ON EQUIVALENCES OF DERIVED AND SINGULAR CATEGORIES

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Abstract. Let $\mathcal{X}$ and $\mathcal{Y}$ be two smooth Deligne-Mumford stacks and consider a pair of functions $f : \mathcal{X} \to \mathbb{A}^1$, $g : \mathcal{Y} \to \mathbb{A}^1$. Assuming that there exists a complex of sheaves on $\mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y}$ which induces an equivalence of $D^b(\mathcal{X})$ and $D^b(\mathcal{Y})$, we show that there is also an equivalence of the singular derived categories of the fibers $f^{-1}(0)$ and $g^{-1}(0)$. We apply this statement in the setting of McKay correspondence, and generalize a theorem of Orlov on the derived category of a Calabi-Yau hypersurface in a weighted projective space, to products of Calabi-Yau hypersurfaces in simplicial toric varieties with nef anticanonical class.

1. Introduction

Mirror symmetry is a conjectural equivalence between holomorphic data on one Calabi-Yau threefold and the symplectic data on another Calabi-Yau threefold. More precisely, Kontsevich conjectured [Ko] given a pair of mirror Calabi-Yau threefolds $X, \widehat{X}$ there is an equivalence between the derived category of coherent sheaves on X and a suitably defined Fukaya category on $\widehat{X}$. Later this was extended to involve Fano varieties. The mirror to a Fano variety, X is no longer just a variety but a pair $(\widehat{X}, W)$ where $\widehat{X}$ is a regular scheme and $W : \widehat{X} \to \mathbb{A}^1$ is a regular morphism. Such a pair is called a Landau-Ginzburg model in the physics literature. When $\widehat{X}$ is affine Kontsevich suggested replacing the derived category of coherent sheaves with a category of 2-periodic complexes where the composition is no longer 0 but multiplication by W. Orlov defined this category and showed it is a triangulated category [O1]. This category is denoted by $DB(W)$ and mirror symmetry is now conjectured to be an equivalence between $DB(W)$ and a Fukaya type category.

The bounded derived category of coherent sheaves $D^b(X)$ has a triangulated subcategory $\mathfrak{Perf}(X)$ consisting of perfect complexes i.e. complexes which are locally quasi-isomorphic to a bounded complex of locally free sheaves. If X is non-singular then every bounded complex of coherent sheaves admits a locally free resolution. This means that $\mathfrak{Perf}(X)$ is equivalent to $D^b(X)$. When X is singular this is no longer true. Orlov [O1] introduced the singular derived category $D_{sg}(X)$ as the quotient of $D^b(X)$ by the full triangulated subcategory $\mathfrak{Perf}(X)$. He showed that for affine X the category $DB(W)$ is equivalent to the product of $D_{sg}(X_w)$ over the critical values w of $W : X \to \mathbb{A}^1$. Thus, one can say that $DB(W)$ reflects the singularities of the fibers of W.

In this paper we study $D^b(\cdot)$ and $D_{sg}(\cdot)$ for Deligne-Mumford stacks.

Convention. The term *stack* will mean a Deligne-Mumford stack $\mathcal{X}$ of finite type over a field k of characteristic zero ($k = \mathbb{C}$ in applications) which has finite stabilizers. In addition we will always assume that $\mathcal{X}$ is *quasi-projective*, i.e. admits a locally closed embedding in a smooth Deligne-Mumford stack $\mathcal{W}$, such that $\mathcal{W}$ is proper over *Spec* k and has a projective coarse moduli space. If we can choose a closed embedding into such $\mathcal{W}$ then $\mathcal{X}$ is called *projective*. Quasi-projectivity ensures that $\mathcal{X}$ has a quasi-projective moduli space and that every coherent sheaf on $\mathcal{X}$ is a quotient of a vector bundle (the so-called *resolution property*). See [Kr] for an excellent overview of related results. In particular, if $\mathcal{X}$ is smooth then every coherent sheaf admits a finite locally free resolution. For a quasi-projective stack $\mathcal{X}$, we denote by $D^b(\mathcal{X})$ the bounded derived category of coherent sheaves. All functors between such categories (pullbacks, pushforwards and tensor products) are assumed to be derived (otherwise they are not well defined on D^b) hence we drop the usual letters R and L from the notation.

The main technical result of this paper, proved in Section 2, is as follows.

Theorem 1.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be two smooth stacks, $f : \mathcal{X} \to \mathbb{A}^1$, $g : \mathcal{Y} \to \mathbb{A}^1$ two morphisms and $\widetilde{\mathscr{F}}$ a complex on $\mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y}$ with support proper over $\mathcal{X}$ and $\mathcal{Y}$. Let $\Phi_{\mathscr{F}} : D^b(\mathcal{X}) \to D^b(\mathcal{Y})$ be the Fourier-Mukai transform with the kernel $\mathscr{F} = h_*\widetilde{\mathscr{F}}$, where $h : \mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ is the closed embedding. If $\Phi_{\mathscr{F}}$ is an equivalence of categories, and $\mathcal{X}_0$, $\mathcal{Y}_0$ are the fibers of f, g over $0 \in \mathbb{A}^1$, respectively, then the pullback of $\widetilde{\mathscr{F}}$ to $\mathcal{X}_0 \times \mathcal{Y}_0$ induces an equivalence of derived categories $D^b(\mathcal{X}_0) \simeq D^b(\mathcal{Y}_0)$ which descends to an equivalence of singular derived categories $D_{sg}(\mathcal{X}_0) \to D_{sg}(\mathcal{Y}_0)$.*

This theorem can be applied to the situation when one has the following diagram of stacks and proper birational morphisms

(1.1)
$$\begin{array}{ccccc} & & \mathcal{W} & & \\ & \swarrow^{\mu} & & \searrow^{\nu} & \\ \mathcal{X} & & & & \mathcal{Y} \\ & \searrow_{\phi} & & \swarrow_{\psi} & \\ & & Z & & \end{array}$$

We assume that $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are smooth stacks, Z is an affine variety, $\mu^*(K_{\mathcal{X}}) = \nu^*(K_{\mathcal{Y}})$ and in addition ϕ and ψ induce projective morphisms of coarse moduli spaces. For instance, we can assume that Z is a singular quotient V/Γ of a finite dimensional vector space V by a linear action of a finite group $\Gamma \subset SL(V)$, $\mathcal{X} = [V/\Gamma]$ is the same quotient considered as a smooth stack and $\mathcal{Y}$ is a crepant resolution of the variety V/Γ.

By a strengthened version of derived McKay Correspondence conjecture, such a diagram (1.1) should imply existence of a Fourier-Mukai equivalence with a kernel obtained by direct image from $\mathcal{X} \times_Z \mathcal{Y}$. In particular, $\widetilde{\mathscr{F}}$ exists whenever the morphisms $f : \mathcal{X} \to \mathbb{A}^1$, $g : \mathcal{Y} \to \mathbb{A}^1$ are obtained by pulling back the same regular function $Z \to \mathbb{A}^1$. This holds in the cases of the derived McKay correspondence considered by Bezrukavnikov and Kaledin; Bridgeland, King, and Reid; and Kawamata [BK, BKR, K1]. Thus, our first application, Theorem 2.10, is in these three cases.

Our Theorem 2.10 generalizes an earlier result of Quintero-Vélez, cf. [QV], who proves the statement under the assumption that $\mathcal{Y}$ is given by $G - Hilb(V)$, the Hilbert scheme of G-clusters, and that Φ is given by the structure sheaf of the universal subscheme. In particular, case (1) of Theorem 2.10 follows from *loc. cit.*. As observed in the same paper, the assumption also holds when $\Gamma = \mathbb{Z}/n\mathbb{Z}$ where $n = \dim V$, Γ acts diagonally on V and $\mathcal{Y}$ is the total space of the canonical bundle on the projective space $\mathbb{P}(V)$. After the first version of this paper appeared as preprint, we have learned about an even earlier work by Mehrotra, cf. [Me], who have constructed a full and faithful embedding $D_{sg}(\mathcal{X}_0) \to D_{sg}(\mathcal{Y}_0)$ under the same assumption (i.e. that $\mathcal{X} = [V/\Gamma]$ and $\mathcal{Y} = G - Hilb(V)$). We also mention here that in the case of schemes, but for arbitrary base change and without any flatness assumption, the results of Section 2.2 were proved earlier by Kuznetsov, cf. Section 2.7 of [Ku].

