Overlapping Schwarz Waveform Relaxation for the Heat Equation in n-Dimensions

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December 29, 1999

Abstract

We analyze overlapping Schwarz waveform relaxation for the heat equation in n spatial dimensions. We prove linear convergence of the algorithm on unbounded time intervals and superlinear convergence on bounded time intervals. In both cases the convergence rates are shown to depend on the size of the overlap. The linear convergence result depends also on the number of subdomains because it is limited by the classical steady state result of overlapping Schwarz for elliptic problems. However the superlinear convergence result is independent of the number of subdomains. Thus overlapping Schwarz waveform relaxation does not need a coarse space for robust convergence independent of the number of subdomains, if the algorithm is in the superlinear convergence regime. Numerical experiments confirm our analysis. We also briefly describe how our results can be extended to more general parabolic problems.

1 Introduction

Overlapping Schwarz waveform relaxation is a class of parallel algorithms for evolution problems. The distribution of the computation is achieved by partitioning the spatial domain into overlapping subdomains, like in the classical Schwarz method. However on subdomains, time dependent problems are solved in the iteration and thus the algorithm is of waveform relaxation type. Overlapping Schwarz waveform relaxation algorithms have been introduced in [GZ97] for the solution of evolution problems in a parallel environment with slow communication links, since they permit to solve over several time steps before communicating information to the neighboring subdomains. These algorithms have been further studied in [Gan97, Gan98a], [GS98], [Gan98b] and independently in [GK97].

Overlapping Schwarz waveform relaxation inherits convergence properties from both the classical overlapping Schwarz method and from waveform relaxation methods. There are two classical convergence results for waveform relaxation algorithms for ordinary differential equations (ODEs): (i) for linear systems of ODEs on unbounded time intervals one can show linear convergence of the algorithm under

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some dissipation assumptions on the splitting ([Nev89a, Nev89b], [MN87], [Bur95] and [JP95]); (ii) for nonlinear systems of ODEs (including linear ones) on bounded time intervals one can show superlinear convergence assuming a Lipschitz condition on the splitting function ([Nev89a, Nev89b], [BZ93] and [Bjø95]). If waveform relaxation is applied to partial differential equations (PDEs) discretized in space, the convergence results (i) and (ii) may depend on the discretization parameter for classical matrix splittings. The convergence rates then deteriorate as one refines the mesh. One possible remedy is to use a multi-grid splitting, proposed by Lubich and Ostermann in [LO87] and further studied by Vandewalle and Horton in [VH95] and by Janssen and Vandewalle in [JV96a, JV96b]. In a different approach Jeltsch and Pohl propose in [JP95] a multi-splitting algorithm with overlap. They prove results (i) and (ii) for their algorithm, but the convergence rates are again mesh dependent. They show however numerically for a discretized one dimensional heat equation that increasing the overlap accelerates the convergence of the waveform relaxation algorithm. Similar observations were made by Burrage et al [BJNR96] in higher dimensions. Gander and Stuart [GS98] relate the overlap in the multi-splitting algorithm to physical overlap of subdomains for the one dimensional heat equation thus linking waveform relaxation with overlapping splittings (or multi-splittings) to domain decomposition. They quantify how the overlap affects the convergence rate and prove result (i) independent of the mesh parameter, provided the physical overlap is hold constant. Independently Giladi and Keller [GK97] proved result (ii) for an overlapping domain decomposition approach and a one dimensional convection diffusion equation. Interestingly the superlinear convergence rate in that case differs from the classical one, the diffusion leads to a faster superlinear rate.

From the domain decomposition side a different approach has been analyzed for evolution problems. One discretizes time and solves at each time level elliptic problems using domain decomposition. This algorithm can only converge linearly, but the underlying evolution permits interesting additional properties. In special cases Cai shows for example that a coarse grid is not needed in [Cai91, Cai94].

In this paper we study overlapping Schwarz waveform relaxation for space decompositions in all generality for the heat equation in n dimensions. We prove both results (i) and (ii) at the continuous level, which leads to algorithms that converge independently of the mesh size if the overlap is hold constant. The superlinear rate in (ii) is found to be faster than for classical waveform relaxation, generalizing the result in [GK97] to n dimensions and arbitrary decompositions. In addition we show why no method of global information propagation is needed in the evolution case, thus generalizing results in [Cai91, Cai94]. It is due to the fact that for evolution problems, the initial conditions are known exactly for each subdomain and thus as long as one does not evolve too far in time, the initial conditions determine mainly the solution away from the artificial boundaries and thus the boundary conditions become less important.

2 Problem Description

We are interested to solve a parabolic partial differential equation in n dimensions using waveform relaxation. We consider the heat equation on a bounded domain

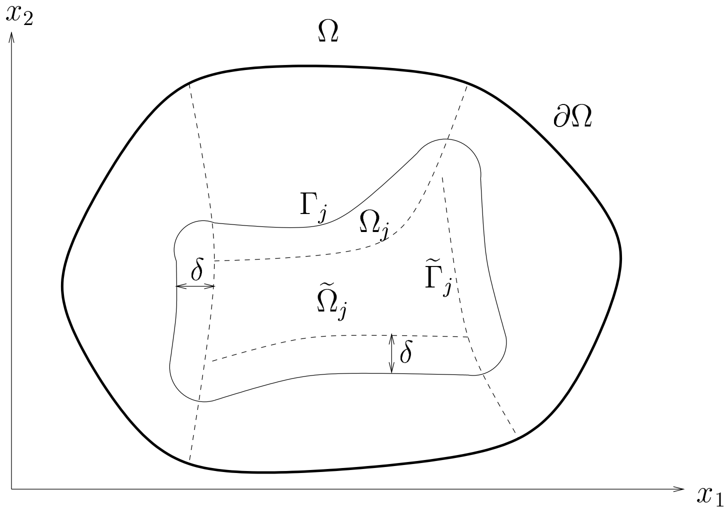

Figure 1: Decomposition into overlapping subdomains such that an overlap of δ is guaranteed.

