Algebraic Geometry and String Theory: Enumeration of Rational Curves on $\mathbb{P}^2$

Jeremy Pecharich

Introduction

Question: How many lines are there through two distinct points?
Answer: 1 (This is usually attributed to the Greeks.)

Question: A natural extension of this problem is to determine the number, N_d, of rational curves of degree d passing through $3d-1$ points in general position in the complex plane?
Answer: Not so easy!!

Known up to 1990

1. $N_1 = N_2 = 1$ (Antiquity)
2. $N_3 = 12$ (Steiner -1848)
3. $N_4 = 620$ (Zeuthen -1873)

The solution is surprisingly simple. It relies on two very basic mathematical properties.

1. $a\star(b\star c) = (a\star b)\star c$ (Associativity of a certain product.)
2. The fact that there is one line through two distinct points in the plane.

Warning

1. Mathematicians would more than likely not have been able to get such a solution without physics intervening.

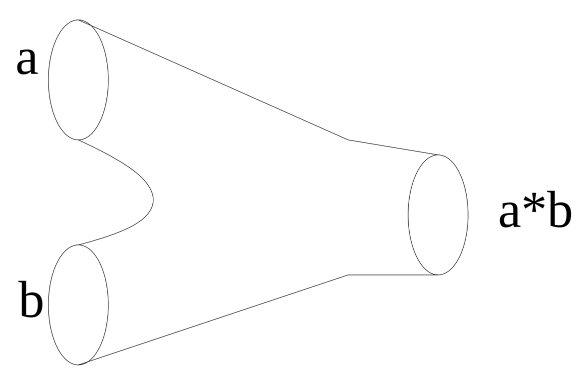
Topological Quantum Field Theory (TQFT)

Definition of a TQFT

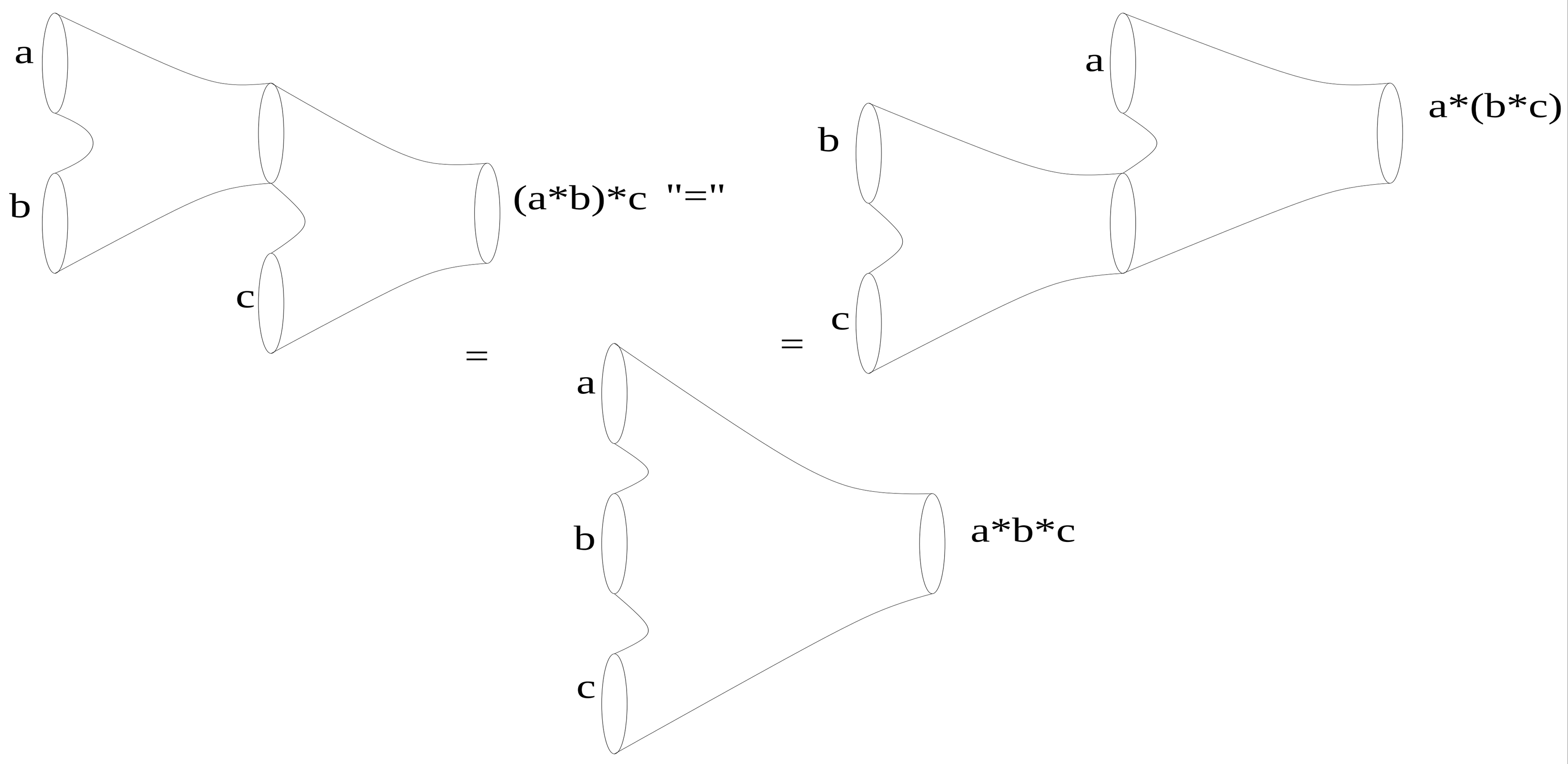
- To each closed oriented d-dimensional manifold Y is associated a finite-dimensional complex vector space $Z(Y)$. This vector behaves functorially under isomorphisms of Y.
- To each $(d+1)$-dimensional oriented manifold X whose boundary ∂X is a closed oriented d-dimensional manifold is associated an element $Z_X \in Z(\partial X)$. This element behaves functorially under isomorphisms of X.

