Coupled Parabolic Equations for Wave Propagation

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Abstract

We develop an algorithm using two coupled parabolic equations for numerical simulation of wave propagation over long distances. The coupled parabolic equations are derived from a two mode wave decomposition. An iterative procedure was used in our numerical algorithm. The coupling between the two parabolic equations allow us to deal with inhomogeneities in the medium and capture the back scattering that are usually neglected in a one way parabolic approximation. We study stability issues of our numerical algorithm and present numerical examples.

1 Introduction

Efficient numerical algorithms are important in order to understand wave propagation in complex media. Resolving the wavelength is one of the basic sampling conditions for numerical algorithms. However, in many applications, such as underwater acoustics, communications and remote sensing, the wave propagates over a long distance which may be several order of magnitudes compared to the wavelength. If the medium is inhomogeneous there is an additional characteristic scale corresponding to the correlation length of the medium inhomogeneities. Wave propagation in a heterogeneous medium is thus a multi-scale problem in space and time, which poses a great challenge for numerical simulations. Here we focus on the simulation of time harmonic waves, that is, we reduce the full wave equation in space and time to the Helmholtz equation in space only by Fourier transformation in time. The Helmholtz equation gives a boundary value problem

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and it is very expensive to solve with a computational domain that is large compared to the wavelength. If the wave propagates in a weakly inhomogeneous medium, e.g., underwater acoustics or electromagnetic waves in the atmosphere [7, 8, 11] the Helmholtz equation can be further simplified. In a situation where the propagation distance is much larger than the transverse dimension corresponding to a narrow angle geometry and when the back scattering can be neglected the Helmholtz equation can be reduced to a parabolic equation. The main advantage of the parabolic approximation is that it gives an initial value problem which is much easier to analyze and much cheaper to compute. The parabolic equation (PE) approximation provides an important tool for analysis and computation of wave propagation. More advanced parabolic approximations have been developed to deal with back scattering and wide angle geometries. For example the two way parabolic approximation by Collins [4]. In this important paper the authors consider a generalization of the one way or outgoing parabolic approximation to handle medium variations in the range, that is, the propagation direction. Their approximation is motivated by applications to ocean waveguides where often the depth variation of the medium parameters is stronger than the variation in range. The range-dependent medium is approximated by a sequence of range-independent sections. At each interface the reflected field is approximated via an iterative technique using the parabolic wave operator. The reflected field components are then propagated back via the incoming parabolic approximation. Here our focus is rather on small scale scatterers with arbitrary geometry and we do not decompose the medium into range independent sections, thus the parabolic modes couple at all depths, moreover, we carry out many iterations in range. We start with the Helmholtz equation and use a wave decomposition as in [1] to derive a system of coupled parabolic equations. We use a bidirectional wave decomposition and get two coupled parabolic equations. The coupling comes from backscattering and transverse spreading of the wave. Based on this decomposition, a straightforward Jacobi type of iterative algorithm can be applied. However, this iterative algorithm is not stable and will not converge due to the lagged coupling in the iterative procedure. In this paper we modify these two equations and develop an efficient algorithm that is as simple as the standard PE approximation but can better deal with heterogeneities and back scattering. We consider the Helmholtz equation in 2-dimensions for simplicity. However the algorithm can easily be generalized to the 3-dimensional case. Numerical experiments are used to compare our method with the standard parabolic approximation.

The outline of the paper is as follows, in Section 2 we give a brief review of the standard parabolic approximation. The coupled parabolic system and the absorbing boundary condition will be derived in Sections 3-5. We present our numerical algorithm and numerical results in Section 6. In the appendix we give the stability analysis.

2 The Parabolic Approximation

We consider the problem of propagation of acoustic signals over large distances. Let $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ be the acoustic velocity and pressure satisfying the equation of continuity of momentum and mass

$$\rho \mathbf{u}_t + \nabla p = \mathbf{F}(\mathbf{x}, t)\,, \tag{1}$$

$$K^{-1} p_t + \nabla \cdot \mathbf{u} = 0, \tag{2}$$

where t is time, z is depth into the medium and defined so as to increase with depth, $(\mathbf{x}, z) = (x, y, z)$ are the space coordinates, ρ is the density, K is bulk modulus, and the source is $\mathbf{F}(\mathbf{x}$,t). We model the medium by $\rho = \rho_0$ constant and

$$K^{-1}(\mathbf{x}, z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_p^{-1}(\mathbf{x}, z)(1 + \nu(\mathbf{x}, z)) & z \in (0, \infty). \end{cases}$$

The function ν modulating the compliance corresponds the medium fluctuations. In the case of a stationary random medium it is a zero-mean, stationary stochastic process whose statistics take on particular forms depending on the assumptions about the medium, whether it is locally layered, strongly or weakly heterogeneous media and so on [1].

At this stage, the source term is omitted, but it will be taken into account through the initial conditions for the parabolic equation.