In Section 3 we give another application in the setting when all four spaces and morphisms are toric with respect to the action of the same (split) torus T. We apply Theorem 1.1 and results of Kawamata, cf. [K1], to the case when $\Gamma \subset SL(V)$ is abelian and f is given by a Γ-invariant polynomial on V. We mention here that the work [HLdS] on Fourier-Mukai transform for Gorenstein schemes also deals with a similar situation, although from a somewhat different angle. To formulate the second application, cf. Corollaries 3.3 and 3.7, first let $\mathbb{P}(\overline{a}) := \mathbb{P}(a_0, \cdots, a_n)$ with $a_i > 0$ for all i, be the weighted projective space. Given f a quasi-homogeneous polynomial that is invariant under the action of $\mathbb{Z}_N$ where $N = \sum a_i$ the zero set Y is a Calabi-Yau hypersurface of $\mathbb{P}(\overline{a})$. Orlov gave an algebraic proof of an equivalence $D^b(Y) \cong D_{sg}^{\mathbb{C}^*}(\mathbb{C}^{n+1}, f)$ [O2, Thm 3.12], where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ with weights $(a_0, \dots, a_n)$ and $D_{sg}^{\mathbb{C}^*}$ stands for the equivariant version of the singular category. As a corollary of Theorem 1.1 and Kawamata's theorem on toric crepant resolutions [K1, Prop. 4.2] we give a geometric proof of this statement and also generalize it to products of Calabi-Yau hypersurfaces in simplicial toric varieties with nef anticanonical class.

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2. Proof

In this section we consider the following commutative diagram in which all horizontal arrows are regular closed embeddings of codimension one, and all other arrows are the canonical projections:

(2.1)
$$\begin{array}{ccccc}
\mathcal{Y}_0 & \xrightarrow{i_0} & \mathcal{Y} & \xleftarrow{\pi_{\mathcal{X}}} & \\
\uparrow{\scriptstyle \pi_{\mathcal{Y}_0}} & & \uparrow{\scriptstyle p_{\mathcal{Y}}} & & \\
\mathcal{X}_0 \times \mathcal{Y}_0 & \xrightarrow{k_0} & \mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y} & \xrightarrow{h} & \mathcal{X} \times \mathcal{Y} \\
\downarrow{\scriptstyle \pi_{\mathcal{X}_0}} & & \downarrow{\scriptstyle p_{\mathcal{X}}} & & \\
\mathcal{X}_0 & \xrightarrow{j_0} & \mathcal{X} & \xleftarrow{\pi_{\mathcal{X}}} &
\end{array}$$

Note that the both squares are cartesian and the vertical arrows represent flat morphisms. We will also consider the shifted line bundles

(2.2)
$$\omega := \pi_{\mathcal{X}}^* K_{\mathcal{X}}[n], \qquad \widetilde{\omega} := h^* \omega[-1], \qquad \omega_0 = \mathbb{C}_0^* \widetilde{\omega}$$

on $\mathcal{X} \times \mathcal{Y}$, $\mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y}$ and $\mathcal{X}_0 \times \mathcal{Y}_0$, respectively.

2.1. **Generalities on sheaves and stacks.** To prove Theorem 1.1 we first need a few lemmas. The reader may wish to skip to Section 2.2 and return to the statements of this section as they are quoted in the proof.

Lemma 2.1. [LMB, Prop. 13.1.9] *Let*

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{v} & \mathcal{Y} \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \zeta} \\ \mathcal{X}' & \xrightarrow[u]{} & \mathcal{X} \end{array}$$

be a cartesian diagram of Deligne-Mumford stacks with ζ quasi-compact and u flat and $\mathscr{N} \in D^+_{qc}(\mathcal{Y})$. Then the canonical morphism

$$u^*\zeta_*\mathscr{N} \to \xi_* v^* \mathscr{N}$$

is an isomorphism in $D^+_{qc}(\mathcal{X}')$.

Lemma 2.2. *Let $\mathcal{X}$ be a smooth quasi-projective stack. There exists a smooth projective stack $\overline{\mathcal{X}}$ and an open embedding $\mathcal{X} \to \overline{\mathcal{X}}$.*

Proof. By assumption $\mathcal{X}$ admits a locally closed embedding in a smooth projective stack $\mathcal{W}$. By [Kr] we can assume that $\mathcal{W} = P/GL(n)$ where P is a smooth quasi-projective variety and $GL(n)$ acts on P linearly and with finite stabilizers. Then also $\mathcal{X} \simeq Q/GL(n)$ where $Q \subset P$ is smooth and locally closed. The closure $\overline{Q}$ of Q in P is $GL(n)$-invariant, although it may be singular. Using canonical desingularization (i.e. versions due to Bierston-Millman, Villamayor and Włodarczyk) by blowing up a maximal stratum of a certain local invariant which is automatically $GL(n)$-invariant, we can find an iterated $GL(n)$-equivariant blowup $\widetilde{P} \to P$ with smooth $GL(n)$-invariant closed centers, such that the proper transform $\widetilde{Q}$ is smooth and the morphism $\widetilde{Q} \to \overline{Q}$ restricts to an isomorphism over Q. Then $\widetilde{Q}$ is quasi-projective, and the stabilizers of the G-action on $\widetilde{Q}$ are still finite (since they embed into the stabilizers of points in $\overline{Q}$). Moreover, the moduli space $\widetilde{Q}/GL(n)$ of the quotient stack $[\widetilde{Q}/GL(n)] =: \overline{\mathcal{X}}$ is projective. In fact, since $\widetilde{Q}/GL(n)$ is closed in $\widetilde{P}/GL(n)$, it suffices to show that the morphism $\widetilde{P}/GL(n) \to P/GL(n)$ is projective. By induction we can assume that $\widetilde{P}$ is a single blowup of P at a smooth $GL(n)$-invariant center R. We can find an ample line bundle L on P and a finite-dimensional G-invariant subspace of sections $U \subset \Gamma(P, L)$ such that the common zero scheme of these sections is R. Then $\widetilde{P}$ can be identified with a closed subvariety of $P \times \mathbb{P}(V^*)$ (i.e. closure of the graph of the rational map defined by the linear system $|U|$) thus it suffices to show that $P \times \mathbb{P}(V^*)/GL(n) \to P/GL(n)$ is a projective morphism, which follows easily by a GIT-type argument. □

The next lemma has a relatively quick proof due to the (quasi)-projectivity condition which imposed on stacks.

Lemma 2.3. *(Relative Serre Duality: smooth projective case) Let $\mathcal{X}$ be a smooth projective stack of dimension n and $\mathcal{Y}$ a quasi-projective stack. The functor $\pi_{\mathcal{Y}*} : D^b(\mathcal{X} \times \mathcal{Y}) \to D^b(\mathcal{Y})$ has a right adjoint*

$$\pi^!_{\mathcal{Y}}(\cdot) \simeq \pi^*_{\mathcal{X}} K_{\mathcal{X}}[n] \otimes \pi^*_{\mathcal{Y}}(\cdot)$$

Proof. We first observe that our stacks satisfy the resolution property and morphisms are separated. Hence by Proposition 1.9 in [Ni] the right adjoint $\pi^!_{\mathcal{Y}}$ of $\pi_{\mathcal{Y}*}$ exists although apriori it is defined as a functor $D(\mathcal{Y}) \to D(\mathcal{X} \times \mathcal{Y})$ on unbounded derived categories of complexes of $\mathcal{O}$-modules with quasi-coherent cohomology. By base change Lemma 2.1, existence of $\pi^!_{\mathcal{Y}}$ and the proof of Theorem 5.4 in [Ne] we can conclude that

$$\pi^!_{\mathcal{Y}}(\cdot) \simeq \pi^!_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}}) \otimes \pi^*_{\mathcal{Y}}(\cdot).$$

It remains to establish

$$\pi^!_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}}) \simeq \pi^*_{\mathcal{X}} K_{\mathcal{X}}[n]$$

When $\mathcal{Y} \simeq Spec(k)$, this is the contents of Theorem 1.32 (Smooth Serre Duality) in [Ni]. For general $\mathcal{X}$ we consider the diagram

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{Y} & \xrightarrow{\pi_{\mathcal{X}}} & \mathcal{X} \\ {\scriptstyle \pi_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{Y} & \xrightarrow[q]{} & Spec(k) \end{array}$$

