On the Geometry of the Moduli Space of Calabi-Yau Manifolds

by

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Chapter 1

Introductions

In this paper, we study the moduli space of the complex structure and the classifying space of polarized Calabi-Yau manifolds. Let $X$ be a Calabi-Yau manifold. That is, $X$ is an $n$-dimensional compact Kähler manifold with zero first Chern class. By the famous theorem of Yau [28], there is a Kähler metric on $X$ such that the Ricci curvature of such a Kähler metric is zero.

Suppose $\Theta$ is the holomorphic tangent bundle of $X$. In [24], Tian proved that the moduli space of the complex structure is smooth, at least locally. The complex dimension of the moduli space is $\text{dim} \, H^1(X, \Theta)$. In other words, there are no obstructions towards the complex deformation of the complex structure of Calabi-Yau manifold. A good reference for the proof is in [9].

Let $n = 3$ for example. A natural question is that to what extent the Hodge structure, namely, the decomposition of $H^3(X, C)$, into the sum of $H^{p,q}$'s (p+q=3), determines a Calabi-Yau threefold. Let’s recall the concept of classifying spaces in [10], which is a generalization of the classical period domains. In the case of Calabi-Yau threefold, the classifying space $D$ is defined as the set of the filtrations of $H = H^3(X, C)$ by

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H$$
with $\dim F^3 = 1$, $\dim F^2 = n = \dim H^1(X, \Theta)$, $\dim F^1 = 2n + 1$, and $H^{p,q} = F^p \cap F^q$, $H = F^p \oplus F^{4-p}$ ($p+q=3$) together with a quadratic form $Q$ such that

1. $\iota Q(x, x) < 0$ if $0 \neq x \in H^{3,0}$

2. $\iota Q(x, x) > 0$ if $0 \neq x \in H^{2,1}$

Here $\iota = \sqrt{-1}$.

Griffiths [10] proved that $D$ is a homogeneous complex manifold and $D$ is a Kähler manifold. But generally, $D$ is not a homogeneous Kähler manifold (i.e. there is no Kähler metric on $D$ which is $G$-invariant).

Using this terminology, we see that there is a natural map from the moduli space into the classifying space. Intuitively, this is because $D$ is just the set of all the possible “Hodge decompositions”. Such a map is called a period map. In the case of Calabi-Yau, the map is a holomorphic immersion; or in other word, the infinitesimal Torelli theorem is valid.

It is not hard to see that $D$ fibers over a symmetric space $D_1$. But such a symmetric space needs not to be Hermitian. Even $D_1$ is Hermitian symmetric, $D$ still needs not fiber holomorphically over $D_1$, although in that case, there is a complex structure on $D$ such that $D$ becomes homogeneous Kähler [20].

Griffiths [10] introduced the concept of horizontal distribution, which is a holomorphic distribution on $D$. He also proved that the moduli space, via the period map to the classifying space, is an integral submanifold of the horizontal distribution. An horizontal slice is an integral complex submanifold of the horizontal distribution. In this terminology the moduli space is an horizontal slice of the classifying space. It is thus also interesting to study the properties of the horizontal slice alone.

The more important reason to study the horizontal slice alone is that usually the moduli space itself is not a complete submanifold in the classifying space. But, it is generally believed that the moduli space can be extended to a complete horizontal slice [9].
We are seeking a differential geometric proof of the quasi-projectivity of the moduli space, cf. Viehweg [26]. This paper is the first attempt to attack this problem.

There is a natural projection \( p \) from \( D \) to its base symmetric space. Here we define the base symmetric space \( D_1 \) of \( D \) to be \( G/K \), where \( K \) is the maximal compact subgroup of \( G \) containing \( V \).

This is our basic setting in this paper.

The organization of the paper is as follows:

In Chapter 3, we studied the canonical map from the moduli space or horizontal slice to the classifying space, we have

**Theorem 1.1.** Suppose \( \mathcal{M} \to D \) is a horizontal slice and an immersion. Then

\[
\mathcal{M} \to D_1
\]

defined from the composition of the immersion and the natural projection \( D \to D_1 \) is again an immersion. Furthermore, it is a pluriharmonic map. i.e. It restricts to a harmonic map from any holomorphic curve of \( \mathcal{M} \). In our notations, it satisfies

\[
\nabla_{p_*X}p_*X + \nabla_{p_*JX}p_*JX + p_*J[X,JX] = 0
\]

for \( X \in \mathcal{M} \), where \( J \) is the complex structure on \( \mathcal{M} \) and \( \nabla \) is the Riemannian connection on \( G/K \).