Eliminating $\mathbf{u}$ from equations (1) and (2), we get

$$\triangle p - \frac{\rho}{K} p_{tt} = 0. \tag{3}$$

The time-harmonic version of (3) is the Helmholtz Equation:

$$\triangle \hat{p} + (1 + \nu)\omega^2 \gamma^2 \hat{p} = 0\,, \tag{4}$$

where

$$\gamma(\mathbf{x}, z) = \sqrt{K_p^{-1}(\mathbf{x}, z)\rho}\,,$$

and $\hat{p}$ is the Fourier transform of p with respect to time:

$$\hat{p} \;=\; \int p e^{i\omega t}\, dt. \tag{5}$$

Note that below we suppress the 'hat' and that γ represents an effective slowness.

The Helmholtz equation is associated with some specific boundary values. If the computational domain of this boundary value problem is large compared to the wavelength, solving the discretized linear system using a direct method may be impossible due to memory constraints. Moreover, since the linear system is not positive definite, usual iterative methods typically converge slowly, if at all. The parabolic approximation can be used to deal with this problem in various settings. The main idea is to neglect the back scattering and only consider forward going waves. The PE approximation becomes an initial value problem which significantly reduces the complexity for both analysis and computation. This approximation is accurate in many scenarios such as in range dependent ocean wave-guides or in the case of atmospheric wave propagation.

We are interested in wave propagation mainly in one direction, along the z axis in our notation. We use the plane wave ansatz

$$p(\mathbf{x}, z) = A(\mathbf{x}, z)\exp(ik_0 z)\,, \tag{6}$$

for the solution of Helmholtz Equation:

$$\triangle p + \omega^2\gamma^2(\mathbf{x}, z)(1+\nu)p = 0\,, \tag{7}$$

where k_0 is a reference wave number. The factor $\exp(ik_0 z)$ in (6) represents a plane wave travelling in the positive z direction and is supposed to take out the rapid oscillations of p in the z direction; the function $A(\mathbf{x}, z)$ captures the modulation of the plane wave phase and usually varies slowly with z.

Substitution of Equation (6) into Equation (7) gives

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0\frac{\partial A}{\partial z} + \triangle_\perp A + [\omega^2\gamma^2(1+\nu) - k_0^2]A = 0\,, \tag{8}$$

with $\triangle_\perp$ being the Laplacian in the lateral coordinates $\mathbf{x}$. We next make the crucial paraxial approximation (small angle approximation) corresponding to the situation with

$$\frac{\partial^2 A}{\partial z^2} << 2ik_0\frac{\partial A}{\partial z}\,, \tag{9}$$

so that we have

$$2ik_0\frac{\partial A}{\partial z} + \triangle_\perp A + [\omega^2\gamma^2(1+\nu) - k_0^2]A = 0. \tag{10}$$

This approximation requires that we consider wave propagation in a narrow beam geometry, not close to the source and that the medium is weakly inhomogeneous. The resulting equation is called the narrow-angle parabolic equation (PE). In the PE method we take into account only waves travelling in the positive z direction; back scattering is neglected, see [7, 8].

3 Decomposition Of Waves In One Dimensional Case

We next aim to extend the parabolic approximation to a strongly range or 'depth' dependent medium and consider first the layered or one-dimensional case. The field is decomposed into outgoing and incoming components. Our equations are based on a decomposition of the field, instead of an operator factorization. The parabolic approximation is a special case of our approach.

In the one dimensional case the bulk modulus $K = K(z)$ depends on z only, the reduced wave equation is now:

$$p_{zz} + (1+\nu)\omega^2\gamma^2 p = 0\,, \tag{11}$$

with $\gamma = \gamma(z)$. Define

$$\tau(z) = \int_{z_s}^{z} \gamma(s)ds\,,$$

where z_s is the location of source signal. The phase $\tau(z)$ is the travel time from the source to depth z for a plane wave travelling in the depth direction. Equation (11) can now be written as:

$$p_{zz} + (1+\nu)\omega^2\tau_z^2 p = 0.$$

Let the pressure p be decomposed as:

$$p = A(z)e^{i\omega\tau} + B(z)e^{-i\omega\tau}. \tag{12}$$

Note that by writing the pressure in this way we introduced two degrees of freedom. Therefore, we need an additional constraint on the amplitudes beyond (12).

The velocity is

$$u = \frac{\gamma}{\rho}(Ae^{i\omega\tau} - Be^{-i\omega\tau}) - \frac{i}{\omega\rho}(A_z e^{i\omega\tau} + B_z e^{-i\omega\tau}),$$

and we make the ansatz

$$A_z e^{i\omega\tau} + B_z e^{-i\omega\tau} = 0. \tag{13}$$

This is the additional constraint on the amplitudes. With it, the velocity becomes:

$$u = \frac{\gamma}{\rho}(Ae^{i\omega\tau} - Be^{-i\omega\tau}).$$

From (13), it follows that

$$\begin{aligned} A_z &= -B_z e^{-2i\omega\tau}\,, & (14)\\ B_z &= -A_z e^{2i\omega\tau}\,, & (15) \end{aligned}$$

and moreover

$$(A_{zz} + i\omega\gamma A_z)e^{i\omega\tau} + (B_{zz} - i\omega\gamma B_z)e^{-i\omega\tau} = 0. \tag{16}$$

Combining the above relations and Helmholtz equation (11) we find the following equations for the amplitudes A and B:

$$\begin{aligned} 2\tau_z A_z + \tau_{zz} A &= i\omega^2 \tau_z^2 \nu (A + Be^{-2i\omega\tau}) + \tau_{zz} Be^{-2i\omega\tau}\,, & (17)\\ 2\tau_z B_z + \tau_{zz} B &= -i\omega^2 \tau_z^2 \nu (Ae^{2i\omega\tau} + B) + \tau_{zz} Ae^{2i\omega\tau}. & (18) \end{aligned}$$

We may interpret A and B in the decomposition (12) as amplitudes of the down- and up-propagating wave modes respectively.