$\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$ as our guiding example,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(\boldsymbol{x}, t) && \boldsymbol{x} \in \Omega, \quad 0 < t < T \\ u(\boldsymbol{x}, t) &= g(\boldsymbol{x}, t) && \boldsymbol{x} \in \partial\Omega, \quad 0 < t < T \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) && \boldsymbol{x} \in \Omega. \end{aligned} \tag{2.1}$$

We assume that the initial condition $u_0(\boldsymbol{x})$ and the boundary condition $g(\boldsymbol{x}, t)$ are bounded piecewise continuous and $f(\boldsymbol{x}, t)$ is continuous. This gives existence and uniqueness of a solution [Fri64, pp 40]. Central in our analysis is the maximum principle satisfied by the solution $u(\boldsymbol{x}, t)$ of (2.1):

Theorem 2.1 (Maximum Principle) *For $f(\boldsymbol{x}, t) \equiv 0$, if the weak solution $u(\boldsymbol{x}, t)$ of (2.1) attains its maximum or minimum value in the interior of $\Omega \times [0, T]$ then $u(\boldsymbol{x}, t)$ is a constant.*

Proof The proof for piecewise continuous data can be found in Liberman [Lib96, pp 128]. ∎

To construct an overlapping decomposition of the domain Ω, we first decompose Ω into N non-overlapping subdomains $\widetilde{\Omega}_j$ with boundaries $\partial\widetilde{\Omega}_j$, $j = 1, 2, \ldots, N$, as shown in Figure 1 for a two dimensional example. We denote the boundaries of the subdomain $\widetilde{\Omega}_j$ interior to the domain Ω by $\widetilde{\Gamma}_j$. Then we construct an overlapping decomposition Ω_j with boundary $\partial\Omega_j$ by enlarging each $\widetilde{\Omega}_j$ so that the boundaries of the new subdomains Γ_j interior to Ω are at least a distance δ away from $\widetilde{\Gamma}_j$ as shown in Figure 1 for the two dimensional example. To solve the parabolic problem (2.1) the overlapping Schwarz waveform relaxation iteration constructs iteratively u_j^{k+1} on each subdomain Ω_j using as the boundary condition the values from the neighboring subdomains u_l^k at the previous iteration. To pass the boundary information, the boundary of Ω_j is decomposed into disjoint subsets Γ_{jl}, $l = 1, \ldots, N$ such that the Euclidean distance of $\boldsymbol{x} \in \Gamma_{jl}$ from the boundary of Ω_l is at least δ. This is possible because of the way the overlapping decomposition was constructed: we simply use

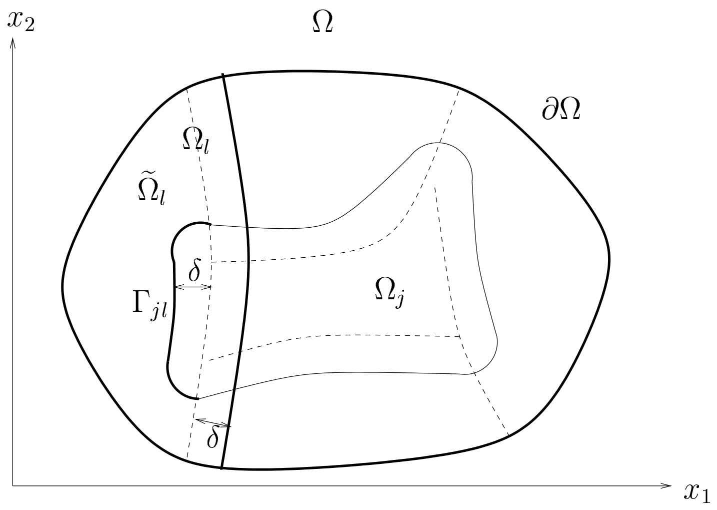

Figure 2: Information for Ω_j from the neighboring subdomain Ω_l is only used from within $\widetilde{\Omega}_l$.

the solutions obtained in Ω_l only within the smaller region $\widetilde{\Omega}_l$. An example for this information exchange is shown in Figure 2. Doing this for each subdomain, we define a complete solution at step k on the whole of Ω which can be used at step $k+1$ as boundary condition for the subdomain solves. We denote also by Γ_{j0} the part of the boundary that Ω_j shares with Ω.

3 Linear Convergence for Unbounded Time Domains

We first consider the case where $T = \infty$. On each subdomain Ω_j we solve at each step $k+1$ of the waveform relaxation iteration the subproblem

$$\begin{array}{rcll} \dfrac{\partial u_j^{k+1}}{\partial t} & = & \Delta u_j^{k+1} + f(\boldsymbol{x}, t) & \boldsymbol{x} \in \Omega_j, \quad 0 < t < T \\ u_j^{k+1}(\boldsymbol{x}, t) & = & u_l^k(\boldsymbol{x}, t) & \boldsymbol{x} \in \Gamma_{jl}, \quad 0 < t < T \\ u_j^{k+1}(\boldsymbol{x}, t) & = & g(\boldsymbol{x}, t) & \boldsymbol{x} \in \Gamma_{j0}, \quad 0 < t < T \\ u(\boldsymbol{x}, 0) & = & u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega_j, \end{array} \tag{3.1}$$

for $j = 1, 2, \ldots, N$, using the boundary information from the neighboring subdomains at step k. This corresponds to an additive Schwarz or Jacobi iteration which can be done in parallel. One can also consider a multiplicative Schwarz or Gauss Seidel iteration which would need a special coloring of subdomains to remain a parallel algorithm. If subdomains with the same color do not touch each other, then subdomains of the same color can be solved in parallel using the boundary values coming from subdomains with different colors. We focus in the analysis on the additive version, the results for the multiplicative versions can be obtained similarly.

