.4) holds then

$$\begin{aligned}\alpha_B(b_x b_y b_z, x_1) &= b_y \alpha_B(b_x b_z, x_1) + b_x b_z \alpha_B(b_y, x_1) - b_x b_y b_z \alpha_B(1, x_1) \\ &= b_y \alpha_B(b_x b_z, x_1) + b_z \big[b_x \alpha_B(n_y, x_1) - b_x b_z \alpha_B(1, x_1)\big] \\ &= b_y \alpha_B(b_x b_z, x_1) + b_z \big[\alpha_B(b_y b_x, x_1) - b_y \alpha_B(b_x, x_1)\big]\end{aligned}$$

as required.

The calculation for (3.2.5) is similar: for terms involving $\alpha_B : B \otimes A \to B$ we get precisely (3.2.4). Comparing the terms involving $\alpha_C : A \otimes C \to C$ and $\alpha_C^t : C \otimes A \to C$ we get the condition (3.2.4) for α_C^t plus the equation

$$c_3 \alpha_C(a, c_1 c_2) - c_3 c_2 \alpha_C(a, c_1) = -\alpha_C^t(c_1 c_2 c_3, a) + c_2 \alpha_C^t(c_1 c_3, a)$$

Since α_C^t is transposed to α_C this equation can also be reduced to (3.2.4) for α_C^t. □

4. Proofs of main results

4.1. **Proof of Theorem 1.1.2: BV differentials.** Once the differential δ is defined globally as in Section 3.1 we can check its properties locally using the affine setting of Sections 3.2-3.3. For $X = Spec(A)$ we split the deformation over $\mathbb{C}$, then choose cocycles α_A, α_B, α_C as in Section 3.2 interpreting δ in terms of the bar construction.

Now by Proposition 2.3.1 (*iii*) and Lemma 3.2.3 with appropriate choice of α_B, α_C the Poisson identity holds already in $T_\bullet$ and hence in the Tor algebra as well.

By Proposition 2.3.1 (*ii*), Lemma 3.1.1 and Lemma 3.2.1 the Jacobi identity will hold on Tor which finishes the proof.

Essentially the same proof works for the BV structure on Ext. □

4.2. **Gerstenhaber structures.** Let X, Y, Z, α_X be as in Section 3.1. We do not assume here the existence of $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

Theorem 4.2.1. *The sheaf of graded algebras* $\mathscr{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ *admits a canonical structure of a Gerstenhaber algebra, i.e. a graded symmetric bracket of degree (-1) which satisfies Poisson and Jacobi identities. Similarly, the sheaf* $\mathscr{E}xt^\bullet_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ *has a canonical structure of a Gerstenhaber module, i.e. a bracket* $[\cdot,\cdot]' : \mathscr{T}or_i \otimes \mathscr{E}xt^j \to \mathscr{E}xt^{j-i+1}$ *such that*

$$[xy, m]' = x[y, m]' + (-1)^{\deg y \deg x} y[x, m]';$$

$$[x, ym]' = [x, y]m + (-1)^{\deg y \deg x} y[x, m]'$$

$$[[x, y], m]' = [x, [y, m]']' + (-1)^{\deg y \deg x}[y, [x, m]']'$$

Proof. By Section 2 on elements of affine open covering $\{U_i\}$ we can find deformations $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$. Applying Theorem 1.1.2 we get a BV structure on $\mathrm{Tor}_\bullet^{A_i}(B_i, C_i)$. By Part 2 of Lemma 3.1.1 the BV differential δ on the Tor algebra stays the same if we do not change the cohomology classes of $\alpha_A \oplus \alpha_B$ and $\alpha_A \oplus \alpha_C$. On a double intersection $U_i \cap U_j = Spec(A_{ij})$ we can find $((\beta_A)_{ij}, (\beta_B)_{ij})$ such that

$$(0 \oplus (\alpha_C)_i - (\alpha_C)_j) = d((\beta_A)_{ij} \oplus (\beta_B)_{ij})$$

