$$\boxed{\textbf{Math2E - Practice Final}}$$

December 6, 2008

1. Evaluate $\int_C xydx + ydy$, C is the sine curve $y = \sin x, 0 \le x \le \pi/2$.
   Answer: $\frac{3}{2}$

Solution: $C:\ y = \sin x,\quad 0 \le x \le \frac{\pi}{2}.$

$$\int_C xydx + ydy = \int_0^{\frac{\pi}{2}} (x\sin x + \sin x\cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(x\sin x + \tfrac{1}{2}\sin 2x\right) dx$$

$$= -\left(x\cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos x\,dx\right) + \left(-\tfrac{1}{4}\cos 2x\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 1 - \tfrac{1}{4}\cdot(-1-1) = 1 + \tfrac{1}{2} = \tfrac{3}{2}.$$

2. $\mathbf{F}(x,y,z) = e^y\mathbf{i} + (xe^y + e^z)\mathbf{j} + ye^z\mathbf{k}$,
(a): Show that $\mathbf{F}$ is conservative,
(b): Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment from $(0,2,0)$ to (4,0,3).
Answer: (a) $\mathbf{F}$ is conservative, (b) 2.

Solution: (a): $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y+e^z & ye^z \end{vmatrix} = (e^z - e^z)\vec{i}$

$- (0 - 0)\vec{j} + (e^y - e^y)\vec{k}$

$= \vec{0}$

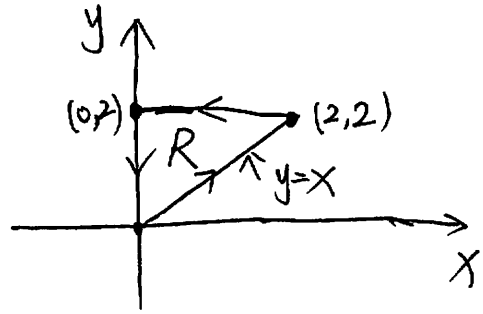
(b): $\begin{cases} f_x = e^y \Rightarrow f = xe^y + g(y,z) = xe^y + ye^z + h(z) = \overbrace{\qquad}^{xe^y+ye^z+C} \\ f_y = xe^y + e^z \quad f_y = xe^y + g_y \Rightarrow g_y = e^z \Rightarrow g = ye^z + h(z) \\ f_z = ye^z \quad f_z = ye^z + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \end{cases}$

$$\int_C \vec{F}\cdot d\vec{r} = (xe^y + ye^z + c)\Big|_{(0,2,0)}^{(4,0,3)}$$

$$= (4) - 2 = 2$$

3. Evaluate $\int_C x^2y\,dx + \ln\sqrt{1+y^2}\,dy$, where C is the triangle from $(0,0)$ to $(2,2)$ to $(0,2)$ to $(0,0)$.
Answer: $-\frac{4}{3}$.

Solution:



$$\int_C x^2y\,dx + \ln\sqrt{1+y^2}\,dy \overset{\text{Green's Thm}}{=\!=} \iint_R (-x^2)\,dA$$

$$= \int_0^2 \left(\int_x^2 -x^2\,dy \right) dx$$

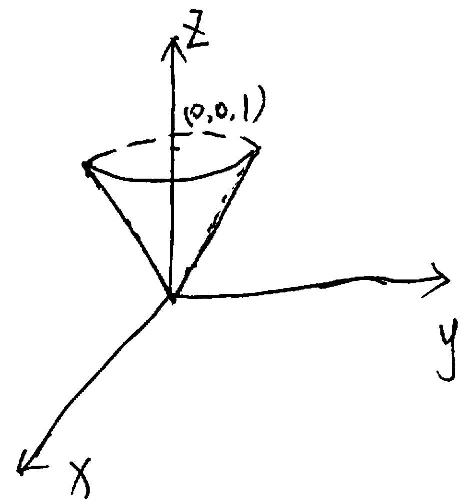
$$= \int_0^2 -x^2(2-x)\,dx$$

$$= -\frac{2}{3}\cdot 2^3 + \frac{1}{4}\cdot 2^4$$

$$= -\frac{16}{3} + 4 = -\frac{4}{3}$$

4. Evaluate $\int\int_S \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation.
Answer: $\frac{\pi}{2}$.

Solution: $S: z = x^2+y^2, R=\{x^2+y^2 \le 1\}.$



$$\vec{n}\, ds = \langle -2x, -2y, 1 \rangle dA$$

$$\iint_S \vec{F} \cdot \vec{n}\, ds = \iint_R (-2x^3 - 2xy^2 + z)\, dA$$

$$= \iint_R (-2x^3 - 2xy^2 + x^2 + y^2)\, dA$$

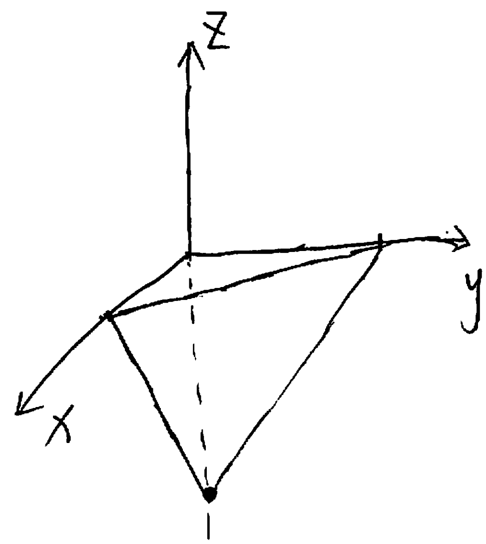
$\underline{x = r\cos\theta,\ y = r\sin\theta,\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi}$ $\int_0^1 \int_0^{2\pi} (-2 r\cos\theta \cdot r^2 + r^2)\, r\, d\theta\, dr$

$$= \frac{\pi}{2}$$

5. Evaluate $\int\int_{\partial Q} \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F} = < x^2 - y^2 z, x \sin z, 4y^2 >$, Q is bounded by $4x + 2y - z = 4$ $(z \leq 0)$ and the coordinate planes.
Answer: $-\frac{2}{3}$.