The error $e_j^{k+1}(\boldsymbol{x}, t)$ between the real solution $u(\boldsymbol{x}, t)$ of (2.1) and the iterates

$u_j^{k+1}(\boldsymbol{x},t)$ of (3.1) satisfies the homogeneous equation

$$\begin{array}{rcll} \frac{\partial e_j^{k+1}}{\partial t} & = & \Delta e_j^{k+1} & \boldsymbol{x} \in \Omega_j, \quad 0<t<T \\ e_j^{k+1}(\boldsymbol{x},t) & = & e_l^k(\boldsymbol{x},t) & \boldsymbol{x} \in \Gamma_{jl}, \quad 0<t<T \\ e_j^{k+1}(\boldsymbol{x},t) & = & 0 & \boldsymbol{x} \in \Gamma_{j0}, \quad 0<t<T \\ e_j^{k+1}(\boldsymbol{x},0) & = & 0 & \boldsymbol{x} \in \Omega_j. \end{array} \tag{3.2}$$

Define the integer distance quantity m_j for each subdomain Ω_j to be the least number of subdomains one has to pass through to touch the boundary $\partial\Omega$, and define $m := \max_j m_j$. Furthermore define the index sets $I_l := \{j : m_j = l\}$ so that the index set I_l contains the indices of all the subdomains which are within distance l of the boundary. Defining for bounded functions $g(\boldsymbol{x},t) : \Omega \times [0,\infty) \to \mathbb{R}$ the norm

$$||g(\cdot,\cdot)||_\infty := \sup_{\boldsymbol{x}\in\Omega, t>0} |g(\boldsymbol{x},t)|$$

we have the following

Lemma 3.1 *The error of the waveform relaxation algorithm decays at the rate*

$$\max_j ||e_j^{k+m+2}(\cdot,\cdot)||_\infty \leq \gamma(m,\delta) \max_j ||e_j^k(\cdot,\cdot)||_\infty \tag{3.3}$$

where $\gamma(m,\delta)$ is a number strictly less than one, independently of k.

Proof The idea of the proof is to construct a sequence of elliptic upper bounds on the iterates and then to apply the convergence analysis based on the maximum principle for the elliptic upper bounds in Lions [Lio88]. Fix k and define $E^k := \max_j ||e_j^k(\cdot,\cdot)||_\infty$ and note that on each subdomain the solution $\tilde{e}_j^{k+1}$ of the elliptic problem

$$\begin{array}{rcll} 0 & = & \Delta \tilde{e}_j^{k+1} & \boldsymbol{x} \in \Omega_j \\ \tilde{e}_j^{k+1}(\boldsymbol{x}) & = & E^k & \boldsymbol{x} \in \Gamma_{jl} \\ \tilde{e}_j^{k+1}(\boldsymbol{x}) & = & 0 & \boldsymbol{x} \in \Gamma_{j0} \end{array} \tag{3.4}$$

is an upper bound on the modulus of e_j^{k+1}. Now $\tilde{e}_j^{k+1}$ satisfies a maximum principle and for $j \in I_0$ $\tilde{e}_j^{k+1} < E^k$ in the interior of $\widetilde{\Omega}_j$, since $\tilde{e}_j^{k+1}$ satisfies on part of the boundary of Ω_j a homogeneous boundary condition. Note that for $j \notin I_0$ we have $\tilde{e}_j^{k+1}$ not necessarily strictly less than E^k since $\tilde{e}_j^{k+1}$ might have the value E^k on all its boundaries and thus by the maximum principle $\tilde{e}_j^{k+1} \equiv E^k$. Define

$$E^{k+1} := \sup_{\boldsymbol{x}\in\widetilde{\Omega}_l, l\in I_0} \tilde{e}_l^{k+1} \leq \gamma_1(\delta) E^k$$

for some constant $\gamma_1(\delta) < 1$. Note that γ_1 depends on the size of the overlap, but not on k since the error $\tilde{e}_j^{k+1}$ is a linear function of the boundary condition. Now for the next iteration by definition part of the boundary of subdomains Ω_j with $j \in I_1$ lie strictly within $\widetilde{\Omega}_l$ with $l \in I_0$ and therefore for $j \in I_1$ the solution $\tilde{e}_j^{k+2}$ of the elliptic problem

$$\begin{array}{rcll} 0 & = & \Delta \tilde{e}_j^{k+2} & \boldsymbol{x} \in \Omega_j \\ \tilde{e}_j^{k+2}(\boldsymbol{x}) & = & E^k & \boldsymbol{x} \in \Gamma_{jl}, \ l \notin I_0 \\ \tilde{e}_j^{k+2}(\boldsymbol{x}) & = & E^{k+1} & \boldsymbol{x} \in \Gamma_{jl}, \ l \in I_0 \end{array} \tag{3.5}$$

is an upper bound on the modulus of e_j^{k+2}. Since $E^{k+1} \leq \gamma_1(\delta)E^k$ we have by the maximum principle $\tilde{e}_j^{k+2} < E^k$ in $\widetilde{\Omega}_j$ and defining E^{k+2} similarly to E^{k+1} before, we find $E^{k+2} \leq \gamma_2(\delta)E^k$ for some constant $\gamma_1(\delta) \leq \gamma_2(\delta) < 1$ independent of k. By induction we find at step $k+m+1$ for the error in the subdomains Ω_j with $j \in I_m$ the elliptic upper bound

$$\begin{array}{rcll} 0 & = & \Delta\tilde{e}_j^{k+m+1} & \boldsymbol{x} \in \Omega_j \\ \tilde{e}_j^{k+m+1}(\boldsymbol{x}) & = & E^k & \boldsymbol{x} \in \Gamma_{jl},\ l \notin I_{m-1} \\ \tilde{e}_j^{k+m+1}(\boldsymbol{x}) & = & E^{k+m} & \boldsymbol{x} \in \Gamma_{jl},\ l \in I_{m-1} \end{array} \tag{3.6}$$

and $\tilde{e}_j^{k+m+1} < E^k$ in $\widetilde{\Omega}_j$. Defining E^{k+m+1} as before we find $E^{k+m+1} \leq \gamma_{m+1}(\delta)E^k$ for some constant $\gamma_1(\delta) \leq \gamma_2(\delta) \leq \ldots \leq \gamma_{m+1}(\delta) < 1$ independent of k. Now for the next iteration step $k+m+2$ all the errors e_j^{k+m+2} have boundary values less than or equal to $E^{k+m+1} \leq \gamma_{m+1}(\delta)E^k$, since they come from iteration step $k+m+1$ in the interior of neighboring subdomains. Defining $\gamma(m,\delta) := \gamma_{m+1}(\delta)$ the result follows. ∎

Note that the above estimate for $\gamma(m,\delta)$ is quite conservative. In a practical implementation the measured convergence rate is expected to be better, since $\gamma(m,\delta)$ was derived assuming worst case behavior of the error. $\gamma(m,\delta)$ will also depend on the shape of the subdomains, which is hidden in the above argument because of the generality of the domain decomposition employed.