The decomposition that we introduced above means that we replaced the problem of solving (11) to that of solving the system (17) and (18) and in order to do so we need to introduce boundary conditions for the amplitudes. Note that in the case where the medium is homogeneous, with $\nu \equiv 0$ and γ constant, the amplitude equations decouple. The term associated with A then corresponds exactly to the wave component travelling in the positive range or depth direction and the term B with the wave component travelling in the opposite direction. In the general case the wave components couple and correspond to approximate locally up and down travelling wave terms.

In Section 6 we will introduce an iterative based numerical scheme for computing approximate solutions to the system (17) and (18).

4 Decomposition In Terms Of Generalized Plane Wave Components

We consider next the case with waves propagating in a three dimensional medium with a constant background slowness γ, but with general three dimensional medium fluctuations modelled by ν. The governing Helmholtz equation is

$$\triangle p + (1+\nu)\omega^2\gamma^2 p = 0\,,$$

and we now define

$$\tau(z) = \int_{z_s}^{z} \sqrt{\frac{\rho}{K_p} - |\kappa|^2}ds = (z - z_s)\sqrt{\gamma^2 - |\kappa|^2}\,, \quad \text{and} \quad S^{\pm} = \kappa \cdot \mathbf{x} \pm \tau\,,$$

with κ being the lateral slowness vector. Note that S^+ is a plane wave phase corresponding to waves travelling in the spatial direction $(\kappa, \sqrt{\gamma^2 - |\kappa|^2})$. In the case with a general three dimensional background the phase terms $S^{\pm}$ will be solutions of the Eiconal equation associated with the slowness $\gamma(\mathbf{x}, z)$, see [1].

We decompose the wave into up-ward and down-ward modes as above:

$$p = Ae^{i\omega S^+} + Be^{i\omega S^-}\,, \tag{19}$$

$$0 = A_z e^{i\omega S^+} + B_z e^{i\omega S^-}. \tag{20}$$

In the general case the mode coupling transport equations become:

$$\begin{aligned} 2\nabla S^+ \cdot \nabla A + \triangle S^+ A - i\omega\gamma^2\nu(A + Be^{i\omega(S^- - S^+)}) \\ = \frac{i}{\omega}\triangle_\perp A - R^- e^{i\omega(S^- - S^+)}\,, \end{aligned} \tag{21}$$

$$\begin{aligned} 2\nabla S^- \cdot \nabla B + \triangle S^- B - i\omega\gamma^2\nu(B + Ae^{i\omega(S^+ - S^-)}) \\ = \frac{i}{\omega}\triangle_\perp B - R^+ e^{i\omega(S^+ - S^-)}\,, \end{aligned} \tag{22}$$

with

$$R^+ = 2\nabla_\perp S^+ \cdot \nabla_\perp A + \triangle S^+ A - \frac{i}{\omega}\triangle_\perp A\,,$$

$$R^- = 2\nabla_\perp S^- \cdot \nabla_\perp B + \triangle S^- B - \frac{i}{\omega}\triangle_\perp B\,,$$

where $\triangle_\perp$ is the transverse Laplacian. In the case that the fluctuations and the reflected field vanish, $(\nu \equiv 0, B \equiv 0)$, (21) becomes

$$2\nabla S^+ \cdot \nabla A + \triangle S^+ A = \frac{i}{\omega}\triangle_\perp A\,,$$

which in the high frequency limit gives

$$2\nabla S^{+}\cdot\nabla A_0+\triangle S^{+}A_0=0\,,$$

that is, the leading order transport equation of geometrical optics.

Here, we will consider the generalization of the parabolic case with waves propagating primarily in the z direction and set $\kappa=0$. Then the coupling transport equations become

$$\begin{aligned} 2ikA_z+\triangle_{\perp}A+k^2\nu A &= -(k^2\nu B+\triangle_{\perp}B)e^{-2ikz}\,, && (23)\\ -2ikB_z+\triangle_{\perp}B+k^2\nu B &= -(k^2\nu A+\triangle_{\perp}A)e^{2ikz}\,, && (24)\end{aligned}$$

with $k=\gamma\omega$.