Since $p^!\mathcal{O}_{Spec(k)} \simeq K_{\mathcal{X}}[n]$ by *loc. cit.* it remains to show that $\pi^*_{\mathcal{X}} p^! \simeq \pi^!_{\mathcal{Y}} q^*$. Following Verdier, we consider the composition

$$\pi^*_{\mathcal{X}} p^! \to \pi^!_{\mathcal{Y}} \pi_{\mathcal{Y}*} \pi^*_{\mathcal{X}} p^! \simeq \pi^!_{\mathcal{Y}} q^* p_* p^! \to \pi^!_{\mathcal{Y}} q^*,$$

where we have used the morphisms induced by adjunctions $Id \to \pi^!_{\mathcal{Y}} \pi_{\mathcal{Y}*}$, $p_* p^! \to Id$ and the base change formula from Lemma 2.1. The morphism of functors $\pi^*_{\mathcal{X}} p^! \to \pi^!_{\mathcal{Y}} q^*$ is functorial with respect to $\mathcal{Y}$ hence in proving that it is an isomorphism it suffices to replace $\mathcal{Y}$ by a scheme Z admitting a finite flat surjective morphism $Z \to \mathcal{Y}$, which exists by [Kr]. Now we apply Theorem 1.23 in [Ni]. This finishes the proof. □

Corollary 2.4. *Recall the notation of* (2.1) *and* (2.2) *and let* $\mathscr{H}$*, resp.* $\mathscr{H}_0$ *be a complex on* $\mathcal{X} \times \mathcal{Y}$*, resp.* $\mathcal{X}_0 \times \mathcal{Y}_0$*, which has proper support over* $\mathcal{Y}$*, resp.* $\mathcal{Y}_0$*. Let also* b*, resp.* b_0*, be a complex on* $\mathcal{Y}$*, resp.* $\mathcal{Y}_0$*. Then there are isomorphisms of bifunctors*

$$Hom_{\mathcal{Y}}(\pi_{\mathcal{Y}*}\mathscr{H}, b) \simeq Hom_{\mathcal{X}\times\mathcal{Y}}(\mathscr{H}, \omega \otimes \pi^*_{\mathcal{Y}}(b)) \tag{2.3}$$

$$Hom_{\mathcal{Y}_0}(\pi_{\mathcal{Y}_0*}\mathscr{H}_0, b_0) \simeq Hom_{\mathcal{X}_0\times\mathcal{Y}_0}(\mathscr{H}, \omega_0 \otimes \pi^*_{\mathcal{Y}_0}(b_0)). \tag{2.4}$$

Proof. For the first isomorphism, choose a smooth compactification $\mathcal{X} \to \overline{\mathcal{X}}$ as in Lemma 2.2, view $\mathscr{H}$ as an object on $\overline{\mathcal{X}} \times \mathcal{Y}$ and then apply Lemma 2.3.

For the second isomorphism consider the upper cartesian square in (2.1). Replacing $\mathcal{X}$ by a larger open substack in the compactification $\overline{\mathcal{X}}$ (and possibly blowing up $\overline{\mathcal{X}}$ but without changing $\mathcal{X}$) we can assume that $f : \mathcal{X} \to \mathbb{A}^1$ is proper (then $\mathscr{H}_0$ will be replaced by its direct image to this larger open substack). Therefore we can assume that $p_{\mathcal{Y}}$ and $\pi_{\mathcal{Y}_0}$ are proper. As in the end of the proof of Lemma 2.3 we have $k_0^* p^!_{\mathcal{Y}} \simeq \pi^!_{\mathcal{Y}_0} i_0^*$. Then

$$Hom_{\mathcal{Y}_0}(\pi_{\mathcal{Y}_0*}\mathscr{H}_0, b_0) \simeq Hom_{\mathcal{X}_0\times\mathcal{Y}_0}(\mathscr{H}_0, \pi^!_{\mathcal{Y}_0}(b_0)) \simeq Hom_{\mathcal{X}_0\times\mathcal{Y}_0}(\mathscr{H}_0, \pi^!_{\mathcal{Y}_0}(\mathcal{O}_{\mathcal{Y}_0}) \otimes \pi^*_{\mathcal{Y}}(b_0))$$

and we only need to apply

$$\pi^!_{\mathcal{Y}_0}(\mathcal{O}_{\mathcal{Y}_0}) \simeq \pi^!_{\mathcal{Y}_0} i_0^*(\mathcal{O}_{\mathcal{Y}}) \simeq k_0^* p^!_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}}) \simeq \omega_0.$$

□

Lemma 2.5. *The complex* $\mathscr{F}_0 = k_0^* \widetilde{\mathscr{F}}$ *is bounded and has proper support over* $\mathcal{X}_0$ *and* $\mathcal{Y}_0$*. Moreover, the following properties hold for* $\mathscr{F}_0$*:*

(1) *for any perfect complex $\mathscr{K}$ on $\mathcal{X}_0 \times \mathcal{Y}_0$ the direct image $\pi_{\mathcal{X}_0 *}(\mathscr{F}_0 \otimes \mathscr{K})$, resp. $\pi_{\mathcal{Y}_0 *}(\mathscr{F}_0 \otimes \mathscr{K})$ is a perfect complex on $\mathcal{X}_0$, resp. $\mathcal{Y}_0$;*
(2) *for any complex $\mathscr{H}$ in $D^b(\mathcal{X}_0)$, resp. $D^b(\mathcal{Y}_0)$, the complex $\mathscr{F}_0 \otimes \pi^*_{\mathcal{X}_0}\mathscr{H}$, resp. $\mathscr{F}_0 \otimes \pi^*_{\mathcal{Y}_0}\mathscr{H}$ is an object of $D^b(\mathcal{X}_0 \times Y_0)$.*

Proof. The support property is obvious since we assume that $\widetilde{\mathscr{F}}$ has proper support over $\mathcal{X}$ and $\mathcal{Y}$. Boundedness of $\mathscr{F}_0$ holds since k_0 is a regular closed embedding of codimension one, hence has finite tor dimension. As for the properties (1) and (2), it suffices to prove those which involve $\pi_{\mathcal{X}_0}$.

Assume that $\widetilde{\mathscr{F}}$ has cohomology in degrees $[-n, 0]$. Since $\mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y}$ has the resolution property, we can construct a complex of vector bundles on it

$$E_{-N} \to E_{-N+1} \to \ldots \to E_{-1} \to E_0 \to 0$$

with $N > n + \dim \mathcal{X}$ which is quasi-isomorphic to $\widetilde{\mathscr{F}}$ except possibly in degree $-N$. By a standard argument using the smoothness of $\mathcal{X}$, the kernel of $E_{-N} \to E_{-N+1}$ is projective over $\mathcal{O}_{\mathcal{X}}$. Thus $\widetilde{\mathscr{F}}$ is quasi-isomorphic to a finite complex of coherent sheaves which are projective over $\mathcal{X}$. Similarly, its pullback $\mathscr{F}_0$ is isomorphic to a finite complex of coherent sheaves which are projective over $\mathcal{O}_{\mathcal{X}_0}$. This proves (2).

To show that $\pi_{\mathcal{X}_0 *}(\mathscr{F}_0 \otimes \mathscr{K})$ is perfect is suffices to show that $\pi_{\mathcal{X}_0 *}(\mathscr{F}_0 \otimes \mathscr{K}) \otimes \mathscr{H}$ is bounded for any bounded $\mathscr{H}$ in $D^b(\mathcal{X}_0)$. By projection formula it suffices to show that $\mathscr{F}_0 \otimes \mathscr{K} \otimes \pi^*_{\mathcal{X}_0}\mathscr{H}$ is bounded which is immediate from the finite $\mathcal{O}_{\mathcal{X}_0}$-projective resolution of $\mathscr{F}_0$. □

2.2. **Main argument.** Lemma 2.5 implies that we have a well-defined Fourier-Mukai transform $F_0 := \pi_{\mathcal{Y}_0 *}(\mathscr{F}_0 \otimes \pi^*_{\mathcal{X}_0}(\cdot)) : D^b(\mathcal{X}_0) \to D^b(\mathcal{Y}_0)$, cf. [Hu]. There is also a similar Fourier-Mukai transform $F : D^b(\mathcal{X}) \to D^b(\mathcal{Y})$ with the kernel $\mathscr{F} = h_* \widetilde{\mathscr{F}}$ (note that $\mathscr{F}$ is a perfect complex since $\mathcal{X}$ and $\mathcal{Y}$ are smooth).