Theorem 1.1 is of its own interest. But the most important application of theorem 1.1 is the following:

**Theorem 1.2.** The metric on the horizontal slice as a Riemannian submanifold of \( D \) is actually Kähler. Moreover, the holomorphic bisectional curvature of such a metric is nonpositive. And the Ricci curvature is negative away from zero.

Another result in this chapter is the non-existence of invariant Kähler metric on some classifying spaces.
Suppose $\Gamma$ is a lattice of the group $G$. We call that $\Gamma$ is cocompact, if $\Gamma \backslash G$ is a compact topological space.

**Theorem 1.3.** There are no Kähler metrics on the classifying space which are $\Gamma$ invariant if $\Gamma$ is cocompact.

In fact, there is a version of the above theorem in the case that $\Gamma \backslash G$ has finite volume. We omit it for the sake of simplification.

Let’s give some remarks on the first theorem. The similar pluriharmonicity was studied in Bryant [3], Burstall and Salamon [5], Black [1]. Those papers only considered the compact cases and cannot apply to the cases we are interested in this paper.

In Chapter 4, we studied the moduli space of Calabi-Yau threefolds. Since the structure of the classifying space and the local structure of the moduli space are relatively easy in this case, we write out the invariant complex structure on the classifying space and the canonical map explicitly. Using this, we not only reproved the theorems in Chapter 3 in the case of dimension 3, but got a formula between the Weil-Petersson metric and the metric of the variation of Hodge Structure as well. In fact, we proved:

**Theorem 1.4.** Suppose $\omega$ is the Kähler form of the Weil-Petersson metric. Let

$$\omega_1 = (n + 3)\omega + Ric(\omega)$$

then $\omega_1$ is a constant multiple of the VHS metric.

There are several applications towards the above theorem. First, using a gradient estimate, we proved that the completeness of the Weil-Petersson metric is equivalent to the boundness of the cubic form:

**Theorem 1.5.** Suppose $M$ is the horizontal slice defined above. If the Weil Petersson metric on $M$ is complete, then the normal of the cubic form with respect to the Weil-Petersson metric is bounded. On the other hand, if the cubic form with respect to the
Weil-Petersson metric is bounded, then the Weil-Petersson completeness is equivalent to the VHS completeness.

We also write out explicitly the curvature tensor of the curvature of the VHS metric. Thus we got a sharp upper bound of the Ricci curvature:

**Theorem 1.6.** Let \( c(n) = ((\sqrt{n} + 1)^2 + 1) \), then

\[
Ric(\omega_1) \leq -\frac{1}{c(n)} \omega
\]

\[
R \leq -\frac{1}{c(n)}
\]

where \( R \) is the superium of the holomorphic sectional curvature. The constant here is optimal. Furthermore, the bisectional curvature is nonpositive.

We also proved that if the Ricci curvature is bounded, so is the sectional curvature. From this, we know that if the Ricci curvature is bounded, then the moduli space or the horizontal slice can be compactified.

**Theorem 1.7.** Suppose \( p \in M \) is a fixed point such that the Ricci curvature has a bound \( C_p \) at \( p \). That is

\[
|Ric(\omega_1)_p| \leq C_p \omega_1
\]

Then the Riemannian sectional curvature has a bound

\[
|R(X,Y,X,Y)| \leq (2 + C_p) ||X||^2 ||Y||^2
\]

where \( X,Y \in T_pM \) and \( X \perp Y \).

Finally, we give an asymptotic behavior of the Weil Petersson metric near the degeneration of the Calabi-Yau threefold:

**Theorem 1.8.** Suppose \( X \to \Delta \) is a degeneration of Calabi-Yau threefolds. Suppose \( \omega \) be the Weil-Petersson metric on \( \Delta^* \). Then if

\[
\lim_{r \to 0} \frac{\log \omega}{\log \frac{1}{r}} = 0
\]
Then

\[ \lambda(z) \leq C_1 (\log \frac{1}{r})^{4c(n)} \]

where \( c(n) = ((\sqrt{n} + 1)^2 + 1) \) and \( C_1 \) is a constant.

In chapter 5 we proved some rigidity theorems. First, we proved that for concave horizontal slices, the VHS metric and the Weil-Petersson metrics are intrinsically defined in certain cases:

**Theorem 1.9.** If the moduli space \( \Gamma \backslash \mathcal{M} \) is a concave manifold. Then the VHS metric is intrinsically defined. Furthermore, in the case of dimension 3, the Weil-Petersson metric is intrinsically defined.