Theorem 3.2 (Linear Convergence) *The waveform relaxation algorithm converges linearly on unbounded time intervals in the infinity norm. The error decays like*

$$\max_j ||e_j^{k(m+2)}(\cdot,\cdot)||_\infty \leq (\gamma(m,\delta))^k \max_j ||e_j^0(\cdot,\cdot)||_\infty \tag{3.7}$$

where $\gamma(m,\delta) < 1$ *as in Lemma 3.1.*

Proof The proof follows by induction from Lemma 3.1. ∎

Note that the convergence result on unbounded time domains depends on the number of subdomains. The more subdomains one uses, the longer it takes for information to propagate from the outer boundary of Ω to the inner subdomains. This is because the steady state solution is limiting the convergence rate, and the steady state solution does not see the zero initial condition. This is different in the superlinear convergence analysis of the next section.

4 Superlinear Convergence for Bounded Time Domains

We now consider a bounded time interval, $T < \infty$. Like in the unbounded time domain case we are solving at each step $k+1$ of the waveform relaxation iteration the subproblem (3.1) using the boundary information from the neighboring subdomains at step k. We are interested in estimating the decay of the error e_j^{k+1} in (3.2) for short time t. Define the infinity norm of a function $g(\boldsymbol{x},t)$ on the boundary Γ_j of subdomain Ω_j by

$$||g(\cdot,\cdot)||_{\Gamma_j} := \sup_{\boldsymbol{x}\in\Gamma_j, 0<t<T} |g(\boldsymbol{x},t)|.$$

We first estimate the decay of the error on the boundary of the subdomains over one step of the iteration.

Lemma 4.1 *In n dimensions, the maximum error e_j^{k+1} on the boundary of all subdomains Ω_j decays in the infinity norm at the rate*

$$\max_j ||e_j^{k+1}||_{\Gamma_j} \leq 2n\, erfc(\frac{\delta}{2\sqrt{nT}}) \max_j ||e_j^k||_{\Gamma_j}.$$

Proof Define $E_k(t) := \max_j(\sup_{\boldsymbol{x}\in\Gamma_j} |e_j^k(\boldsymbol{x},t)|)$. Because of the overlapping property we are interested in the magnitude of $e_j^k(\boldsymbol{x},t)$ at distance δ from the boundary Γ_j for short time. An upper bound for all e_j^k can be obtained from the decay of $\overline{e}$ which satisfies

$$\begin{array}{rcll} \frac{\partial \overline{e}}{\partial t} &=& \Delta \overline{e} & \boldsymbol{x}\in B(\delta), \quad 0<t<T \\ \overline{e}(\boldsymbol{x},t) &=& E_k(t) & \boldsymbol{x}\in \partial B(\delta), \quad 0<t<T \\ \overline{e}(\boldsymbol{x},0) &=& 0 & \boldsymbol{x}\in B(\delta) \end{array} \tag{4.1}$$

where $B(\delta)$ denotes a ball with radius δ in $\mathbb{R}^n$. To get an explicit bound, inscribe a hypercube $Q(\delta)$ in $\mathbb{R}^n$ with side $\frac{2\delta}{\sqrt{n}}$ into the ball $B(\delta)$ and consider the decay of $\tilde{e}$ in the hypercube $Q(\delta)$,

$$\begin{array}{rcll} \frac{\partial \tilde{e}}{\partial t} &=& \Delta \tilde{e} & \boldsymbol{x}\in Q(\delta), \quad 0<t<T \\ \tilde{e}(\boldsymbol{x},t) &=& E_k(t) & \boldsymbol{x}\in \partial Q(\delta), \quad 0<t<T \\ \tilde{e}(\boldsymbol{x},0) &=& 0 & \boldsymbol{x}\in Q(\delta). \end{array} \tag{4.2}$$

Evaluating $\tilde{e}$ at the center of the hypercube $Q(\delta)$ we obtain an upper bound on the error e_j^k at distance δ from the boundary Γ_j. An upper bound on $\tilde{e}$ at the center of the hypercube $Q(\delta)$ can be obtained by summing the half space solution v of the heat equation for each of the $2n$ faces of the hypercube,

$$\begin{array}{rcll} \frac{\partial v}{\partial t} &=& \Delta v & \boldsymbol{x}\in \mathbb{R}^n,\ x_1>0, \quad 0<t<T \\ v(\boldsymbol{x},t) &=& E_k(t) & \boldsymbol{x}\in \mathbb{R}^n,\ x_1=0, \quad 0<t<T \\ v(\boldsymbol{x},0) &=& 0 & \boldsymbol{x}\in \mathbb{R}^n, x_1\geq 0, \end{array} \tag{4.3}$$

where x_1 denotes the first component of $\boldsymbol{x}\in\mathbb{R}^n$. The solution of (4.3) is [Can84]

$$v(\boldsymbol{x},t) = \int_0^t K_x(x_1, t-\tau)E_k(\tau)d\tau, \tag{4.4}$$

with the kernel

$$K_x(x,t) = \frac{x}{2\sqrt{\pi}t^{3/2}}e^{-\frac{x^2}{4t}}. \tag{4.5}$$

Hence an upper bound on e_j^k at distance δ from the boundary Γ_j is given by $2n$ times (4.4). This value is by the formulation of the iteration an upper bound on the boundary values for e_j^{k+1} and thus

$$E_{k+1}(t) := \max_j(\sup_{\boldsymbol{x}\in\Gamma_j} |e_j^{k+1}(\boldsymbol{x},t)|) \leq 2n\int_0^t K_x(\frac{\delta}{\sqrt{n}}, t-\tau)E_k(\tau)d\tau. \tag{4.6}$$

Now take the supremum of $E_k(t)$, $0 < t < T$ out of the integral and apply the variable transform

$$y := \frac{\delta}{2\sqrt{n(t-\tau)}}$$

to the integral. This leads to

$$E_{k+1}(t) \leq 2n\text{erfc}(\frac{\delta}{2\sqrt{nt}}) \max_j ||e_j^k||_{\Gamma_j}.$$

Since $\text{erfc}(\delta/(2\sqrt{nt}))$ is monotonically increasing with t we can replace t with the upper bound T on the right. Now the bound is independent of t so we can take the supremum over t on the left and the result follows. ■

Note that Lemma 4.1 can be used to derive an arbitrary fast linear upper bound on the convergence rate by shortening the time interval $[0, T)$ appropriately, since $\lim_{x\to\infty} \text{erfc}(x) = 0$. We derive however an upper bound on the decay of the error over k steps of the iteration which leads to a superlinear convergence result.