(which means in particular that $(\beta_A)_{ij} : A_{ij} \to A_{ij}$ is a derivation). However, for $(\alpha_B)_i - (\alpha_B)_j$ we may need a completely different choice of $(\beta_A)_{ij}$. Hence adjusting by a coboundary allows to assume that $(\alpha_C)_i = (\alpha_C)_j$ but not the corresponding equality for α_B. However, we can assume that $(\alpha_B)_i - (\alpha_B)_j$ is given by $(b \otimes a) \mapsto b\mu(a)$ for a certain derivation $\mu : A_{ij} \to A_{ij}$. Thus, the difference of two BV differentials $\delta_i - \delta_j$ on $U_i \cap U_j$ is induced by the operator

$$\tilde{\delta}(b \otimes a_1 \otimes \ldots \otimes a_i \otimes c) = b\mu(a_1) \otimes a_2 \otimes \ldots \otimes a_i \otimes c$$

But such a $\tilde{\delta}$ induces the zero bracket on Tor, hence the brackets induced by δ_i and δ_j agree on $U_i \cap U_j$. This means that the Gersenhaber bracket is independent on the local choice of α_B and α_C, which finishes the proof.

For the Gerstenhaber module part the proof we observe that the module bracket can be defined via

$$[x, m]' = \delta'(xm) - \delta(x)m - (-1)^{\deg x} x\delta'(m).$$

Both versions of Poisson identity are equivalent to (3.2.5). The Jacobi identity on the module follows from $(\delta')^2 = 0$, once the Poisson identity is established. The remaning part of the proof is the same as for Tor. ☐

5. Examples.

The proof of the previous Theorem shows that sometimes the BV structure on Tor or Ext is well-defined globally.

5.1. **Lagrangian intersections.** For Lagrangian submanifolds in a symplectic manifold Behrend and Fantechi in [BF] prove that one can define a canonical BV differential δ'. In our situation this would mean that the derivation μ of the proof of Theorem 4.2.1 is identically zero, i.e. α_B and α_C^t are not chosen independently. It appears that such a coherent choice is induced by a Lagrangian foliation transversal to Y and Z.

5.2. **Koszul bracket.** We can also see that if α_C *and* α_B are defined globally then both δ and δ' are also defined globally. One instance is as follows. Take $X = Y \times Y$.

$$\mathscr{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Omega_Y^\bullet; \qquad \mathscr{E}xt^\bullet_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Lambda^\bullet \Omega_Y^\vee$$

Choose and fix a Poisson bivector field $P \in H^0(Y, \Lambda^2_Y)$.

Proposition 5.2.1. *The induced BV differential* δ *on* $\Omega^\bullet_{B/k}$ *is given by the graded commutator* $\delta = Pd_{DR} + d_{DR}P$ *of the degree 1 De Rham differential* d_{DR} *and degree (-2) contraction with the bivector* P*. The BV differential* δ' *on* $\Lambda^\bullet \Omega^\vee_{B/k}$ *is given by the Schouten bracket* $[P, \cdot]$*.*

Proof. To unload notation we will work in the affine case although all formulas make sense globally. Thus we consider $A = B \otimes B$ with the quotient map $m : B \otimes B \to B$ given by the product. Also, take $C = B$. By standard results, e.g. [We], we have a Hochschild cocycle on A:

$$\alpha_A(x \otimes x', y \otimes y') = \frac{1}{2}P(dx \wedge dy) \otimes x'y' - xy \otimes \frac{1}{2}P(dx' \wedge dy')$$

and the diagonal $Y_\Delta \subset X = Y \times Y$ is coisotropic with respect to the corresponding Poisson structure. We also have a right deformation of B induced by

$$\alpha_B(b, x \otimes y) = -\frac{1}{2}P(d(bx) \wedge dy) + \frac{1}{2}P(db \wedge dx)y : B \otimes A \to B$$

For the second argument of $\mathrm{Tor}^A_\bullet(\cdot, \cdot)$ we use the transposed map $\alpha_B^t : A \otimes B \to B$. Our goal is to compute the induced BV differential on $\Omega^\bullet_{B/k}$. To that end, we need explicit quasi-isomorphisms between $\Omega^\bullet_{B/k}$ and $T_\bullet$.