Lemma 2.6. *Let $i_0 : \mathcal{Y}_0 \to \mathcal{Y}$ and $j_0 : \mathcal{X}_0 \to \mathcal{X}$ be the closed immersions of the fibers then there is a functorial isomorphism*

$$i_{0*}F_0 \cong F j_{0*}$$

Proof. Similarly to Theorem 6.1 in [Ch] we use a series of isomorphisms

$$\begin{aligned}
F j_{0*}(\cdot) &= \pi_{\mathcal{Y}*}(\pi^*_{\mathcal{X}}(j_{0*}(\cdot)) \otimes \mathscr{G}) \\
&= \pi_{\mathcal{Y}*}(\pi^*_{\mathcal{X}}(j_{0*}(\cdot)) \otimes j_* \mathscr{F}) \\
&= \pi_{\mathcal{Y}*}(h_*(h^* \pi^*_{\mathcal{X}} j_{0*}(\cdot) \otimes \mathscr{F})) \\
&= \pi_{\mathcal{Y}*}(h_*(p^*_{\mathcal{X}} j_{0*}(\cdot) \otimes \mathscr{F})) \\
&= \pi_{\mathcal{Y}*}(h_*(k_{0*} \pi^*_{\mathcal{X}_0}(\cdot) \otimes \mathscr{F})) \\
&= \pi_{\mathcal{Y}*}(h_* k_{0*}(\pi^*_{\mathcal{X}_0}(\cdot) \otimes k_0^* \mathscr{F})) \\
&= \pi_{\mathcal{Y}*}(i_0 \times j_0)_*(\pi^*_{\mathcal{X}_0}(\cdot) \otimes \mathscr{F}_0) \\
&= i_{0*} \pi_{\mathcal{Y}_0 *}(\pi^*_{\mathcal{X}_0}(\cdot) \otimes \mathscr{F}_0) \\
&= i_{0*} F_0(\cdot)
\end{aligned}$$

The third and sixth isomorphisms are due to the projection formula, forth is by commutativity of the lower triangle in (2.1), and the fifth is the base change isomorphism of Lemma 2.1 applied to the lower square of the same diagram. □

Lemma 2.7. *In the notation of* (2.1), (2.2), *define a complex in* $D^b(\mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y})$:

$$\widetilde{\mathscr{G}} = \mathcal{H}om(\widetilde{\mathscr{F}}, \widetilde{\omega}).$$

Then the right adjoint $G : D^b(\mathcal{Y}) \to D^b(\mathcal{X})$ *to* F *is given by the Fourier-Mukai transform with the kernel* $\mathscr{G} = h_*\widetilde{\mathscr{G}}$. *Similarly, the right adjoint* $G_0 : D^b(\mathcal{Y}_0) \to D^b(\mathcal{X}_0)$ *to* F_0 *is given by the Fourier-Mukai transform with the kernel* $\mathscr{G}_0 = k_0^*\widetilde{\mathscr{G}}$.

Proof. We first note that boundedness of $\widetilde{\mathscr{G}}$ follows by repeating the argument of Lemma 2.5 and using smoothness of $\mathcal{X}$.

To prove the assertion about F, assume that a, resp. b, is a object in $D^b(\mathcal{X})$, resp. $D^b(\mathcal{Y})$, and recall that we denoted $\mathscr{F} = h_*\widetilde{\mathscr{F}}$. Therefore by (2.3)

$$\begin{aligned} Hom_{\mathcal{Y}}(F(a), b) &= Hom_{\mathcal{Y}}(\pi_{\mathcal{Y}*}(\mathscr{F} \otimes \pi_{\mathcal{X}}^*(a)), b) && \simeq Hom_{\mathcal{X}\times\mathcal{Y}}(\mathscr{F} \otimes \pi_{\mathcal{X}}^*(a), \omega \otimes \pi_{\mathcal{Y}}^*(b)) \\ &\simeq Hom_{\mathcal{X}\times\mathcal{Y}}(\pi_{\mathcal{X}}^*(a), \mathscr{F}^\vee \otimes \omega \otimes \pi_{\mathcal{Y}}^*(b)) && \simeq Hom_{\mathcal{X}}(a, \pi_{\mathcal{X}*}(\mathscr{F}^\vee \otimes \omega \otimes \pi_{\mathcal{Y}}^*(b))) \end{aligned}$$

In the second line we use the fact that $\mathscr{F}$ is perfect and the adjunction between $\pi_{\mathcal{X}}^*$ and $\pi_{\mathcal{X}*}$. Observe that the second argument of the last expression is a Fourier-Mukai transform of b with the kernel $\mathscr{F}^\vee \otimes \omega \simeq \mathcal{H}om(\mathscr{F}, \omega)$. Thus by definition on $\mathscr{F}$ and $\widetilde{\omega}$ it suffices to show that

$$\mathcal{H}om(h_*\widetilde{\mathscr{F}}, \omega) \simeq h_*\mathcal{H}om(\widetilde{\mathscr{F}}, h^*\omega[-1])$$

Since $h^!(\cdot) \simeq h^*(\cdot)[-1]$ this follows from Corollary 1.22 of [Ni] (we may even reduce to the case of schemes constructing a morphism from the LHS to the RHS as in *loc. cit.* and then checking that it is an isomorphism on etale local affine charts). This finishes the proof for the functor G.

The assertion about G_0 is proved similarly: let $\mathscr{H}_0 = \mathscr{F}_0 \otimes \pi_{\mathcal{X}_0}^*(b)$ and apply (2.4) to obtain

$$\begin{aligned} Hom_{\mathcal{Y}_0}(F_0(a), b) &= Hom_{\mathcal{Y}}(\pi_{\mathcal{Y}_0*}(\mathscr{F}_0 \otimes \pi_{\mathcal{X}_0}^*(a)), b) \\ &\simeq Hom_{\mathcal{X}_0\times\mathcal{Y}_0}(\mathscr{F}_0 \otimes \pi_{\mathcal{X}_0}^*(a), \omega_0 \otimes \pi_{\mathcal{Y}_0}^*(b)) \\ &\simeq Hom_{\mathcal{X}_0\times\mathcal{Y}_0}(\pi_{\mathcal{X}_0}^*(a), \mathcal{H}om(\mathscr{F}_0, \omega_0 \otimes \pi_{\mathcal{Y}_0}^*(b))) \end{aligned}$$

Repeating the proof of Lemma 3.5 in [Ba] we derive from our Lemma 2.5 that

$$\mathcal{H}om(\mathscr{F}_0, \omega_0 \otimes \pi_{\mathcal{Y}_0}^*(b)) \simeq \mathcal{H}om(\mathscr{F}_0, \omega_0) \otimes \pi_{\mathcal{Y}_0}^*(b),$$

thus our assertion reduces to

$$k_0^*\mathcal{H}om(\widetilde{\mathscr{F}}, \widetilde{\omega}) \simeq \mathcal{H}om(\mathscr{F}_0, \omega_0) = \mathcal{H}om(\mathbb{C}_0^*\widetilde{\mathscr{F}}, k_0^*\widetilde{\omega}).$$

Since $\widetilde{\omega}$ is a shift of a line bundle, the last isomorphism follows immediately by replacing $\widetilde{\mathscr{F}}$ with an $\mathcal{O}_{\mathcal{X}}$-projective resolution as in the proof of Lemma 2.5. □

Lemma 2.8. *With the notation as in Lemma 2.7 there is a functorial isomorphism*

$$j_{0*}G_0 \cong Gi_{0*}$$

Proof. The proof is exactly the same as Lemma 2.6 therefore we omit it. □

To finish the proof of Theorem 1.1 we need a category theory lemma:

Lemma 2.9. [O1, Lemma 1.2] *Let $\mathscr{N}$ and $\mathscr{N}'$ be full triangulated subcategories of triangulated categories $\mathscr{D}$ and $\mathscr{D}'$ respectively. Let $F : \mathscr{D} \to \mathscr{D}'$ and $G : \mathscr{D}' \to \mathscr{D}$ be an adjoint pair of exact functors such that $F(\mathscr{N}) \subset \mathscr{N}'$ and $G(\mathscr{N}') \subset \mathscr{N}$. Then they induce functors*

$$\overline{F} : \mathscr{D}/\mathscr{N} \to \mathscr{D}'/\mathscr{N}' \quad \overline{G} : \mathscr{D}'/\mathscr{N}' \to \mathscr{D}/\mathscr{N}$$

which are adjoints. Moreover, if the functor $F : \mathscr{D} \to \mathscr{D}'$ is fully faithful, the functor $\overline{F} : \mathscr{D}/\mathscr{N} \to \mathscr{D}'/\mathscr{N}'$ is also fully faithful.