Lemma 4.2 *In n dimensions, the maximum error e_j^k on the boundary of all subdomains Ω_j decays in the infinity norm at least at the rate*

$$\max_j ||e_j^k||_{\Gamma_j} \leq (2n)^k erfc(\frac{k\delta}{2\sqrt{nT}}) \max_j ||e_j^0||_{\Gamma_j}.$$

Proof By iteration of inequality (4.6) in Lemma 4.1 we get a bound in form of a convolution, namely

$$\begin{aligned} E_k(t) \quad \leq \quad & (2n)^k \int_0^t K_x(\frac{\delta}{\sqrt{n}}, t - s_1) \cdots \\ & \int_0^{s_{k-1}} K_x(\frac{\delta}{\sqrt{n}}, s_{k-1} - s_k) ds_k \cdots ds_1 \max_j ||e_j^0||_{\Gamma_j}. \end{aligned}$$

To unfold the convolutions, note that the Laplace transform of a convolution is the product of the Laplace transformed kernels. In our case the Laplace transform of the kernel is [AS64]

$$\int_0^\infty e^{-st} K_x(\frac{\delta}{\sqrt{n}}, t) dt = e^{-\frac{\delta}{2\sqrt{n}}\sqrt{s}}$$

and thus the k-fold convolution is the product of identical exponentials in the Laplace transformed domain,

$$e^{-\frac{k\delta}{2\sqrt{n}}\sqrt{s}}.$$

Back-transforming this expression, we find

$$E_k(t) \leq (2n)^k \int_0^t K_x(\frac{k\delta}{\sqrt{n}}, t - \tau) d\tau \max_j ||e_j^0||_{\Gamma_j}.$$

Using a similar variable transform as in Lemma 4.1 the result follows. ■

Defining for bounded functions $g(\boldsymbol{x}, t) : \Omega \times [0, T) \to \mathbb{R}$ the norm

$$||g(\cdot, \cdot)||_T := \sup_{\boldsymbol{x}\in\Omega, 0<t<T} |g(\boldsymbol{x}, t)|$$

we have the following

Theorem 4.3 (Superlinear Convergence) *The waveform relaxation algorithm converges superlinearly on bounded time intervals in the infinity norm. The error decays like*

$$\max_j ||e_j^k(\cdot,\cdot)||_T \leq (2n)^k erfc(\frac{k\delta}{2\sqrt{nT}}) \max_j ||e_j^0(\cdot,\cdot)||_T. \tag{4.7}$$

Proof The proof follows from Lemma 4.2 and the maximum principle. ∎

Note that this superlinear convergence rate is faster than the superlinear convergence rate found for classical waveform relaxation algorithms. The classical result gives a contraction governed by a factorial [MN87] with asymptotic expansion

$$\frac{(CT)^k}{k!} = \left(\frac{1}{\sqrt{2\pi}} + O(k^{-1})\right) e^{-k\ln k + (1+\ln(CT))k - \frac{1}{2}\ln k} \sim e^{-k\ln k}$$

whereas the new result (4.7) gives a contraction with asymptotic expansion

$$C_1^k \operatorname{erfc}(\frac{C_2 k}{\sqrt{T}}) = \left(\frac{\sqrt{T}}{C_2\sqrt{\pi}} + O(k^{-2})\right) e^{-\frac{C_2^2}{T}k^2 + \ln(C_1)k - \ln k} \sim e^{-k^2}.$$

The numerical experiments in the next section show that it is indeed the second result which is observed numerically.

5 Numerical Experiments

We show three sets of numerical experiments. First a one dimensional experiment to show that the bounds we derived are sharp in one dimension. Then a two dimensional experiment on a square with 2×2 subdomains to illustrate the linear and superlinear convergence behavior of the algorithm depending on the length of the time interval. Finally we show a scaling experiment in the number of subdomains which confirms that the superlinear convergence rate is indeed independent of the number of subdomains. We also illustrate this fact graphically with a sequence of iterates which shows the intuitive reason for the independence of the number of subdomains when the algorithm is in the superlinear convergence regime.

5.1 One Dimensional Example

We solve the one dimensional heat equation

$$\begin{array}{rcll} \frac{\partial u}{\partial t} &=& \frac{\partial^2 u}{\partial x^2} + 5e^{-(t-2)^2-(x-\frac{1}{4})^2} & 0 < x < 1,\ 0 < t < T \\ u(0,t) &=& 0 & 0 < t < T \\ u(1,t) &=& e^{-t} & 0 < t < T \\ u(x,0) &=& x^2 & 0 \leq x \leq 1 \end{array} \tag{5.1}$$

splitting the domain $\Omega = [0,1] \times [0,T]$ into the two subdomains $\Omega_1 = [0,\beta] \times [0,T]$ and $\Omega_2 = [\alpha,1] \times [0,T]$ and applying the overlapping Schwarz waveform Relaxation algorithm for for three pairs of values $(\alpha,\beta) \in \{(0.4,0.6),(0.45,0.55),(0.48,0.52)\}$. We obtain from Theorem 3.2 for this special case the