We also proved a local rigidity theorem for the monodromy group representation which is an intermediate step in proving the global (super)-rigidity of the moduli spaces.
Chapter 2

Preminliaries

In this chapter, we shall review and collect some basic facts about the structure of semisimple Lie algebras, symmetric and homogeneous spaces and the theory of classifying space that we will use in this paper. The materials are basically from [10] and [11].

Throughout, $g_c$ will denote a complex semisimple Lie algebra, $m_0$ a compact real form of $g_c$, and $\tau$ complex conjugation of $g_c$ with respect to $m_0$. We choose a maximal abelian subalgebra $h_0$ of $m_0$, complexifying $h_0$, one obtains a $\tau$-invariant Cartan subalgebra $h$ of $g_c$. The adjoint representation of $h$ on $g_c$ determines a decomposition

$$g_c = h \oplus \sum_{\alpha \in \Delta} g^\alpha$$

where $\Delta$, the set of nonzero roots, is a subset of the dual space of $h$, and each root space

$$g^\alpha = \{ x \in g_c | [h, x] = \langle \alpha, h \rangle x, \text{ for all } h \in h \}$$

is one dimensional. If $\alpha, \beta, \alpha + \beta \in \Delta$, then

$$[g^\alpha, g^\beta] = g^{\alpha+\beta}$$

Since $m_0$ is a compact real form, all roots assume real value on $h_R = \sqrt{-1} h_0$. We shall regard $\Delta$ as a subset of $h_R^*$, the dual space of $h_R$. For every $\alpha \in \Delta$

$$\tau(g^\alpha) = g^{-\alpha}$$
because $\mathfrak{h}_R$ is purely imaginary with respect to $\mathfrak{m}_0$.

The Cartan-Killing form

$$B(x, y) = Tr(ad x ad y), \quad x, y \in \mathfrak{g}_c$$

restricts to a positive definite bilinear form on $\mathfrak{h}_R$, and by duality determines an inner product $(\ , \ )$ on $\mathfrak{h}_R^*$. The hyperplanes $P_\alpha = \{ \mu \in \mathfrak{h}_R^* | (\mu, \alpha) = 0 \}, \alpha \in \Delta$ divide $\mathfrak{b}_R^*$ in to a finite number of closed convex cones, the so-called Weyl chambers. The reflections about the hyperplanes $P_\alpha$ generates a group of linear transformations $W$, the Weyl group, which leaves $\Delta$ invariant and permutes the Weyl chambers simply and transitively. A system of positive roots is a subset $\Delta_+ \subset \Delta$ such that

a). for every $\alpha \in \Delta$, either $\alpha$ or $-\alpha$, but not both, belongs to $\Delta_+$;

b). if $\alpha, \beta \in \Delta_+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_+$

Equivalently, such a set $\Delta_+$ can be described as the set of all elements of $\Delta$ which are positive with respect to some suitably chosen linear order of $\mathfrak{b}_R^*$. To each system of positive roots $\Delta_+$, there corresponds a distinguished Weyl chamber, the highest Weyl chamber

$$C = \{ \mu \in \mathfrak{b}_R^* | (\alpha, \mu) \geq 0 \quad for \quad every \quad \alpha \in \Delta_+ \}$$

This correspondence between systems of positive roots and Weyl chamber is bijective. Consequently $W$ act simply and transitively also on the collection of systems of positive roots.

For the future use, we will study the symmetric space in the Lie algebra level. Let $\mathfrak{b}$ be a parabolic subalgebra of $\mathfrak{g}_c$, $\mathfrak{g}$ a noncompact real form of $\mathfrak{g}_c$, with a maximal compactly embedded subalgebra $\mathfrak{f}_0$ such that $\mathfrak{v}_0 = \mathfrak{g} \cap \mathfrak{b} \subset \mathfrak{f}_0$. The complexification $\mathfrak{f}$ of $\mathfrak{f}_0$ has a unique $\text{ad} \mathfrak{f}$-invariant complement $\mathfrak{p}$. We set $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{f}_0 + \mathfrak{p}_0$ is a Cartan decomposition, and $\mathfrak{f}_0 + \sqrt{-1} \mathfrak{p}_0$ is a compact real form of $\mathfrak{g}_c$ which contains $\mathfrak{v}_0$. Henceforth, $\mathfrak{m}_0$ will designate this particular real form. Let $\sigma$ and $\tau$ be complex
conjugation of $g_c$ with respect to $g$ and $m_0$ respectively. They commute, and $\theta = \sigma \tau$ is an involutive automorphism of $g_c$ whose $(+1)$ and $(-1)$ eigenvalues are $f$ and $p$. Since $v_0$ has the same rank as $m_0$, we may assume that the Cartan subalgebra $h_0 \subset m_0$, chosen at the beginning of this section lies in $v_0$, and hence in $f_0$. Then $\theta$ commutes with the adjoint action of $h$, and every root space $g^\alpha$ is contained either in $f$ or in $p$. The root $\alpha$ is said to be compact if the former is the case, and noncompact otherwise. We denote the set of compact and noncompact roots by $\Delta_f$ and $\Delta_p$.