Proof of Theorem 1.1. Let $\mathscr{E} \in D^b(\mathcal{X}_0)$ and consider the exact triangle

$$\mathscr{E} \to G_0F_0\mathscr{E} \to \mathscr{C} \to \mathscr{E}[1]$$

where $\mathscr{C}$ is a cone over the morphism $\mathscr{E} \to G_0F_0\mathscr{E}$. Applying j_{0*} yields an exact triangle

$$j_{0*}\mathscr{E} \to j_{0*}G_0F_0\mathscr{E} \to j_{0*}\mathscr{C} \to j_{0*}\mathscr{E}[1]$$

But $j_{0*}G_0F_0 \cong Gi_{0*}F_0 \cong GFj_{0*}$ where the first isomorphism is by Lemma 2.8 and the second by Lemma 2.6. Since GF is isomorphic to the identity morphism and j_0 is a closed immersion, $\mathcal{C} \cong 0$. This implies that $\mathscr{E} \to G_0F_0\mathscr{E}$ is an isomorphism.

By Lemma 2.5 the functor $F_0 := \pi_{\mathcal{Y}_0}(\mathscr{F}_0 \otimes \pi^*_{\mathcal{X}_0}(\cdot))$ takes perfect complexes to perfect complexes, and similarly for G_0. By Lemma 2.7 these functors induce a pair of adjoint functors $\overline{F}_0 : D_{sg}(\mathcal{X}_0) \to D_{sg}(\mathcal{Y}_0)$ and $\overline{G}_0 : D_{sg}(\mathcal{Y}_0) \to D_{sg}(\mathcal{X}_0)$. Moreover, the composition $\overline{G}_0\overline{F}_0$ is isomorphic to the identity on $D_{sg}(\mathcal{X}_0)$ and similarly $\overline{F}_0\overline{G}_0$ is isomorphic to the identity on $D_{sg}(\mathcal{Y}_0)$. □

2.3. **Applications to McKay correspondence.** In this subsection we assume that V is a finite dimensional vector space of $k = \mathbb{C}$ and $\mathcal{X}$ is the quotient stack $[V/\Gamma]$, where $\Gamma \subset SL(V)$ is a finite subgroup acting freely in codimension one. We take $\mathcal{Y}$ to be a crepant resolution of the singular quotient variety $Z = V/\Gamma$ (thus, we implicitly assume that $\mathcal{Y}$ exists which is not always the case). Finally, let $t : Z \to \mathbb{A}^1$ be a morphism inducing the compositions $f : \mathcal{X} \to \mathbb{A}^1$, $g : \mathcal{Y} \to \mathbb{A}^1$ and denote by $\mathcal{X}_0$, $\mathcal{Y}_0$ the fibers of f and g over $0 \in \mathbb{A}^1$, respectively.

Theorem 2.10. *In the above setting, suppose that there exists a complex $\overline{\mathscr{F}}$ in $D^b(\mathcal{X} \times_Z \mathcal{Y})$ such that its direct image $\mathscr{F}$ to $\mathcal{X} \times \mathcal{Y}$ induced a Fourier-Mukai transform which is an equivalence. Let $\mathscr{F}_0 = k_0^* j_* \overline{\mathscr{F}}$ where $j : \mathcal{X} \times_Z \mathcal{Y} \to \mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y}$ and $k_0 : \mathcal{X}_0 \times \mathcal{Y}_0 \to \mathcal{X} \times_{\mathbb{A}^1} \mathcal{Y}$ are the natural closed embeddings. Then the Fourier-Mukai transform with the kernel $\mathscr{F}_0$ induces equivalences*

$$D^b(\mathcal{X}_0) \simeq D^b(\mathcal{Y}_0), \quad \mathfrak{Perf}(\mathcal{X}_0) \simeq \mathfrak{Perf}(\mathcal{Y}_0), \quad D_{sg}(\mathcal{X}_0) \simeq D_{sg}(\mathcal{Y}_0)$$

In particular, such equivalences hold either of the three cases:

(1) $\dim V = 2$ *or* 3 *and* $\mathcal{Y} = G - Hilb$
(2) V *is a symplectic vector space and* $\Gamma \subset Sp(V)$
(3) Γ *is a finite abelian subgroup of* $SL(V)$ *and* $\mathcal{Y}$ *is projective over* Z.

Proof. The first part of the statement is an immediate consequence of Theorem 1.1. In the second part, we only need to establish existence of appropriate $\widetilde{\mathscr{F}}$. For (1) and (2) this follows from the main results in [BKR] and [BK], respectively. For (3) we observe that since Γ is abelian, there exists an n-dimensional torus T such that Z is a toric variety with the torus T, and that by the proof of Corollary 3.5 in [K2] the crepant resolution $\mathcal{Y}$ is automatically toric with the same torus T. Note that the corollary quoted is stated for projective varieties, and in our case we need to repeat its proof invoking the relative toric version of MMP from [FS]. Existence of $\widetilde{\mathscr{F}}$ is proved in Theorem 3.1 by a simple extension of results in [K1]. □

Remark 2.11. We could require that the action of Γ is generically free (i.e. stabilizers *are* allowed in codimension one). In this case consider the normal subgroup $\Pi \subset \Gamma$ generated by complex reflections in V. Then the quotient $V' = V/\Pi$ is a smooth variety and in Theorem 2.10 one can consider $\mathcal{X} = [V'/\Gamma']$ with $\Gamma' = \Gamma/\Pi$. Alternatively, we can keep $\mathcal{X} = V/\Gamma$ but view $\mathcal{Y}$ not as a smooth variety but as a smooth stack with non-trivial cyclic stabilizers in codimension one (see the beginning of the next section).

3. Applications to toric geometry

3.1. **Toric stacks and Kawamata's Theorem.** The applications in this section come from toric geometry. We will assume $k = \mathbb{C}$. Let X, Y be quasi-smooth (i.e. simplicial) toric varieties with the action of the same split torus T, with two effective T-invariant $\mathbb{Q}$-divisors B, C respectively, and assume that the coefficients of these divisors are in the set $\{1 - \frac{1}{r} | r \in \mathbb{N}\}$. To this data one can associate two smooth Deligne-Mumford stacks $\mathcal{X}, \mathcal{Y}$, respectively, as in [K1]. One reason why considering B, C is useful is that a quotient of a smooth variety V by an effective action of a finite group Γ may be smooth in codimension one yet have non-trivial stabilizers there (automatically cyclic). Then the multiplicities of B and C will encode the information about the sizes of such stabilizers.

If B and C are zero, the stacks $\mathcal{X}$, $\mathcal{Y}$ may be interpreted in terms of the Batyrev-Cox quotient construction which we briefly recall. Let $\mathbb{P}(\Sigma)$ be a quasi-smooth toric variety associated to a simplicial fan Σ which has n one-dimensional cones $\rho_1, \ldots, \rho_n$. Then, cf. e.g. [CLS], $\mathbb{P}(\Sigma)$ it can be realized as a quotient $(\mathbb{C}^n \setminus \mathbb{B})/G$ where the basis $u_1, \ldots, u_n$ in $\mathbb{C}^n$ is in bijective correspondence with $\rho_1, \ldots, \rho_n$, $\mathbb{B}$ is a union of some coordinate subspaces of codimension at least 2, and G is an algebraic subgroup of $(\mathbb{C}^*)^n$ with its natural action on $\mathbb{C}^n$. Thus G itself is isomorphic to a product of several copies of $(\mathbb{C}^*)$ and a finite abelian group. The assumption that Σ is simplicial ensures, cf. *loc. cit.*, that G-action on $\mathbb{C}^n \setminus \mathbb{B}$ has finite stabilizers and thus there exists a smooth Deligne-Mumford quotient stack $\mathcal{P}(\Sigma)$ with coarse moduli space isomorphic to $\mathbb{P}(\Sigma)$.

We will use a version of the Kawamata's theorem on equivalence of derived categories for toric stacks. In our setup, we assume that in the diagram (1.1) all stacks are toric with the action of the same torus T, and that all morphisms are T-equivariant.