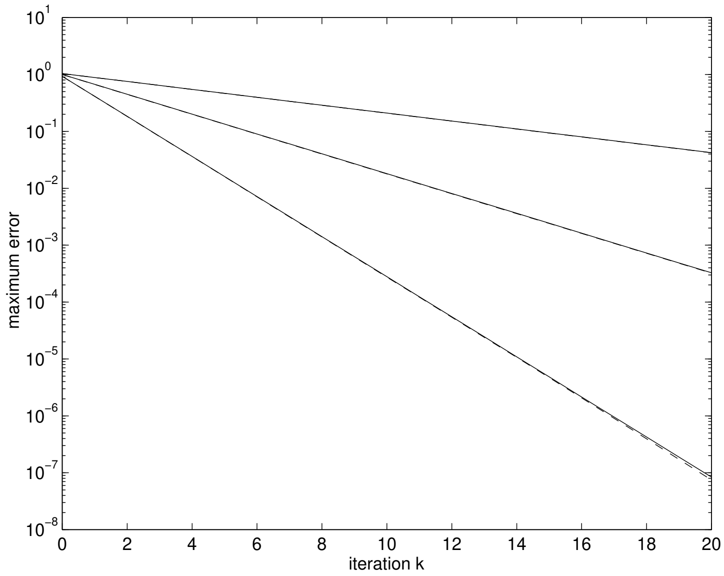

Figure 3: Theoretical and measured decay rate of the error for two subdomains and three different sizes of the overlap.

Corollary 5.1 *The Overlapping Schwarz waveform relaxation iteration for the one dimensional heat equation with two subdomains* $\Omega_1 = [0, \alpha] \times [0, T]$ *and* $\Omega_2 = [\beta, 1] \times [0, T]$ *converges at least at the linear rate*

$$\max_{j=1,2} ||e_j^{2k}(\cdot, \cdot)||_\infty \leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)} \right)^k \max_{j=1,2} ||e_j^0(\cdot, \cdot)||_\infty.$$

Proof The proof follows from Theorem 3.2 using the special overlap structure in one dimension. A direct proof can be found in [GS98]. ■

As a numerical scheme on the subdomains we use centered finite difference in space with $\Delta x = 0.01$ and backward Euler in time with $\Delta t = 0.01$. Figure 3 shows the convergence of the algorithm in the maximum norm for a long time interval $T = 3$. The solid line is the predicted convergence rate according to Corollary 5.1 and the dashed line is the measured one. The measured error displayed is the difference between the numerical solution on the whole domain and the solution obtained from the domain decomposition algorithm. Clearly it is the steady state which limits the convergence of the overlapping Schwarz waveform relaxation algorithm, the steady state convergence bounds are sharp.

To see the algorithm in the superlinear convergence regime, we shorten the time interval to $[0, 0.05]$ and apply the same overlapping Schwarz waveform relaxation algorithm to the above problem. We get for this special one dimensional example from Theorem 4.3 the

Corollary 5.2 *The overlapping Schwarz waveform relaxation iteration for the one dimensional heat equation with two subdomains* $\Omega_1 = [0, \alpha] \times [0, T]$ *and* $\Omega_2 = [\beta, 1] \times [0, T]$ *converges superlinearly on bounded time intervals* $t \in [0, T]$ *with at least the*

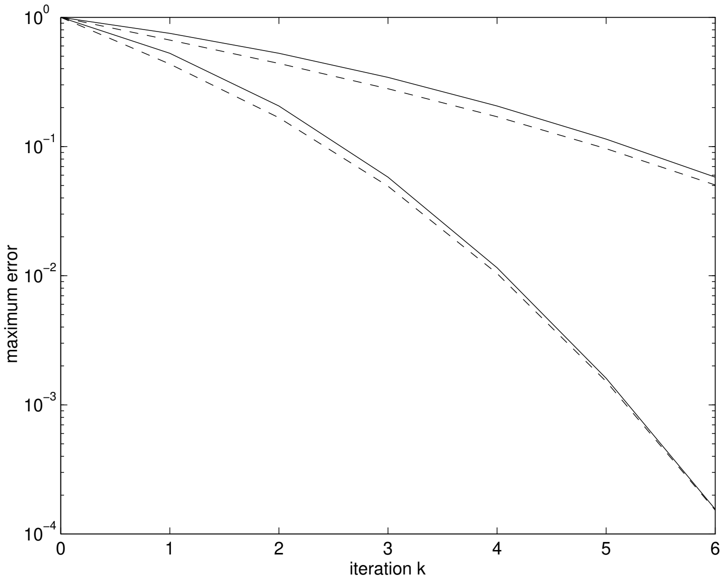

Figure 4: Superlinear decay of the error for a small time interval and two different sizes of overlap.

rate

$$\max_{j=1,2} ||e_j^k(\cdot,\cdot)||_T \leq erfc(\frac{k(\beta-\alpha)}{2\sqrt{T}}) \max_{j=1,2} ||e_j^0(\cdot,\cdot)||_T.$$

Proof The proof follows from Theorem 4.3 without the geometric factor from the sphere and inscribed hypercube. ■

Note that the algebraic term 2^k is absent in the one dimensional result because we can directly estimate the decay in one dimension which leads to a sharp upper bound on the convergence rate. We use the first and second of the splittings of the previous experiment, namely $(\alpha, \beta) \in \{(0.4, 0.6), (0.45, 0.55)\}$. Figure 4 shows the convergence of the algorithm, where again the solid line is the predicted convergence rate and the dashed line is the measured one. Clearly now the algorithm converges superlinearly, as predicted by our analysis.

As a last example we perform the computation on a medium time interval $[0, 0.5]$ to see what happens between the linear and the superlinear convergence regime. We chose for the overlap $(\alpha, \beta) = (0.4, 0.6)$ as in the previous experiments. Since both the linear and the superlinear convergence result are upper bounds on the convergence of the algorithm, the actual convergence behavior needs to stay below both bounds. Figure 5 shows both the linear and the superlinear bounds as solid lines and the actual computation as a dashed line. The algorithm is first in the linear convergence regime, but once the superlinear bound becomes dominant, it goes through a transition into the superlinear convergence regime. To benefit from the superlinear convergence, it is therefore important to choose time windows such that the algorithm converges superlinearly.