Observe that usually $\Omega^\bullet_{B/k}$ is identified with the cohomology of $C_\bullet(A, A) = A^{\otimes(\bullet+1)}$ and the standard Hochschild differential. Our complex $T_\bullet$ is slightly different although quasi-isomorphic to $C_\bullet(A, A)$ by the map:

$$b \otimes a_1 \otimes \ldots \otimes a_n \otimes b' \mapsto (b'b) \otimes m(a_1) \otimes \ldots \otimes m(a_n)$$

Combining we the Hochschild-Kostant-Rosenberg isomorphism we have a pair of mutually inverse quasi-isomorphisms:

$$b \otimes a_1 \otimes \ldots \otimes a_n \otimes b' \mapsto \frac{1}{n!} b'b \cdot dm(a_1) \wedge \ldots \wedge dm(a_n) : T_\bullet \to \Omega^\bullet_{B/k}$$

and the map

$$b_0 db_1 \mapsto b_0 \otimes (b_1 \otimes 1) \otimes 1 : \Omega^1_{B/k} \to T_1$$

extended multiplicatively. For example, $b_0 db_1 \wedge db_2 \wedge db_3$ maps to the antisymmetrization of

$$b_0 \otimes (b_1 \otimes 1) \otimes (b_2 \otimes 1) \otimes (b_3 \otimes 1) \otimes 1 \in T_3$$

in the three middle terms. The assertion follows from the above definitions of α_A and α_B and a straightforward computation.

The case of δ' is entirely similar. □

We observe here that the differential δ of the above proposition was first constructed from a Poisson bivector P by Koszul in [Kos]. Also, the differential δ (and not just the induced Gerstenhaber bracket) is canonically defined since the two arguments of $\mathrm{Tor}^A_\bullet(B, B)$ are taken with their conjugate deformations. We also remark that the (co)homology of the differentials δ' and δ in this case are called Poisson cohomology and homology, respectively.

5.3. **Self-intersection of a coisotropic submanifold.** Finally, assume that $B = C$ and consider $\mathscr{E}xt^\bullet_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \wedge^\bullet(N_{X/Y})$. The proof of Theorem 4.2.1 shows that the differential δ' is well-defined globally: although $\alpha_B(b, a)$ exist in general only locally, on double intersections of an affine cover the difference between two choice of α_B is a coboundary (since $B = C$) hence these two choices give the same δ' on cohomology. The subsheaf $\mathcal{K} \subset \wedge^\bullet(T_X)$ formed by vector fields which project to zero in $\wedge^\bullet(N_{X/Y})$, is a Lie subalgebra with respect to the Schouten bracket. Then $\mathcal{K}$ acts on $\wedge^\bullet(N_{X/Y})$ since any Lie subalgebra acts on a quotient by itself.

Proposition 5.3.1. *The BV differential on $\wedge^\bullet(N_{X/Y})$ is given by the action of the Poisson bivector.*

Proof. It suffices to prove the statement on an affine open subset. Observe that $\mathrm{Ext}^\bullet_A(B, B)$ is computed by the subcomplex

$$C^{\bullet-1}(A, \mathrm{End}_{\mathbb{C}}(B)) \subset C^\bullet_g.$$

The differential δ' of $\gamma_B : A^{\otimes(n-1)} \to \mathrm{End}_{\mathbb{C}}(B)$ is explicitly given by

$$\delta'(\gamma_B) = [\alpha_A \oplus \alpha_B, 0 \oplus \gamma_B] : A^{\otimes n} \to \mathrm{End}_{\mathbb{C}}(B)$$

where $[\cdot, \cdot]$ is the Lie bracket on C^n_g introduced in Section 2.2. Now a straightforward calculation finishes the proof. □

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