Since the root spaces $g^\alpha$ are of one dimensional, every subalgebra $u_0$ of $g$ which contains $h_0$ is spanned over $R$ by $h_0$ and $g\cap (g^\alpha \oplus \sigma(g^\alpha))$, with $\alpha$ ranging over a suitable subset $\Psi$ of $\Delta$. It is known that the exponential map, restricted to $p_0$, is a diffeomorphism. Moreover, $g\cap (g^\alpha \oplus \sigma(g^\alpha)) \subset p_0$ whenever $\alpha$ is noncompact.

For each $\alpha \in \Delta$, one can choose vectors $e_\alpha \in g^\alpha$ and $h_\alpha \in h_R = \sqrt{-1}h_0$ such that

a). $B(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}, [e_\alpha, e_{-\alpha}] = h_\alpha$

b). $B(h_\alpha, x) = \langle \alpha, x \rangle$ for $x \in h$

c). $[e_\alpha, e_\beta] = 0$ if $\alpha \neq -\beta$ and $\alpha + \beta \notin \Delta$.

d). $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$, if $\alpha, \beta, \alpha + \beta \in \Delta$, where $N_{\alpha,\beta}$ are nonzero real constants.

e). $\tau(e_\alpha) = -e_{-\alpha}$.

f). $\sigma(e_\alpha) = \varepsilon_\alpha e_{-\alpha}$, $\varepsilon_\alpha = -1$ if $\alpha$ is compact, $\varepsilon_\alpha = 1$ if $\alpha$ is noncompact.

g). $\varepsilon_{\alpha+\beta} = -\varepsilon_\alpha \varepsilon_\beta$ whenever $\alpha, \beta, \alpha + \beta \in \Delta$

Now we turn to the discussion of some fact of the compact, simply connected, homogeneous complex manifolds.

Let $G_c$ be a connected complex semisimple Lie group, $B$ a parabolic subgroup. The complex analytic quotient space $X = G_c/B$ is then a Kähler C-space, and every Kähler C-space arises in this fashion. The Lie algebras of $G_c$ and $B$ will be referred to as $g_c$ and $b$, both are complex Lie algebras. We choose a maximal compact subgroup $M$ of $G_c$. Its Lie algebra, $m_0$, is a real form of $g_c$, and we denote complex conjugation of $g$ with respect
to $m_0$ by $\tau$, as before. The algebra $b$ has a unique maximal nilpotent ideal $n_-$. Since the subgroup of $G_c$ corresponding to $n_-$ can be realized as a group of upper triangular matrices, with ones along the diagonal, it has no nontrivial compact subgroups. Thus $m_0 \cap n_-$, and hence also $n_- \cap \tau(n_-)$, must be zero. By the Bruhat’s lemma, we can conclude that the parabolic subalgebras $b$ and $\tau(b)$ are opposite each other, i.e., $g$ is spanned by $b$ and $\tau(b)$. Moreover, $v = b \cap \tau(b)$ is a reductive subalgebra such that

$$b = v \oplus n_-$$

and

$$g_c = v \oplus n_- \oplus \tau(n_-)$$

Since the real span of $m_0$ and $b$ is all of $g_c$, the $M$-orbit of $eB \in X = G_c/B$ must be open. On the other hand, this orbit is closed because $M$ is compact. Hence $M$ acts transitively on $X$, with isotropy group $V = M \cap B$, and we can identify the quotient space $M/V$ with $X$. The Lie algebra of $V$ is $v_0 = m_0 \cap b = m_0 \cap b \cap \tau(b) = m_0 \cap b$, $V$ is connected because $X$ is simply connected.

The group $G$ acts on the Kähler space $G_c/B = M/V$. The $G$ orbit is an open subset of $M/V$. We call this manifold the dual manifold of the Kähler C-space.