In the case that the reflected wave vanish, $B\equiv 0$, (23) becomes

$$2ikA_z+\triangle_{\perp}A+k^2\nu A=0\,,$$

which is the standard narrow-angle parabolic approximation. On the other hand, in the case that there is no lateral variation in the amplitudes we find

$$\begin{aligned} 2ikA_z &= -k^2\nu(A+Be^{-2ikz})\,,\\ 2ikB_z &= k^2\nu(Ae^{2ikz}+B). \end{aligned}$$

These are the transport equations associated with the layered three dimensional case which are analyzed in for instance [1] and [10]. A particular κ corresponds to a specific plane wave mode. This variable is the Fourier variable dual to the lateral space variable $\mathbf{x}$ introduced when the wave field in space and time is transformed into plane wave modes via Fourier transformation with respect to the lateral spatial coordinates. We next continue our discussion of the system (23) and (24) by introducing specific boundary conditions and a scheme for numerical approximation of the solution.

5 Boundary Conditions

5.1 Initial Condition For Parabolic Equations

We assume that the scatterer ν is compactly supported and is located in a slab of thickness L, so that k is constant for $z<0$ and $z>L$. The source is located in the homogeneous medium, at $z_s<0$. Recall that with $B=0$ the amplitude equation for A is equivalent to the standard parabolic equation. We impose an initial condition for the down-ward field A at $z=0$ and for the reflected field B at $z = L$. At the top, we use a Gaussian form for the initial data [11]:

$$A(\mathbf{x}, 0) = \sqrt{k_0} e^{-\frac{|\mathbf{x}|^2}{2}}.$$

Since the medium is homogeneous for $z > L$ and there are no sources located in this half space we set

$$B(\mathbf{x}, L) = 0.$$

5.2 Artificial Boundary

The scatterer ν is located in infinite homogeneous medium. We truncate the domain by adding PML (Perfectly Matched Layer) artificial boundary in the lateral dimensions. The idea is to introduce an exterior layer at the artificial boundary in such a way that all plane waves are totally absorbed, and no reflection occurs at the boundary [2]. For simplicity, we now assume two spatial dimensions corresponding to one lateral dimension. In order to introduce the artificial boundary condition we return to the Helmholtz equation:

$$p_{zz} + p_{xx} + k^2(1 + \nu(x, z))p = 0.$$

In the matched layer we change this equation to obtain damping of the plane wave modes. We introduce the governing equations

$$\frac{\partial^2 p}{\partial z^2} + \frac{i\omega}{\sigma - i\omega}\frac{\partial}{\partial x}\left(\frac{i\omega}{\sigma - i\omega}\frac{\partial p}{\partial x}\right) + k^2(1 + \nu)p \;\; = \;\; 0\,,$$

where $\sigma(x) > 0$ in the artificial domain giving damping of the plane wave modes, whereas $\sigma(x) = 0$ in the physical domain giving the Helmholtz equation there. Denote

$$s(x) = \frac{i\omega}{\sigma(x) - i\omega}\,,$$

then with the PML, the coupling transport equations can be written

$$\begin{aligned} &2ikA_z + s^2(z)A_{xx} + k^2\nu A + s(x)s'(x)A_x \\ &\qquad = -(k^2\nu B + s^2(x)B_{xx} + s(x)s'(x)B_x)e^{-2ikz}\,, \end{aligned} \tag{25}$$

$$\begin{aligned} &-2ikB_z + s^2(z)B_{xx} + k^2\nu B + s(x)s'(x)B_x \\ &\qquad = -(k^2\nu A + s^2(x)A_{xx} + s(x)s'(x)A_x)e^{2ikz}. \end{aligned} \tag{26}$$

The ideal case of using these two coupled parabolic equations for the forward wave field A and backward wave field B in numerical computations would be that we can solve them in the form of Jacobi iteration, i.e., solve A with current B and then solve B with current A iteratively. This would reduce the computation of the Helmholtz equation into the computation of a sequence of parabolic equations. However, as we will show in the appendix, this iterative procedure is not stable for evanescent modes in the lateral direction. In another word, without coupling A and B together simultaneously in the propagation direction evanescent modes can grow exponentially. Since we are interested in the narrow angle wave propagation in the z-direction, we drop the lateral scattering terms $\triangle_\perp B$ (or $\triangle_\perp A$) when we solve for $A(x,z)$ (or $B(x,z)$) in (23) (or (24)). Consider the equation (23). In the homogeneous case with $\nu \equiv 0$ the terms involving the reflected field B will be lower order correction terms to the paraxial approximation. We will consider regimes where there is significant back-scattering due to the scatterer ν and therefore retain the term involving ν in the coupling part of equations (23) and (24). With PML included we have the following two coupled parabolic approximations (with one lateral dimension):

$$2ikA_z + s^2(z)A_{xx} + k^2\nu A + s(x)s'(x)A_x = -k^2\nu B e^{-2ikz}\,, \qquad (27)$$
$$-2ikB_z + s^2(z)B_{xx} + k^2\nu B + s(x)s'(x)B_x = -k^2\nu A e^{2ikz}\,, \qquad (28)$$
$$A(x,0) = \sqrt{k_0}e^{-\frac{|x|^2}{2}}; \quad B(x,L) = 0.$$

We next introduce a numerical scheme for approximation of the solution to this system.

6 Numerical Solution

We will solve the system (27, 28) by iteration. In the first step we compute an approximation for A using (27) with $B = 0$. Next, we solve for B using (28) with the computed approximation for A. This procedure is then repeated with the updated values for A and B. Note that it is unstable to solve the system (25, 26) iteratively, we show this in the appendix. Observe also that in the homogeneous case with $\nu = 0$ the iteration converges after the first step with $B \equiv 0$ and A solving the standard narrow-angle parabolic equation.