(1+1)-dimensional TQFT We will describe (1+1)-dimensional TQFT using "pants" diagrams. Since there are only two 1-dimensional manifolds without boundary the description is easy. By definition we associate a complex vector space $\mathcal{H} = Z(S^1)$. For two elements $a, b \in \mathcal{H}$ we define $a \star b$ as:

We define a "3-point correlation" function by $\langle a, b, c\rangle = \langle a\star b, c\rangle$. We now claim that $\langle a\star b, c\rangle = \langle a, b\star c\rangle$. This also says that the product is associative i.e. $(a\star b)\star c = a\star(b\star c)$.



Product



Associativity

Proof

Using an idea from physics we define a function on the cohomology of $\mathbb{P}^2$ by

$$\Phi^{\mathbb{P}^2}(x_0T^0 + x_1T^1 + x_2T^2) := \frac{1}{2}(x_0x_1^2 + x_0^2x_2) + \sum_{d=1}^{\infty} N_d \frac{x_2^{3d-1}}{(3d-1)!}e^{dx_1}$$

where T^0, T^1, T^2 are the generators of the cohomology of $\mathbb{P}^2$. The first part of the equation is "classical"

Define a product on cohomology of $\mathbb{P}^2$

$$T^i \star T^j = \sum_{e+f=2} \Phi_{ije}T^f$$

where

$$\Phi_{ije} := \partial_i\partial_j\partial_e\Phi$$

From here one easily checks that

1. $T^1\star T^1 = T^2 + \Phi_{111}T^1 + \Phi_{112}T^0$
2. $T^1\star T^2 = \quad \Phi_{121}T^1 + \Phi_{122}T^0$
3. $T^2\star T^2 = \quad \Phi_{221}T^1 + \Phi_{222}T^0$

Theorem: The quantum product is associative. That is

$$(T^i\star T^j)\star T^k = T^i\star(T^j\star T^k)$$

There are only two non-trivial associativity relations:

$$(T^1\star T^1)\star T^2 = T^1\star(T^1\star T^2)$$

and

$$(T^1\star T^2)\star T^2 = T^1\star(T^2\star T^2)$$

Working out the first associativity equation gives:

$$(T^1\star T^1)\star T^2 = \Phi_{221}T^1 + \Phi_{222}T^0 + \Phi_{111}(\Phi_{121}T^1 + \Phi_{122}T^0) + \Phi_{112}T^2$$

while

$$T^1\star(T^1\star T^2) = \Phi_{121}(T^2 + \Phi_{111}T^1 + \Phi_{112}T^0) + \Phi_{122}T^1$$

By the Theorem above the two equations must be equal. Therefore the T^0 terms must be equal. Hence

$$\Phi_{222} + \Phi_{111}\Phi_{122} = \Phi_{112}\Phi_{112}$$

This equation is known as the WDVV equation. Plugging Φ from above into this equation and rearranging the terms finally yields ......

$$N_d = \sum_{d_A+d_B=d} N_{d_A}N_{d_B}d_A^2d_B\left(d_B\binom{3d-4}{3d_A-2} - d_A\binom{3d-4}{3d_A-1}\right)$$

We get new numbers: $N_5 = 87,304$, $N_6 = 26,312,976$, $N_7 = 14,616,808,192$

Why this is so Beautiful!!

- The problem was easy to state.
- The answer is easy to understand, they are natural numbers!!!
- The problem that resisted a century of investigation is reduced to the number of lines through two distinct points, $N_1 = 1$.