Now we turn to the variation of the Hodge structure. Let $X$ be a compact Kähler manifold. A $C^\infty$ form on $X$ decomposes into $(p,q)$-components according to the number of $dz$’s and $d\bar{z}$’s. Denoting the $C^\infty$ $n$-forms and the $C^\infty(p,q)$ forms on $X$ by $A^n(X)$ and $A^{p,q}(X)$ respectively, we have the decomposition

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)$$

The cohomology group is defined as

$$H^{p,q}(X) = \{\text{closed}(p,q) - \text{forms}\}/\{\text{exact}(p,q) - \text{forms}\}$$

$$= \{\phi \in A^{p,q}(X)|d\phi = 0\}/dA^{n-1}(X) \cap A^{p,q}(X)$$
We have

**Theorem 2.1 (Hodge Decomposition Theorem).** Let $X$ be a compact Kähler manifold of dimension $n$. Then the $n$-th complex de Rham cohomology group of $X$ can be written as a direct sum

$$H^n_{DR}(X, Z) \otimes C = H^n_{DR}(X, C) = \bigoplus_{p+q=n} H^{p,q}(X)$$

such that $H^{p,q}(X) = H^{q,p}(X)$.

**Remark 2.1.** We can define a filtration of $H^n_{DR}(X, C)$ by

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 = H = H^n_{DR}(X, C)$$

such that

$$H^{p,q}(X) = F^p \cap F^q$$

So the set $\{H^{p,q}(X)\}$ and $\{F^p\}$ are equivalent to define the Hodge decomposition. In the remaining of this paper, we will use both notations interchangeably.

**Definition 2.1.** A Hodge structure of weight $j$, denoted $\{H_Z, H^{p,q}\}$, is given by a lattice $H_Z$ of finite rank together with a decomposition on its complexification $H = H_Z \otimes C$

$$H = \bigoplus_{p+q=j} H^{p,q}$$

such that

$$H^{p,q} = \overline{H_{q,p}}$$

A polarized algebraic manifold is a pair $(X, \omega)$ consisting of an algebraic manifold $X$ together with a Kähler form $\omega$ on $X$. Let

$$L : H^j(X, C) \rightarrow H^{j+2}(X, C)$$

be the multiplication by $\omega$, we recall below two fundamental theorems of Lefschetz:
Theorem 2.2 (Hard Lefschetz Theorem). On a polarized algebraic manifold $(X, \omega)$ of dimension $n$

$$L^k : H^{n-k}(X, C) \rightarrow H^{n+k}(X, C)$$

is an isomorphism for every positive integer $k \leq n$.

Thus

$$L^{n-j} : H^j(X, C) \rightarrow H^{2n-j}(X, C)$$

is an isomorphism. The primitive cohomology $P^j(X, C)$ is defined to be the kernel of $L^{n-j+1}$ on $H^j(X, C)$.

Theorem 2.3 (Lefschetz Decomposition Theorem). On a polarized algebraic manifold $(X, \omega)$, we have for any integer $j$ the following decomposition

$$H^j(X, C) = \bigoplus_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} L^k P^{j-2k}(X, C)$$

It follows that the primitive cohomology groups determine completely the full complex cohomology.

From now on we are only interested in the cohomology group $H^n_{DR}(X, C)$. Define

$$H_Z = P^n(X, C) \cap H^n(X, Z)$$

and

$$H^{p,q} = P^n(X, C) \cap H^{p,q}(X)$$

Now suppose that $Q$ is the quadric form on $H^n_{DR}(X, C)$ induced by the cup product of the cohomology group. In the complex case or real case, $Q$ can be represented by

$$Q(\phi, \psi) = (-1)^{(n-1)/2} \int \phi \wedge \psi$$

$Q$ is a nondegenerated form, and is skewsymmetric if $n$ is odd and is symmetric if $n$ is even. It satisfies the two Hodge-Riemannian relations
1. \( Q(H^{p,q}, H^{p',q'}) = 0 \) unless \( p' = n - p, q' = n - q \)

2. \((\sqrt{-1})^{p-q} Q(\phi, \overline{\phi}) > 0\) for any nonzero element \( \phi \in H^{p,q} \)

Let \( H_Z \) be a fixed lattice, \( n \) an integer, \( Q \) a bilinear form on \( H_Z \), which is symmetric if \( n \) is even and skewsymmetric if \( n \) is odd. And \( \{h^{p,q}\} \) a collection of integers such that \( p + q = n \) and \( \sum h^{p,q} = \text{rank} H_Z \). Let \( H = H_Z \otimes \mathbb{C} \).