We now discretized the above iteration by introducing finite differences for the derivatives. The discretized field variables are denoted A_l^m and B_l^m, where l and m are grid indices in x and z, respectively. Assuming the field A

is known at step m, we proceed to determine the field at step $m+1$. For B, we do it backward, i.e., determine the field at step m from the $m+1$ step.

We use Crank-Nicolson scheme to solve equation (27). At the half grid point $(x_l, z_{m+\frac{1}{2}})$, we have

$$\begin{aligned} &2ikA_z(x_l, z_{m+\frac{1}{2}}) + k^2\nu A(x_l, z_{m+\frac{1}{2}}) + s^2(x_l)A_{xx}(x_l, z_{m+\frac{1}{2}}) \\ &+s(x_l)s'(x_l)A_x(x_l, z_{m+\frac{1}{2}}) = -k^2\nu B(x_l, z_{m+\frac{1}{2}})e^{-2ikz_{m+\frac{1}{2}}}. \end{aligned} \tag{29}$$

We use the following finite difference formulae

$$\begin{aligned} A_x(x_l, z_{m+\frac{1}{2}}) &\approx \frac{A_x(x_l, z_m) + A_x(x_l, z_{m+1})}{2} \\ &\approx \frac{A_{l+1}^m + A_{l+1}^{m+1} - A_{l-1}^m - A_{l-1}^{m+1}}{4h}, \\ A_z(x_l, z_{m+\frac{1}{2}}) &\approx \frac{A_l^{m+1} - A_l^m}{\tau}, \\ A_{xx}(x_l, z_{m+\frac{1}{2}}) &\approx \frac{A_{xx}(x_l, z_{m+1}) + A_{xx}(x_l, z_m)}{2} \\ &\approx \frac{A_{l+1}^{m+1} + A_{l-1}^{m+1} + A_{l+1}^m + A_{l-1}^m - 2A_l^{m+1} - 2A_l^m}{2h^2}, \\ A(x_l, z_{m+\frac{1}{2}}) &\approx \frac{A_l^{m+1} + A_l^m}{2}, \\ B(x_l, z_{m+\frac{1}{2}}) &\approx \frac{B_l^{m+1} + B_l^m}{2}, \end{aligned}$$

with $\tau = z_m - z_{m-1}$ and $h = x_l - x_{l-1}$. By multiplying both sides of (29) by $4h^2$ and using these symmetric finite difference approximations we can express the solution in vector form as

$$\begin{aligned} &[u_l, v_l^{m+\frac{1}{2}}, w_l] \begin{bmatrix} A_{l-1}^{m+1} \\ A_l^{m+1} \\ A_{l+1}^{m+1} \end{bmatrix} \\ = \; &[-u_l, \hat{v}_l^{m+\frac{1}{2}}, -w_l] \begin{bmatrix} A_{l-1}^m \\ A_l^m \\ A_{l+1}^m \end{bmatrix} - 2k^2h^2\nu_l^{m+\frac{1}{2}}(B_l^{m+1} + B_l^m)e^{-2ikz_{m+\frac{1}{2}}}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} u_l &= (2s^2(x_l) - s(x_l)s'(x_l)h), \\ v_l^{m+1/2} &= 8ik\frac{h^2}{\tau} + 2k^2h^2\nu_l^{m+\frac{1}{2}} - 4s^2(x_l), \\ w_l &= (2s^2(x_l) + s(x_l)s'(x_l)h), \\ \hat{v}_l^{m+1/2} &= 8ik\frac{h^2}{\tau} - 2k^2h^2\nu_l^{m+\frac{1}{2}} + 4s^2(x_l). \end{aligned}$$

To complete the formulation we finally collect the local solutions given by (30) into a global matrix solution over the mesh points in depth,

$$\begin{bmatrix} v_1^{m+\frac{1}{2}} & w_1 & & & \\ u_2 & v_2^{m+\frac{1}{2}} & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & w_{N-1} \\ & & & u_N & v_N^{m+\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A_1^{m+1} \\ A_2^{m+1} \\ \vdots \\ A_{N-1}^{m+1} \\ A_N^{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{v}_1^{m+\frac{1}{2}} & -w_1 & & & \\ -u_2 & \hat{v}_2^{m+\frac{1}{2}} & -w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -w_{N-1} \\ & & & -u_N & \hat{v}_N^{m+\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A_1^{m} \\ A_2^{m} \\ \vdots \\ A_{N-1}^{m} \\ A_N^{m} \end{bmatrix} - 2k^2h^2e^{-2ikz_{m+\frac{1}{2}}} \begin{bmatrix} \nu_1^{m+\frac{1}{2}}(B_1^m + B_1^{m+1}) \\ \nu_2^{m+\frac{1}{2}}(B_2^m + B_2^{m+1}) \\ \vdots \\ \nu_{N-1}^{m+\frac{1}{2}}(B_{N-1}^m + B_{N-1}^{m+1}) \\ \nu_N^{m+\frac{1}{2}}(B_N^m + B_N^{m+1}) \end{bmatrix}$$