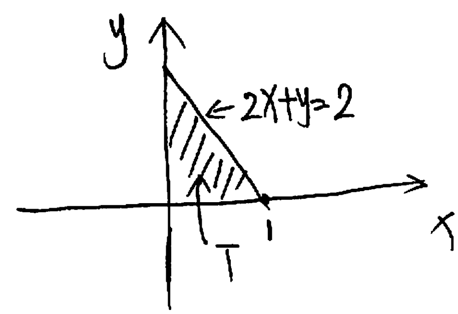
Solution:

$$\iint_{\partial Q} \vec{F} \cdot \vec{n}\, ds = \iiint_V (\operatorname{div} \vec{F})\, dV$$



$$= \iiint_V (2x)\, dV$$

$$= \iint_T \left(\int_{4x+2y-4}^{0} 2x\, dz \right) dA$$



$$= \iint_T 2x(4-4x-2y)\, dA$$

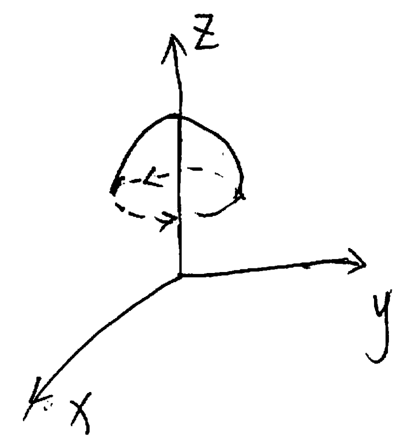
$$= \int_0^1 \left(\int_0^{2-2x} 2x(4-4x-2y)\, dy \right) dx$$

$$= -\frac{2}{3}.$$

6. Evaluate $\int\int_S \operatorname{curl}\mathbf{F} \cdot \mathbf{n}dS$, where $\mathbf{F}(x,y,z) = x^2yz\mathbf{i} + yz^2\mathbf{j} + z^3e^{xy}\mathbf{k}$, S is the part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$, and S is oriented upward.
Answer: -4π.

Solution:

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$$



$C: x^2+y^2=4, \quad z=1.$

$x=2\cos\theta, \; y=2\sin\theta, \; z=1, \quad 0\le\theta\le 2\pi$

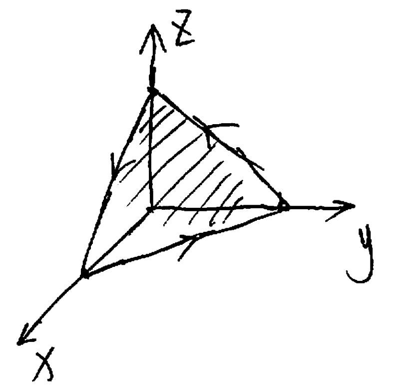
$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left(x^2yz\,dx + yz^2\,dy + z^3e^{xy}\,dz\right)$$

$$= \int_0^{2\pi} \left(4\cos^2\theta\cdot 2\sin\theta\cdot -2\sin\theta + 2\sin\theta\cdot 2\cos\theta\right)d\theta$$

$$= -4\pi.$$

7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed above.
Answer: $-\frac{1}{2}$

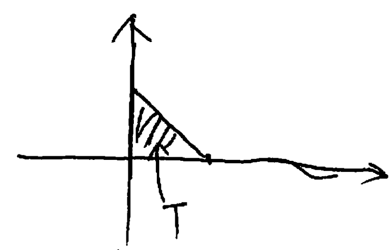
Solution:



$$S: z = 1 - x - y, \quad (x, y) \in T$$

$$C = \partial S$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS$$



$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = (-y)\vec{i} - (z)\vec{j} + (-x)\vec{k}$$

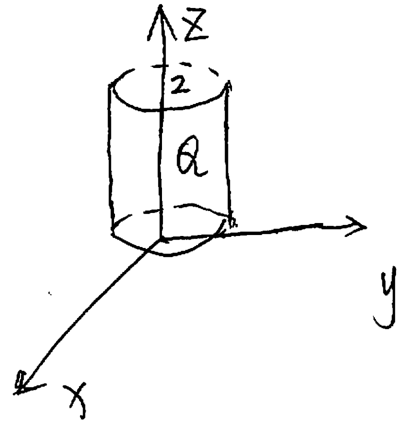
$$\vec{n}\, dS = \langle 1, 1, 1 \rangle$$

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = \iint_T (-y - z - x)\, dA = -\iint_T dA$$

$$= -\frac{1}{2}$$

8. Evaluate $\int\int_S \mathbf{F}\cdot\mathbf{n}dS$, where $\mathbf{F}(x,y,,z) = x^3\mathbf{i}+y^3\mathbf{j}+z^3\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=2$.
Answer: 11π.

Solution:



$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_\Omega \operatorname{div}\vec{F}\,dV$$

$$= \iint_{\{x^2+y^2\le 1\}} \int_0^2 (3x^2+3y^2+3z^2)\,dz\,dA$$

$$= \iint_{\{x^2+y^2\le 1\}} \left(6(x^2+y^2)+8\right)\,dA$$

$$\underline{\underline{x=r\cos\theta,\ y=r\sin\theta,\ 0\le r\le 1,\ 0\le\theta\le 2\pi}} \int_0^1\int_0^{2\pi} 6r^2 \; r\,d\theta\,dr + 8\pi$$

$$= 11\pi.$$