Theorem 3.1. *In the situation described, there exists a T-equivariant kernel $\mathcal{F}$ on $\mathcal{X} \times \mathcal{Y}$ given by a direct image of a T-equivariant object in $D^b(\mathcal{X} \times_Z \mathcal{Y})$, such that the corresponding Fourier-Mukai transform $\Phi_{\mathscr{F}} : D^b(\mathcal{X}) \to D^b(\mathcal{Y})$ is an equivalence.*

Proof. By the relative version of the toric MMP explained in [FS] and a standard argument modeled on the proof of Theorem 12.1.8 in [Ma], the birational isomorphism $X \to Y$ may be decomposed into a finite sequence of T-equivariant divisorial contrations and flips *over* Z which are log crepant. For every such contraction or flip we can apply Theorem 4.2 in [K1]. Observe that the kernels in *loc. cit.* are indeed T-equiariant and given by direct images from $\mathcal{X} \times_Z \mathcal{Y}$. The convolution of several such kernels (giving the composition of equivalences) also satisfies this condition. □

3.2. **Hypersurfaces in simplicial toric varieties.** As another application of Theorem 1.1 we extend a result of Orlov on the derived category of a Calabi-Yau hypersurface in a weighted projective space. From now on the log divisors B, C are zero. Our application is based on the following result due to M.U. Isik, cf. [Is]. For completeness we reproduce a sketch of the proof with the kind permission of the author.

Proposition 3.2. *Let X be a smooth Deligne-Mumford stack over $\mathbb{C}$ and s a regular section of a vector bundle E with determinant dual to the canonical bundle of X. Let $Y \subset X$ be the zero scheme of s and $Z \subset E^\vee$ the zero scheme of s viewed as a fiberwise linear function on the total space of the dual bundle $E^\vee$. Then $D^b(Y)$ is equivalent to the Karoubian completion (or split completion) of the equivariant singular category $D^{\mathbb{C}^*}_{sg}(Z)$, which obtained by adding images of all projectors. The action of $\mathbb{C}^*$ on Z is restricted from the natural action by scalar dilations on the fibers of $E^\vee$.*

Proof. The proof involves sheaves of graded DG-algebras (and modules over them) which have homological degree $\deg_h$ and internal degree $\deg_i$. First one identifies (up to derived equivalence) complexes of graded $\mathcal{O}_Z$-modules with complexes of graded modules over the Koszul DG algebra

$$\mathcal{B} = (\varepsilon Sym^\bullet(E) \to Sym^\bullet(E)); \qquad d(\varepsilon) = s$$

where the formal variable ε satisfies $\varepsilon^2 = 0$ and the differential is linear with respect to $Sym^\bullet(E)$. In both categories the differential increases $\deg_h$ by one and preserves $\deg_i$. Note that $\deg_h(\varepsilon) = -1$, $\deg_i(\varepsilon) = 1$, $\deg_h(Sym^\bullet(E)) = 0$ while the internal degree on $Sym^\bullet(E)$ is given by the usual polynomial degree.

Next use fiber-by-fiber BGG correspondence (or Koszul duality) to construct a derived equivalence between graded DG modules over $\mathcal{B}$ and graded DG-modules over the “dual" algebra

$$\mathcal{A} = \Lambda^\bullet(E^\vee) \otimes_{\mathcal{O}_X} \mathcal{O}_X[t]$$

where the differential is t-linear and satisfies $d(f) = t \cdot \langle s, f\rangle$ for a local section f of $E^\vee$ (this admits a unique extension to the exterior algebra by Leibniz rule). This time $\deg_i(t) = \deg_i(f) = 1$, $\deg_h(t) = 2, \deg_h(E^\vee) = 1$.

The composition of these two equivalences sends perfect complexes of graded $\mathcal{O}_Z$-modules map to complexes of graded $\mathcal{A}$-modules on which t acts nilpotently. Therefore, using a version of Thomason's localization theorem [TT], Lemma 5.5.1, we conclude that the split completion of $D^{\mathbb{C}^*}_{sg}(Z)$ is equivalent to the category of graded modules over the localization $\mathcal{A}_t$. Since t is now invertible, multiplication by it identifies all homogeneous components (with respect to $\deg_i$), i.e. the category is equivalent to non-graded modules over the Koszul resolution $\mathcal{A}'$ of $\mathcal{O}_Y$. Using the derived equivalence between $\mathcal{A}'$-modules and $\mathcal{O}_Y$-modules we obtain the result. □

Now consider a projective simplicial toric stack as in Section 3.1. We fix the fan Σ and drop it from notation, writing simply $\mathcal{P}$ and $\mathbb{P}$. We will require that $K^\vee_{\mathcal{P}}$ is nef, i.e. a positive power of the anti-canonical bundle on $\mathcal{P}$ descends to a nef bundle on $\mathbb{P}$.

The Picard group of the stack $\mathcal{P} = [(\mathbb{C}^n \setminus \mathbb{B})/G]$ can be identified with the $A = Hom(G, \mathbb{C}^*)$ since $\mathbb{B}$ has codimension ≥ 2 and therefore any line bundle on $\mathbb{C}^n \setminus \mathbb{B}$ is trivial. Explicitly, for $a \in A$ we have a line bundle L_a over $\mathcal{P}$ with the total space $[(\mathbb{C}^n \setminus \mathbb{B}) \times \mathbb{C}]/G$ where G acts on the second factor by the character $-a$. The sections of L_a, viewed as regular functions on the total space of L_{-a} linear along the fibers, may be identified with the space $\mathbb{C}[x_1, \ldots, x_n]^a$ of polynomials on $\mathbb{C}^n$, on which G acts via the character a. Here $x_1, \ldots, x_n$ is the basis dual to the basis $u_1, \ldots, u_n$ of Section

3.1. Observe that $\mathbb{C}[x_1, \dots, x_n]^a$ is non-zero if and only if a belongs to the semigroup $A^+ \subset A$ formed by all non-negative integral linear combinations of the G-weights $a_1, \dots, a_n$ of $x_1, \dots, x_n$, respectively. Note that all elements in $A^+ \setminus 0$ have infinite order if $\mathbb{P}$ is projective, otherwise some monomial with non-negative exponents would give a non-constant regular function on $\mathbb{P}$. The line bundle L_a descends to the coarse moduli space $\mathbb{P}$ precisely when a is trivial on all stabilizers of the G-action on $\mathbb{C}^n \setminus B$, hence the Picard group of $\mathbb{P}$ is a subgroup of finite index in A. We also recall that $K^\vee_{\mathcal{P}(\Sigma)}$ is isomorphic to L_c with

$$c = a_1 + \ldots + a_n.$$

In the smooth case this is proved in Section 4.3 of [Fu] which is sufficient for us since $\mathbb{P}$ is smooth in codimension 1. Choose and fix a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]^c$ and consider its zero locus $\mathcal{M}$, a closed substack of $\mathcal{P}$ which has trivial canonical bundle by the adjunction formula. Note that $\mathcal{M}$ may be identified with the quotient stack $\big[Z(f) \cap (\mathbb{C}^n \setminus Z)\big]/G$.

Let $H \subset G$ be the kernel of the character $c : G \to \mathbb{C}^*$. Then the group of characters of H may be identified with $A_H = A/\mathbb{Z} \cdot c$ and we denote by A_H^+ the image of A^+ in A_H. The G-action on $\mathbb{C}^n$ restricts to H and for a generic choice of $a \in A_H^+$ the H-linearization of the trivial bundle, induced by a, will satisfy the property that all semi-stable points are stable (in fact, it suffices to require that a does not belong to any subgroup in A_H which is generated by a finite subset in the image of $\{a_1, \dots, a_n\}$ and has non-maximal rank). Let $\mathcal{Y}^a = (\mathbb{C}^n)^s/H$ be the corresponding quotient stack; its coarse moduli space Y is just the GIT quotient with respect to the linearization induced by a.

Since the character $c : G \to \mathbb{C}^*$ is trivial on H, the polynomial f on $\mathbb{C}^n$ is H-invariant and therefore descends to a morphism $g : \mathcal{Y}^a \to \mathbb{A}^1$. For the same reason, i.e. triviality of c on H, the stack $\mathcal{Y}^a$ has trivial canonical bundle. The action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ descends to the action of the torus $T_H = (\mathbb{C}^*)^n/H$ on $\mathcal{Y}^a$. Observe that there exists a short exact sequence

$$1 \to \mathbb{C}^* \to T_H \to T \to 1$$

where the subgroup $\mathbb{C}^*$ may be identified with G/H.