The coefficient of B_z in (28) is negative of that of A_r in (27), however, since we solve B from bottom to top, we will determine B at level m from level $m+1$. We use the following approximation

$$B_z(x_l, z_{m+\frac{1}{2}}) = \frac{B_l^m - B_l^{m+1}}{-\tau}.$$

Then, we obtain the same matrix formulation as when solving for A (modulo the sign of the phase):

$$\begin{bmatrix} v_1^{m+\frac{1}{2}} & w_1 & & & \\ u_2 & v_2^{m+\frac{1}{2}} & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & w_{N-1} \\ & & & u_N & v_N^{m+\frac{1}{2}} \end{bmatrix} \begin{bmatrix} B_1^{m} \\ B_2^{m} \\ \vdots \\ B_{N-1}^{m} \\ B_N^{m} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{v}_1^{m+\frac{1}{2}} & -w_1 & & & \\ -u_2 & \hat{v}_2^{m+\frac{1}{2}} & -w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -w_{N-1} \\ & & & -u_N & \hat{v}_N^{m+\frac{1}{2}} \end{bmatrix} \begin{bmatrix} B_1^{m+1} \\ B_2^{m+1} \\ \vdots \\ B_{N-1}^{m+1} \\ B_N^{m+1} \end{bmatrix} - 2k^2h^2e^{2ikz_{m+\frac{1}{2}}} \begin{bmatrix} \nu_1^{m+\frac{1}{2}}(A_l^m + A_l^{m+1}) \\ \nu_2^{m+\frac{1}{2}}(A_2^m + A_2^{m+1}) \\ \vdots \\ \nu_{N-1}^{m+\frac{1}{2}}(A_{N-1}^m + A_{N-1}^{m+1}) \\ \nu_N^{m+\frac{1}{2}}(A_N^m + A_N^{m+1}) \end{bmatrix}$$

6.1 Numerical Example

We test our algorithm by considering a sound signal propagating through a homogeneous medium with some imbedded compact scatterers.

Several imbedded scatterers are located in the domain, they are strongly anisotropic with dimensions about 1300m in the vertical direction and 195m in the horizontal dimension. The signal speed in the homogeneous medium is 330m/s and the wavelength is about 13m.

Our computational domain contains 20 wavelength (horizontal) and 200 wavelength (vertical). We use 10 grid points per wavelength in both directions, whereas on the left and the right sides, we use 25 grid points for the PML medium. The medium and the computational domain is shown as Figure 1.

After 15 iterations, we get the solution for the coupled parabolic equations. The solutions of Helmholtz equation, Parabolic approximation and Coupled Parabolic approximation are shown in Figures 2-4 respectively.

In Figure 5 we show p at the center of the domain along the propagation direction in z. The solid line is the numerical solution of the Helmholtz equation. The solution of the parabolic equation is shown by the dotted line in the top plot. The solution of the coupled parabolic equations (C-PE) using the algorithm described in the previous section is shown by the dotted line in the bottom plot. Note that our C-PE solution is very close to the Helmholtz solution whereas the PE solution fails to capture the oscillations in p that is due to the imbedded scatterers which give a significant back-scattering.

7 conclusion

In this paper we develop an algorithm using two coupled parabolic equations to simulate wave propagation over long distance. The two coupled parabolic equations are derived from a wave field decomposition. Both forward propagation field and backward propagation field as well as their interactions are captured. Our method can deal with strong inhomogeneities with arbitrary geometry in the medium and the computational cost is comparable to that of the parabolic approximation.

Appendix

Here we show the potential instability when solving the full system (23, 24) by iteration. We simplify the system by considering only one transverse variable x, i.e.,

$$2ikA_z + k^2\nu A + A_{xx} + (B_{xx} + k^2\nu B)e^{-2ikz} = 0\,, \tag{31}$$
$$-2ikB_z + k^2\nu B + B_{xx} + (A_{xx} + k^2\nu A)e^{2ikz} = 0\,, \tag{32}$$

Introduce a new variable: $D(x,z) = A(x,z) + e^{-2ikz}B(x,z)$. We then get a new system:

$$A_z = \frac{i}{2k}(\partial_{xx} + k^2\nu)D \tag{33}$$
$$D_z = 2ik(A - D) \tag{34}$$

We solve D from the second equation,

$$D = [D(0) + 2ik\int_0^z A(s)e^{2iks}ds]e^{-2ikz}. \tag{35}$$

Plug D into the first equation we have

$$A_z = \frac{i}{2k}(\partial_{xx} + k^2\nu)[A(0) + B(0) + 2ik\int_0^z A(s)e^{2iks}ds]e^{-2ikz}. \tag{36}$$

Differentiate the above equation in z and if the medium is homogeneous, i.e., $\nu \equiv 0$, we get

$$A_{zz} + 2ikA_z + A_{xx} = 0 \tag{37}$$

Use Fourier mode ansatz, $A(x,z) = a(z)e^{inx}$, we get

$$a_{zz} + 2ika_z - n^2 a = 0$$

for which the characteristic equation has two roots:

$$r^{\pm} = -ik \pm \sqrt{n^2 - k^2}$$

We can see that evanescent modes ($n > k$) become unstable when we march in either direction of z. The instability is caused by the transverse Laplacian. In heterogeneous media evanescent modes can be generated by scattering. In numerical computations, roundoff errors or artificial boundary conditions can also generate spurious evanescent modes and cause instability. Only if A and B are solved simultaneously with two appropriate point boundary conditions. (which is the case for Helmholtz equation), the evanescent modes are under control. That is why an iterative procedure for the coupled parabolic equations (23, 24) for A and B is numerically unstable, which can be also shown by a similar but messier Fourier mode analysis on the discretized equations.