**Definition 2.2.** A polarized Hodge structure of weight \( n \), denoted \( \{H_Z, F^p, Q\} \), is given by a filtration of \( H = H_Z \otimes \mathbb{C} \)

\[
0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^0 \subset H
\]

such that

\[
H = F^p \oplus F^{n-p+1}
\]

together with a bilinear form

\[
Q : H_Z \otimes H_Z \to \mathbb{Z}
\]

which is skewsymmetric if \( n \) is odd and symmetric if \( n \) is even such that it satisfies the two Hodge-Riemannian relations:

1. \( Q(F^p, F^{n-p+1}) = 0 \) unless \( p' = n - p, q' = n - q \)

2. \((\sqrt{-1})^{p-q} Q(\phi, \overline{\phi}) > 0\) if \( \phi \in H^{p,q} \) and \( \phi \neq 0 \)

where \( H^{p,q} \) is defined by

\[
H^{p,q} = F^p \cap F^{q'}
\]

**Definition 2.3.** With the notations as above, the classifying space \( D \) for the polarized Hodge structure is the set of all the filtration

\[
0 \subset F^n \subset \cdots \subset F^1 \subset H, \dim F^p = f^p
\]
with \( f^p = h^{n,0} + \cdots + h^{n,n-p} \) on which \( Q \) satisfies the Hodge-Riemannian relations as above.

It is proved by Griffiths that \( D \) actually is a complex manifold. We are going to study this fact a little bit in detail via the Lie group point of view.

Define \( H_R = H_Z \otimes R \) and let

\[
G_R = \text{Aut}(H_R, Q) = \{ g : H_R \to H_R | Q(g\phi, g\psi) = g(\phi, \psi), \phi, \psi \in H_R \}
\]

\[
G_C = \text{Aut}(H_C, Q) = \{ g : H_R \to H_R | Q(g\phi, g\psi) = g(\phi, \psi), \phi, \psi \in H_C \}
\]

Then \( G_R \) acting on \( D \) transitively and thus \( D \) is a homogeneous space.

Let \( V \) be the isotropy group fixing one point of \( G_R \), then \( V \) is a compact group. We will see that \( G_R/V \) is a Kähler manifold. But it is generally not a homogeneous Kähler manifold.

Let \( g = g_R \) be the real semisimple Lie algebra of \( G_R \), then we have the standard Cartan decomposition

\[
g = f_0 + p_0
\]

into the compact part \( f_0 \) and noncompact part \( p_0 \). We assume that the Lie algebra \( v_0 \) of \( V \) is contained in \( f_0 \). Since the Killing form on \( f_0 \) is negative definite, there is a subset \( v_1 \) of \( f \) such that

\[
f_0 = v_0 + v_1
\]

is an orthonormal decomposition of \( f_0 \). Thus we have

\[
g = v_0 \oplus v_1 \oplus p_0
\]

There is a natural representation of \( v_0 \) to \( v_1 \oplus p \) such that the tangent bundle of \( D \) is the associated bundle of the principle bundle \( G \to G/V \) with respect to this representation.
Now we turn to the study of the complex structure on the manifold $G/V$. Suppose that $A^c$ is the complexification of $A$ where $A$ is an object. Then we have

$$g^c = v_0^c \oplus v_1^c \oplus p^c$$

Let $J$ be the invariant complex structure on $v_1^c \oplus p^c$. Here by invariance we mean that $J$ commutes with the adjoint representation of $v_0^c$ on $g^c$. Let

$$v_1^c \oplus p^c = n_- \oplus \tau(n_-)$$

be the splitting as before. Then $n_-$ and $\tau(n_-)$ are the $(\pm \sqrt{-1})$-J eigenspaces respectively. Thus the complex structure $J$ of $D$ is determined by $\tau(n_-)$.
Chapter 3

On the Geometry of the

Horizontal Slices

In this chapter, we studied the canonical map from the moduli space or horizontal slice to the classifying space, we have

**Theorem 3.1.** Suppose \( \mathcal{M} \rightarrow D \) is a horizontal slice and an immersion. Then

\[
\mathcal{M} \rightarrow D_1
\]

defined from the composition of the immersion and the natural projection \( D \rightarrow D_1 \) is again an immersion. Furthermore, it is a pluriharmonic map; i.e., it restricts to a harmonic map from any holomorphic curve of \( \mathcal{M} \). In our notations, it satisfies

\[
\nabla_{p_*X} p_*X + \nabla_{p_*JX} p_*X + p_*J[X, JX] = 0
\]

for \( X \in \mathcal{M} \), where \( J \) is the complex structure on \( \mathcal{M} \) and \( \nabla \) is the Riemannian connection on \( G/K \).

Theorem 3.1 is of its own interest. But the most important application of theorem 3.1 is the following:
**Theorem 3.2.** The metric on the horizontal slice as a Riemannian submanifold is actually Kähler. Moreover, the holomorphic bisectional curvature of such a metric is non-positive, and the Ricci curvature is negative away from zero.