Corollary 3.3. *In the above setting, the derived category $D^b(\mathcal{M})$ is equivalent to the split completion of the equivariant singular category $D_{sg}^{\mathbb{C}^*}(g^{-1}(0))$.*

Proof. First, we can view f as a function, which we denote by the same letter, on the total space of the canonical bundle $K_{\mathcal{P}}$. By Proposition 3.2 the derived category of $\mathcal{M}$ is equivalent to the split completion of the equivariant singular category of the zero fiber of $f : K_{\mathcal{P}} \to \mathbb{A}^1$. Observe that both $K_{\mathcal{P}}$ and $\mathcal{Y}^a$ are toric with the same torus T_H. For $K_{\mathcal{P}}$ this follows from the fact that it is a quotient of an open subset in $\mathbb{C}^n \times \mathbb{C}$ by G, and if we use the character c to lift the natural embedding $G \subset (\mathbb{C}^*)^n$ to an embedding $G \subset (\mathbb{C}^*)^n \times \mathbb{C}$, then the quotient $(\mathbb{C}^*)^n \times \mathbb{C}/G$ is canonically isomorphic to T_H.

Since both $K_{\mathcal{P}}$ and $\mathcal{Y}^a$ have trivial canonical bundles, by Theorem 1.1 and Theorem 3.1 it suffices to find an affine variety Z which is toric with respect to T_H, and two proper equivariant birational morphisms $\phi : K_{\mathcal{P}} \to Z$, $\psi : \mathcal{Y}^a \to Z$ which induce projective morphisms on coarse moduli spaces. Since generic stabilizers of $K_{\mathcal{P}}$ and $\mathcal{Y}^a$ are trivial and the morphisms to the coarse moduli spaces are proper ([Vi, Prop. 2.11], [Ed, Prop. 4.2]), it suffices to construct birational projective toric morphisms from the coarse moduli spaces to Z.

Choose Z to be the spectrum of the ring of invariants $\mathbb{C}[x_1, \dots, x_n]^H$. Then the coarse moduli space Y^a of $\mathcal{Y}^a$ is projective over Z: by GIT it is a $Proj$ of a graded ring $\bigoplus_{l \geq 0} R_l$ with $Z \simeq Spec(R_0)$.

On the other hand, the moduli space of $K_{\mathcal{P}}$ is the geometric quotient of an open subset in $\mathbb{C}^n \times \mathbb{C}$ by G, where the action on the second factor is via the character c of G.

Consider the ring of invariants $\mathbb{C}[x_1, \ldots, x_n, z]^G$ for this action. Since z is dual to the last coordinate vector in $\mathbb{C}^n \times \mathbb{C}$, the character of the G-action on it is $(-c)$. Every G-invariant polynomial is a linear combination of terms of the form $h(x)z^l, l \geq 0$ where G acts on $h(x) \in \mathbb{C}[x_1, \ldots, x_n]$ via the character $l \cdot c$. In particular, each $h(x)$ is H-invariant. Therefore evaluation $z \mapsto 1$ gives a ring homomorphism

$$\mathbb{C}[x_1, \ldots, x_n, z]^G \to \mathbb{C}[x_1, \ldots, x_n]^H$$

To show that it is as isomorphism observe that every H-invariant polynomial is a sum of H-invariant monomials and that of each H-invariant monomial x^α the group G acts via a character with is trivial on H. Since $Ker(A \to A_H) = \mathbb{Z} \cdot c$, such a character is a multiple of c. But a multiple $l \cdot c$ with *negative* l may not be a G-weight of monomial x^α with *positive* exponents, otherwise $x^\alpha (x_1 \ldots x_n)^{-l}$ would give a non-constant regular function on the projective variety $\mathbb{P} = (\mathbb{C}^n \setminus \mathbb{B})/G$. We proved that

$$\mathbb{C}[x_1, \ldots, x_n]^H = \bigoplus_{l \geq 0} \mathbb{C}[x_1, \ldots, x_n]^{l \cdot c}$$

which implies that the above map of invariants is an isomorphism. We also observe that the grading on the left hand side is precisely the grading resulting from the action of $\mathbb{C}^* \subset T_H$ on Z.

It remains to show that the coarse moduli space of $K_{\mathcal{P}}$, i.e. the geometric quotient $\big((\mathbb{C}^n \setminus \mathbb{B}) \times \mathbb{C}\big)/G$, admits a projective morphism to Z. Since the stabilizers of the G-action are finite the assertion would follow from GIT if $(\mathbb{C}^n \setminus \mathbb{B}) \times \mathbb{C}$ is the set of semistable points for some G-linearization of the trivial bundle, coming from a character $a \in A$. Choose any a which gives an ample line bundle in $Pic(\mathbb{P}) \subset A$. We need to prove that $(\mathbb{C}^n \setminus \mathbb{B}) \times \mathbb{C}$ is precisely the set of points for which one can find a non-vanishing polynomial $f(x)z^l$ of G-weight $m \cdot a$, $m \geq 0$.

For any point $(p, q) \in (\mathbb{C}^n \setminus B) \times \mathbb{C}$, p will project to a point $\overline{p} \in \mathbb{P}$ and by ampleness there is a section of $L_a^{\otimes m}$ not vanishing at $\overline{p}$. This section gives a polynomial $f(x)$ of G-weight $a \cdot m$ non-vanishing at p, hence $f(x)z^0$ is a quasi-invariant polynomial not vanishing at (p, q).

On the other hand, we want to show that any $f(x)z^l$ of G-weight $m \cdot c$ will vanish at any point $(p, q) \in \mathbb{B} \times \mathbb{C}$. It suffices to show that any polynomial $f(x)$ of weight $a' = l \cdot c + m \cdot a$ with $m > 0, l \geq 0$, will vanish at $p \in B$. Since we assumed that $K_{\mathcal{P}}^\vee$ is nef, replacing $f(x)$ by its positive power we can assume that a' is an ample class in $Pic(\mathbb{P})$. We can also assume that $f(x)$ is a monomial x^α.

Now take a closer look at $\mathbb{B}$, a union of coordinate subspaces of the form $\mathbb{B}_j = \{x_i = 0 | i \in P_j\}$ where $P_j \subset \{1, \ldots, n\}$ is a subset called a *primitive collection* (and the index j will run over all primitive collections). See Section 5.1 of [CLS] for details. The crucial observation is that each P_j gives a class r_j in the cone of effective curves on $\mathbb{P}$. Since a' is ample and the cone of effective curves is spanned by r_j, see Theorem 6.3.10 in *loc. cit*, the intersection number $a' \cdot r_j$ should be positive for all j, which will imply that for any j there exists $i \in P_j$ which gives a positive exponent α_i in the monomial $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Then x^α must vanish on each B_j, which will imply the assertion about the set of stable points.

In more detail: each variable $x_i, i = 1, \ldots, n$ corresponds to a one-dimensional cone $\rho_i \in N_{\mathbb{R}}$ in the fan Σ defining our toric variety, and hence to a torus-invariant prime divisor D_i in $\mathbb{P}$. One the other hand, the space of numerical classes of curves in $\mathbb{P}$ can be identified with the space of vectors

$C = (\beta_1, \dots, \beta_n)$ such that $\sum b_i \rho_i = 0$ in $N_{\mathbb{R}}$, in such a way that for a divisor $D = \sum \alpha_i D_i$ the intersection number $D \cdot C$ is given by $\sum \alpha_i \beta_i$, cf. Exercise 6.3.3 in *loc. cit.*

Now, for any primitive collection P_j, Batyrev's construction gives a relation in $N_{\mathbb{R}}$ of the type

$$r_j = \sum_{i \in P_j} \rho_i - \sum_{s=1}^{n} c_s \rho_s$$

with $c_s \in \mathbb{Q}_{\geq 0}$, cf. Definition 6.3.9 of *loc. cit.* Moreover, since r_j corresponds to a numerically effective class of curves by ampleness of a' we should have

$$\sum_{i \in P_j} \alpha_i - \sum_{s=1}^{n} c_s \alpha_s > 0.$$

All c_s and α_s are non-negative hence α_i for some $i \in P_j$. That is, the monomial x^α contains the variable x_i with a positive exponent and hence vanishes on the subspace $\mathbb{B}_j \subset \mathbb{B}$. Since this holds for every primitive collection P_j, and $\mathbb{B}$ is the union of $\mathbb{B}_j$, we conclude that x^α vanishes on $\mathbb{B}$.