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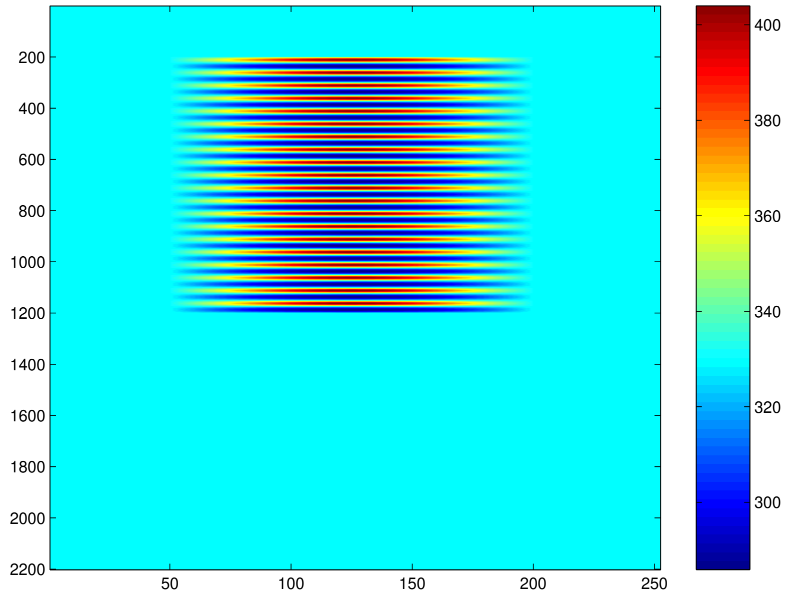