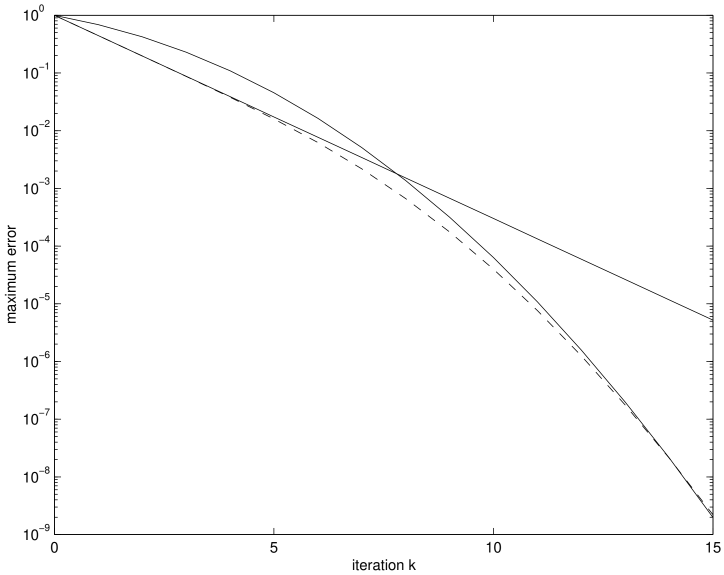

Figure 5: Transition from the linear to the superlinear convergence regime. The convergence rate of the algorithm is first dominated by the linear convergence regime and after some iterations by the superlinear convergence regime.

5.2 Two Dimensional Example

We solve the heat equation in two dimensions

$$\frac{\partial u}{\partial t} = \Delta u, \quad (x_1, x_2) \in [0,1] \times [0,1], \; t \in [0,T] \tag{5.2}$$

with homogeneous initial and boundary conditions which means we are looking for the zero solution and thus are simulating directly the error equations in the analysis. We decompose the unit square into four smaller squares which form overlapping subdomains of equal size. We run the overlapping Schwarz waveform relaxation algorithm in its additive version. For the overlap parameter δ we chose two values, $\delta \in \{0.1, 0.06\}$. We solve the subdomain problems using a centered finite difference scheme with $\Delta x = 0.02$ and integrate in time using backward Euler. Figure 6 shows the algorithm in the linear convergence regime, integrating up to $T = 3$.

Figure 7 shows the algorithm in the superlinear convergence regime, integrating over a shorter time interval, $T = 0.05$. Note the different scale which shows how much faster the superlinear convergence is compared to the linear convergence in the previous experiment.

5.3 Scaling in the Number of Subdomains

We finally analyze the scaling behavior of overlapping Schwarz waveform relaxation numerically. We solve again the heat equation in two dimensions (5.2) but now vary the number of subdomains. We use a discretization in space with $\Delta x = 1/30$. Table 1 shows the number of iterations needed to decrease the error below a given tolerance for different numbers of subdomains and the two time intervals $[0,5]$ and $[0,0.01]$ with fixed overlap parameter $\delta = 1/30$. The table shows that over long

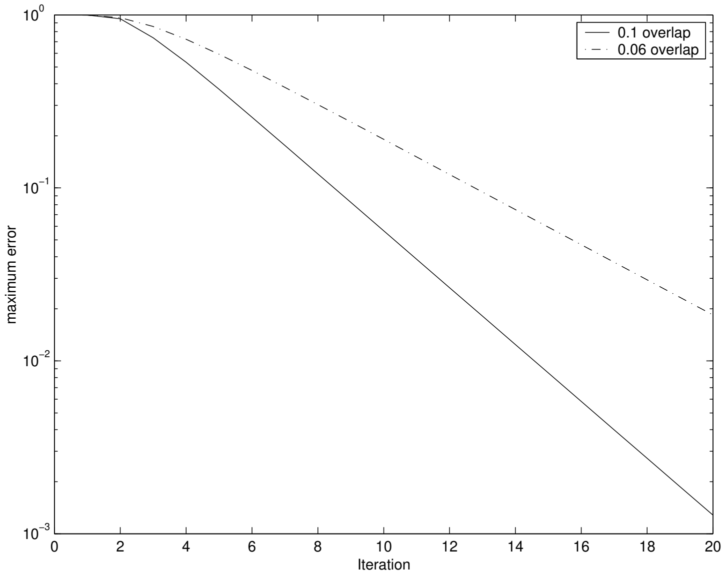

Figure 6: Two dimensional model problem with four subdomains, algorithm in the linear convergence regime for $T = 3$.

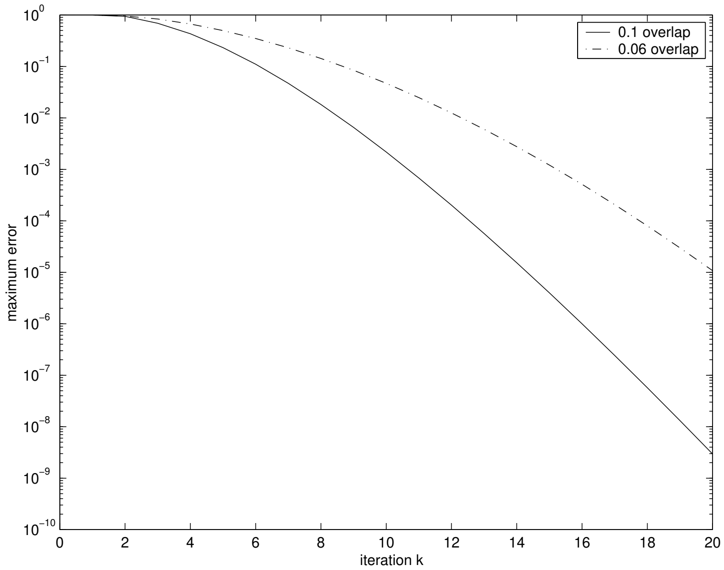

Figure 7: Two dimensional model problem with four subdomains, algorithm in the superlinear convergence regime for $T = 0.05$.