To sum up: we have proved that $\mathcal{X} = K_{\mathcal{P}}$ and $\mathcal{Y}^a$ admit projective toric morphisms (automatically birational) onto the affine variety $Z = Spec(\mathbb{C}[x_1, \dots, x_n]^H)$. Since the functions on $\mathcal{X}$ and $\mathcal{Y}^a$ are pulled back from Z, we can apply Theorem 3.1 and Theorem 1.1 to conclude that the equivariant singular categories of $\mathcal{X}_0$ and $\mathcal{Y}_0^a$ are equivalent, which finishes the proof. □

Example 3.4. [O2, Thm. 3.12, Calabi-Yau case] Take the weighted projective space $\mathbb{P} = \mathbb{P}(\overline{a}) := \mathbb{P}(a_0, \cdots, a_n)$ with $a_i > 0$ for all i. Let f be a quasi-homogeneous polynomial that is invariant under the action of $H = \mathbb{Z}_N$ where $N = \sum a_i$ and $\mathcal{M}$ the zero set of f in the toric stack corresponding to $\mathbb{P}(a)$. Denote by g the induced function on the quotient stack $\mathcal{Y} = [\mathbb{C}^n / \mathbb{Z}_N]$ (here H is finite, and no choice of a is needed). Then there is an equivalence between $D^b(\mathcal{M})$ and the split completion of $D_{sg}^{\mathbb{C}^*}(g^{-1}(0))$.

Example 3.5. Take $\mathcal{P} = \mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$ with $A = \mathbb{Z} \oplus \mathbb{Z}$. The weights of the homogeneous coordinates $x_1, \dots, x_4$ are $a_1 = a_2 = (1, 0)$ and $a_3 = a_4 = (0, 1)$. A quick calculation shows that $H \simeq \mathbb{C}^* \times \mathbb{Z}_2$ hence $\mathcal{Y}^a$ is a GIT quotient of $\mathbb{C}^4$ by H. Restricting the action on $\mathbb{C}^4$ to $\mathbb{C}^* \subset H$ we find that the weights are $(+1, +1, -1, -1)$. There are two essentially different linearizations: one which gives a positive weight when restricted to $\mathbb{C}^*$ and one which gives a negative weight. The two resulting quotients $\mathcal{Y}^+$ and $\mathcal{Y}^-$ differ by a standard toric flop. In particular, they have equivalent derived categories and applying Theorem 1.1 with $\mathcal{X} = \mathcal{Y}^+$, $\mathcal{Y} = \mathcal{Y}^-$ we see that the equivariant singular derived categories of $(g^+)^{-1}(0)$ and $(g^-)^{-1}(0)$ are equivalent. Their split completions are further equivalent to the derived category of the elliptic curve $\mathcal{M}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by Corollary 3.3.

Example 3.6. Take $\mathcal{P} = \mathbb{P} = \mathbb{F}_1$, the Hirzebruch surface, which may be identified with the quotient $(k^4 \setminus \mathbb{B})/(\mathbb{C}^*)^2$ where $\mathbb{B}$ is the union of two coordinate planes $x_1 = x_2 = 0$ and $x_3 = x_4 = 0$. The weights of the action of $G = (\mathbb{C}^*)^2$ on $\mathbb{C}^4$ are $(1, 0), (1, 1), (0, 1), (0, 1)$. The subgroup $H \simeq \mathbb{C}^*$ is given by (t^3, t^{-2}) and hence the weights of the H action on $\mathbb{C}^4$ are $(3, 1, -2, -2)$. Again we have two linearizations for the H action and the corresponding choices of $\mathcal{Y}^a$ differ by a toric flop.

3.3. **Products of hypersurfaces.** Assume that for $i = 1, \dots m$, we are given a Calabi-Yau hypersurface $\mathcal{M}_i$ in a projective simplicial toric stack $\mathcal{P}_i$ with nef anticanonical bundle. By adjunction, $\mathcal{M}_i$ is defined by vanishing of a section t_i of the anticanonical bundle $K^\vee_{\mathcal{P}_i}$. We can view the product $\mathcal{M} = \mathcal{M}_1 \times \dots \mathcal{M}_m$ as a complete intersection in the stack $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_m$.

On the other hand, let $\mathcal{Y}^{a_i}$ be a stack corresponding to a character a_i as in Section 3.2, and $g_i : \mathcal{Y}^{a_i} \to \mathbb{A}^1$ the function induced by the defining equation of $\mathcal{M}_i$. This gives a function $g = g_1 + \dots + g_m$ on the direct product $\mathcal{Y} = \mathcal{Y}^{a_1} \times \dots \times \mathcal{Y}^{a_m}$. The action of $\mathbb{C}^*$ on each $\mathcal{Y}^{a_i}$ induces the diagonal action on $\mathcal{Y}$.

Corollary 3.7. *There exists an equivalence between $D^b(\mathcal{M})$ and the split completion of $D^{\mathbb{C}^*}_{sg}(g^{-1}(0))$.*

Proof. Let $\mathcal{X} = K_{\mathcal{P}_1} \times \dots K_{\mathcal{P}_m}$ be the product of total spaces of canonical bundles on the stacks, and $f : \mathcal{X} \to \mathbb{A}^1$ the fiberwise linear function corresponding to $t_1 \oplus \dots \oplus t_m$. By the same argument as with a single hypersurface, both $\mathcal{X}$ and $\mathcal{Y}$ admit a proper birational morphism to the product affine variety $Z = Z_1 \times \dots \times Z_m$ which induces a projective morphism on moduli spaces. Thus $D^{\mathbb{C}^*}_{sg}(f^{-1}(0)) \simeq D^{\mathbb{C}^*}_{sg}(g^{-1}(0))$ as before. Now we apply Proposition 3.2 to the direct sum of anticanonical bundles on $\mathcal{P}_1 \times \dots \times \mathcal{P}_m$. □

Remark 3.8. One could attempt to apply the same reasoning to general Calabi-Yau complete intersections in a toric variety $\mathcal{P}$ but it leads to a technical difficulty. Let $\beta_1, \dots, \beta_m$ be elements in the stack Picard group A, such that

$$\beta_1 + \dots + \beta_m = a_1 + \dots + a_n$$

Assume in addition that each bundle L_{β_i} has a non-zero regular section t_i and that $t_1, \dots, t_m$ form a regular sequence. This implies, in particular, that each β_j is a linear combination of the a_i with non-negative coefficients. Let A_H be the quotient by the subgroup of A spanned by $\beta_1, \dots, \beta_m$ and $H \subset G$ the dual group. Suppose we wanted to establish a derived equivalence between the total space $\mathcal{X}$ of the bundle $L_{-\beta_1} \oplus \dots \oplus L_{-\beta_m}$ and a quotient $\mathcal{Y}$ of some open subset of H-stable points in $\mathbb{C}^n$, by the action of H. In particular, we would want proper birational morphisms from $\mathcal{X}$ and $\mathcal{Y}$ to the same variety (or stack) Z.

If we define Z as a spectrum of a ring of invariants, then for $\mathcal{Y}$ this ring should be $\mathbb{C}[x_1, \dots, x_n]^H$, while for $\mathcal{X}$ the ring is $\mathbb{C}[x_1, \dots, x_n, z_1, \dots, z_m]^G$ where G acts on the extra variable z_j via the character $-\beta_j$. Sending each z_j to 1 (or to some nonzero constant c_j) we will get a map

$$\mathbb{C}[x_1, \dots, x_n, z_1, \dots, z_m]^G \to \mathbb{C}[x_1, \dots, x_m]^H.$$

We observe that this map is injective if and only if $\beta_1, \dots, \beta_m$ are linearly independent over $\mathbb{Z}$, which is a reasonable condition. However, the surjectivity is less trivial: one would need to ensure that for every equality in A

$$p_1 a_1 + \dots + p_n a_n = q_1 \beta_1 + \dots q_m \beta_m$$

the condition $p_i \geq 0$ for all i implies the condition $q_j \geq 0$ for all j. The reader is invited to check that this fails e.g. when $\mathcal{P} = \mathbb{P}^2 \times \mathbb{P}^2$ with $Pic = \mathbb{Z} \oplus \mathbb{Z}$, and

$$a_1 = a_2 = a_3 = (1,0); \quad a_4 = a_5 = a_6 = (0,1); \quad \beta_1 = a_1 + a_4 + a_5; \quad \beta_2 = a_2 + a_3 + a_6.$$

In such a situation $\mathcal{X}$ and $\mathcal{Y}$ would be proper over non-isomorphic, although birational, affine varieties. Perhaps one could modify our construction, e.g. by passing to a partial compactification of $\mathcal{X}$ or $\mathcal{Y}$, to obtain the analogue of Corollary 3.7 in a more general situation.

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