Figure 1: Medium

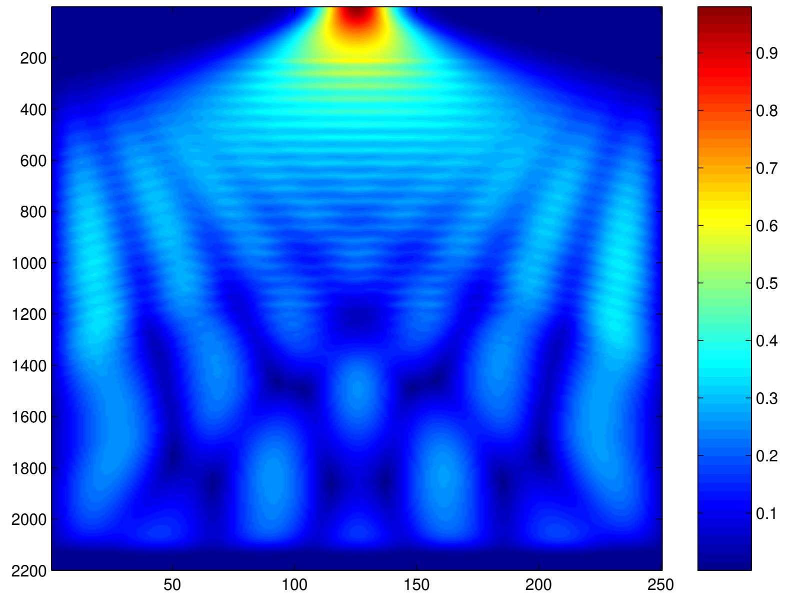

Figure 2: Helmholtz Equation

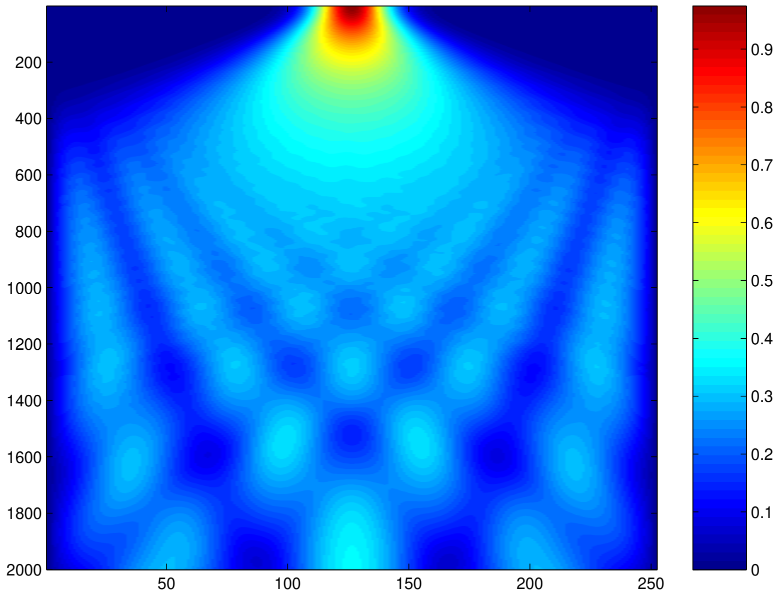

Figure 3: Parabolic Equation

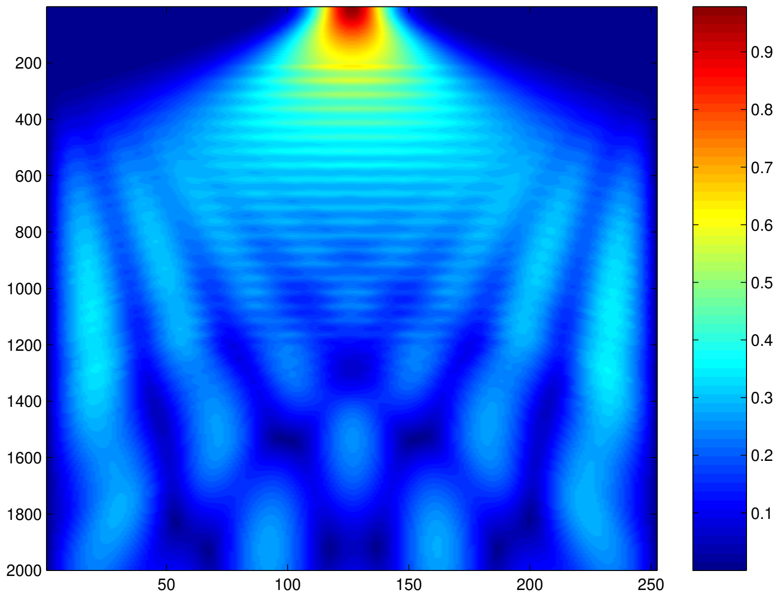

Figure 4: Coupled Parabolic Equation

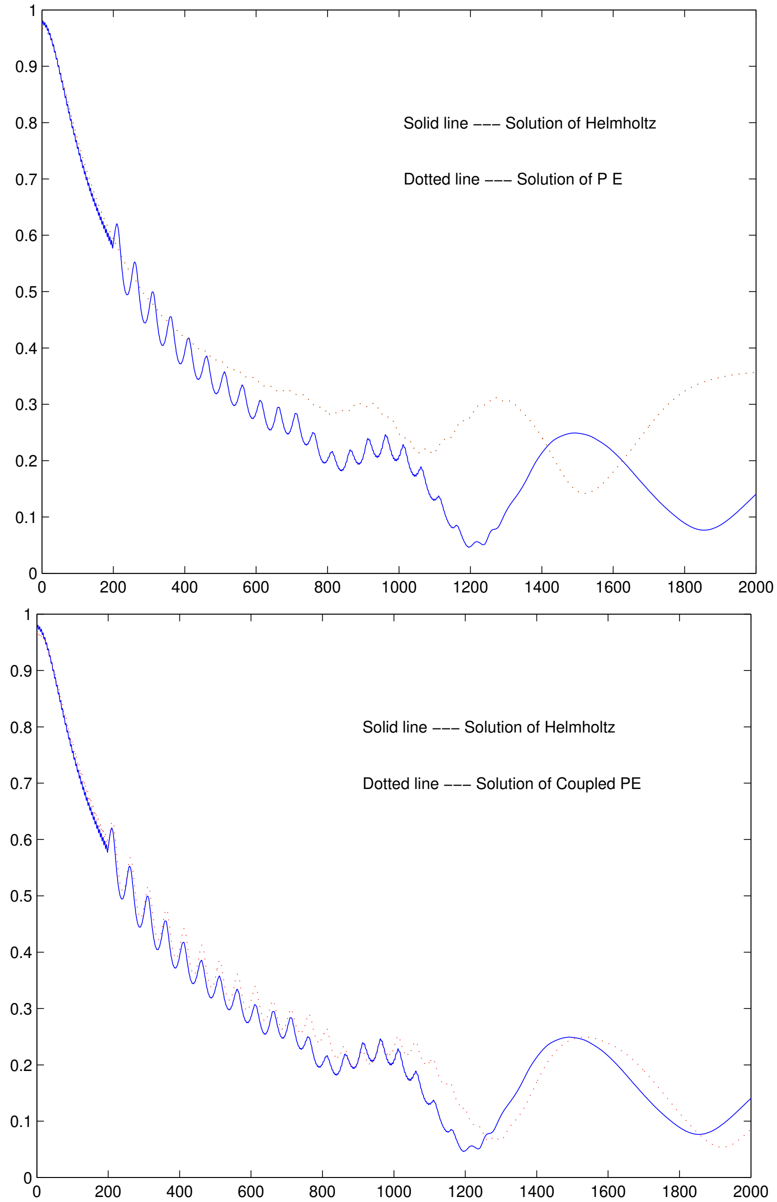

Figure 5: The pressure computed by the C-PE algorithm (dotted line in bottom plot) and the PE algorithm (dotted line in top plot) compared to the Helmholtz solution (solid lines).