Another result in this chapter is the non-existence of invariant Kähler metric on some classifying spaces.

Suppose $\Gamma$ is a lattice of the group $G$. We call that $\Gamma$ is cocompact, if $\Gamma \backslash G$ is a compact topological space.

**Theorem 3.3.** There are no Kähler metrics on the classifying space which are $\Gamma$ invariant if $\Gamma$ is cocompact.

The idea to prove theorem 3.1 and theorem 3.3 is to consider the Kähler form $\omega$ of the classifying space $D$. In fact, the invariant metric on $D$ is actually Hermitian. It is known that if the projection $D \to D_1$ is holomorphic, then $D$ is Kählerian and $\omega$ is the Kähler form. In our case, $D \to D_1$ is in general not holomorphic. However, there is a relation between $\omega$ and the pull back of the invariant metric on $D_1$. From this we concluded that although $d\omega \neq 0$ as a differential form, we see that at some directions, $d\omega$ is indeed zero. In particular, we use the fact that if $X,Y,Z$ are horizontal, then $d\omega(X,Y,Z) = 0$ in theorem 3.1, and we use the fact that if $X,Y$ are vertical and $z$ is horizontal, then $d\omega(X,Y,Z) = 0$ is also zero in theorem 3.3.

Let’s give some remarks on the first theorem. The similar pluriharmonicity was studied in Bryant [3], Burstall and Salamon [5], Black [1]. Those papers only considered the compact cases and can not apply to the cases we are interested in this paper.

### 3.1 The Canonical Map and the Horizontal Distribution

The purpose of this section is, roughly speaking, to write out the Cartan decomposition $\mathfrak{g} = \mathfrak{f}_0 + \mathfrak{p}_0$ explicitly in our context, and at the same time, realize the base symmetric
space by means of “Hodge decomposition”. Most parts of this section are merely linear algebra, but the computations and identifications will be useful in the following sections and in other applications. cf. [16].

Suppose $D = G/V$ is a classifying space. We fix a point of $D$, say $p$, which can be represented by the subvector spaces of $H$

$$0 \subset F^n \subset F^{n-1} \subset \cdots F^1 \subset H$$

or the set

$$\{H^{p,q} | p + q = n\}$$

described in the previous section. Then we can define the subspaces of $H$:

$$H^+ = H^{n,0} + H^{n-2,2} + \cdots$$

$$H^- = H^{n-1,1} + H^{n-3,3} + \cdots$$

Suppose $K$ is the subgroup of $G$ such that $K$ leaves $H^+$ invariant. Then we have

**Lemma 3.1.** The identity component $K_0$ of $K$ is the maximal compact subgroup of $G$ containing $V$. In particular, $V$ itself is a compact subgroup.

**Proof:** Recall that $V \subset G \subset \text{Hom}(H_R, H_R)$ is a real subgroup. Since $V$ fixes $p$, we have

$$VF^p \subset F^p$$

where $p = 1, \cdots, n$. This implies that

$$VF^{p,q} \subset F^{p,q}$$

for $q = 1, \cdots, n$. Thus we have

$$VH^{p,q} = V(F^p \cap F^{p,q}) \subset VF^p \cap V\overline{F^{p,q}} = H^{p,q}$$
So, $V$ leaves $H^+$ invariant and thus $V \subset K$.

Now we fix some $H^+, H^- \subset H$. Note that if $0 \neq x \in H^+$, then from the second Hodge-Riemannian Relation

$$(\sqrt{-1})^n Q(x, \bar{x}) > 0$$

So for any norm on $H^+$, there is a $c > 0$ such that

$$\frac{1}{c} ||x||^2 \geq (\sqrt{-1})^n Q(x, \bar{x}) \geq c ||x||^2$$

For the same reason, we have

$$\frac{1}{c} ||x||^2 \geq -(\sqrt{-1})^n Q(x, \bar{x}) \geq c ||x||^2$$

for $x \in H^-$. Now if $g \in K_0$. For any $x$, let $x = x^+ + x^-$ be the decomposition of $x$ into $H^+$ and $H^-$ parts, then

$$||gx^\pm||^2 \leq \pm \frac{1}{c} (\sqrt{-1})^n Q(gx^\pm, gx^\mp)$$

$$= \pm \frac{1}{c} (\sqrt{-1})^n Q(x^\pm, x^\mp) \leq \frac{1}{c^2} ||x^\pm||^2$$

So

$$||g|| \leq C$$

So the norm of the element of $K_0$ is uniformly bounded. And thus $K_0$ is a compact subgroup.