	2×2	3×3	4×4	6×6
$T = 5$	12	15	19	28
$T = 0.01$	5	5	5	6

Table 1: Number of iterations needed to reach a certain tolerance for different numbers of subdomains and two different time intervals. While the algorithm does not scale in the linear convergence regime with respect to the number of subdomains, it does scale in the superlinear convergence regime.

time intervals, the overlapping Schwarz waveform relaxation algorithm does not scale with respect to the number of subdomains. This is due to the fact that the convergence rate is limited by the steady state solution and corresponds to the m factor in Theorem 3.2. For elliptic problems it is well known that overlapping Schwarz needs a coarse mesh to exhibit convergence independent of the number of subdomains [CM94]. Such a coarse mesh could also be added to the overlapping Schwarz waveform relaxation algorithm. This is however not necessary when the algorithm is in the superlinear convergence regime. Here the algorithm converges independently of the number of subdomains, as predicted by Theorem 4.3 and shown in Table 1 experimentally. This can be understood intuitively by noting that it is the initial condition which determines over short time intervals mainly the solution of a parabolic evolution problem and thus we have error decay away from the interior boundaries even if they are far away from the real boundary, since the initial conditions are known for all subdomains. This is illustrated in Figure 8 which shows the error of three consecutive iterates for 4×4 subdomains at the end of the time interval on the left for $T = 5$ where the algorithm is in the linear convergence regime and on the right for $T = 0.01$ where the algorithm is in the superlinear convergence regime. One can clearly see how the error is diminished in all subdomains on the right due to the initial condition, whereas it has to be eliminated from the original boundaries on the left.

6 Conclusions

Although we used the linear heat equation as our guiding example, the techniques introduced here can be generalized. The linear convergence rate is derived using a maximum principle. Hence similar results can be obtained for more general equations satisfying a maximum principle, like equations with variable coefficients and convection terms. However for evolution problems the important property of the algorithm is the superlinear convergence rate. In our analysis we rely only on a one dimensional result, so that convergence results obtained in [Gan98b] for the one dimensional reaction diffusion equation lead to similar results in higher dimensions using the techniques presented here. Similarly an estimate for a convection diffusion equation is possible as well using the result of [GK97].

Acknowledgment: We would like to thank Joe Keller for suggesting the estimate of the decay within a ball in the overlap and Andrew Stuart for fruitful discussions.

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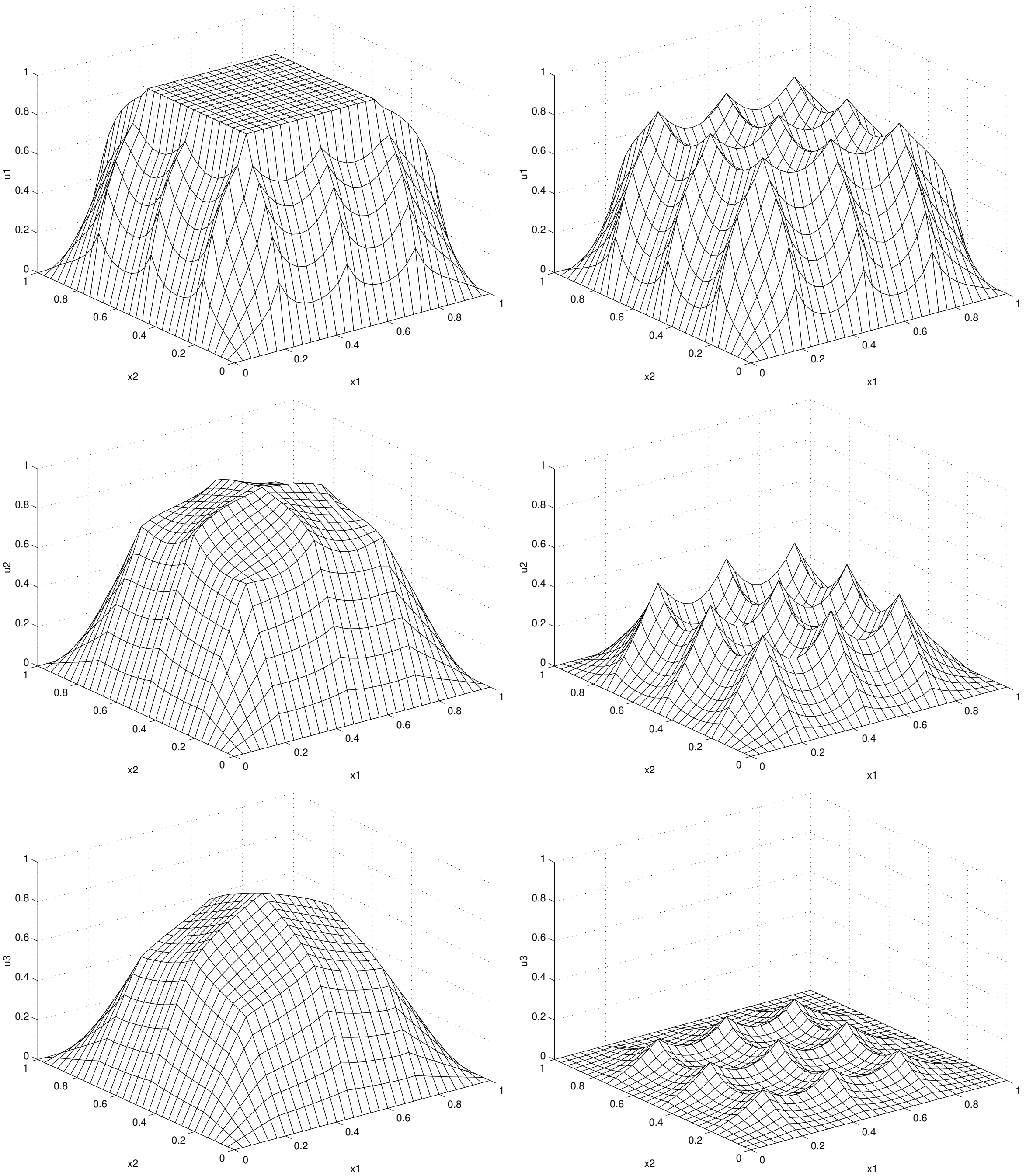

Figure 8: Three consecutive iterates at the end of the time interval on the left at $T = 5$ where the algorithm is in the linear convergence regime and on the right at $T = 0.01$ where the algorithm is in the superlinear convergence regime. Note how the error decays uniformly over all subdomains on the right.

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