Now suppose that $K' \supset K_0$ is a compact connected subgroup. Suppose $\mathfrak{f}'$ is the Lie algebra of $K'$, then if $K_0$ is not maximal, there is a $\xi \in \mathfrak{f}'$ such that $\xi \notin \mathfrak{f}$ for the Lie algebra $\mathfrak{f}$ of $K_0$.

Suppose $\xi = \xi_1 + \xi_2$ is the decomposition for which

$$\xi_1 : H^+ \to H^+, H^- \to H^-$$

$$\xi_2 : H^+ \to H^-, H^- \to H^+$$
Then we have

**Lemma 3.2.** \( \xi_1, \xi_2 \in \mathfrak{g}_R \) for the Lie algebra \( \mathfrak{g}_R \) of \( G \).

**Proof:** First we observe that

\[
Q(H^+, H^+) = Q(H^-, H^-) = 0, \quad n \text{ odd}
\]
\[
Q(H^+, H^-) = Q(H^-, H^+) = 0, \quad n \text{ even}
\]

from the type consideration. Since \( Q \) is invariant under the action of \( G \) by definition, we have

\[
Q(\xi_1 x, y) + Q(x, \xi_1 y) = 0
\]

So

\[
Q(\xi_1 x, y) + Q(x, \xi_1 y) + Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0
\]

If \( n \) is odd then if \( x \in H^+, y \in H^+ \) or \( x \in H^-, y \in H^- \) then

\[
Q(\xi_1 x, y) + Q(x, \xi_1 y) = 0
\]

so in this case

\[
Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0
\]

and if \( x \in H^+, y \in H^- \) or \( x \in H^-, y \in H^+ \) then we have

\[
Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0
\]

automatically. Thus we concluded

\[
Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0
\]

for any \( x, y \in H \). So \( \xi_2 \in \mathfrak{g}_R \) and thus \( \xi_1 \in \mathfrak{g}_R \).

The same goes if \( n \) is even. \( \Box \)

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We define the Weil operator

\[ C : H^{p,q} \rightarrow H^{p,q}, \quad C|_{H^{p,q}} = (\sqrt{-1})^{p-q} \]

Then we have

\[ C|_{H^+} = (\sqrt{-1})^n, \quad C|_{H^-} = -(\sqrt{-1})^n \]

Let

\[ Q_1(x, y) = Q(Cx, \overline{y}) \]

Then we have

**Lemma 3.3.** \( Q_1 \) is an Hermitian inner product.

**Proof:** Let

\[ x = x_1 + x_2 \]

be the decomposition of \( x \) such that \( x_1 \in H^+ \) and \( x_2 \in H^- \).

If \( n \) is odd, then \( \overline{x_2} \in H^+ \). So \( Q(x_1, \overline{x_2}) = 0 \); if \( n \) is even, then \( \overline{x_2} \in H^- \). So \( Q(x_1, \overline{x_2}) = 0 \).

Thus if \( x \neq 0 \) we have

\[
Q_1(x, x) = Q_1(x_1, x_1) + Q_1(x_2, x_2) + Q_1(x_1, x_2) + Q_1(x_2, x_1) \\
= Q_1(x_1, x_1) + Q_1(x_2, x_2) \\
= (\sqrt{-1})^n(Q(x_1, \overline{x_1}) - Q(x_2, \overline{x_2})) > 0
\]

Thus \( Q_1(\cdot, \cdot) \) is a Hermitian product on \( H \). In particular, it is an inner product on \( H_R = H_Z \otimes R \).

Now back to the proof of lemma 3.1, we have

\[
Q_1(\xi_2 x, y) = Q(C\xi_2 x, \overline{y}) = -Q(\xi_2 Cx, \overline{y}) = Q(Cx, \xi_2 \overline{y}) = Q_1(x, \xi_2 y)
\]
Thus $\xi_2$ is a Hermitian metrics under the metric $Q_1$ and since $K'$ is a compact group,
\[
||\exp (t\xi_2)|| \leq C < +\infty
\]
for all $t \in \mathbb{R}$ which implies $\xi_2 = 0$.

**Lemma 3.4.** Let

\[
D_1 = \{H^{n,0} + H^{n-2,2} + \cdots | \{H^{p,q}\} \in D\}
\]

Then the group $G$ acts on $D_1$ transitively with the stable subgroup $K_0$, and $D_1$ is a symmetric space.

**Proof:** For $x, y \in D_1$. Suppose $H^{p,q}_x, H^{p,q}_y$ are the corresponding points in $D$. Then we have a $g \in G$ such that
\[
g\{H^{p,q}_x\} = H^{p,q}_y
\]
So $gx = y$. By definition, $K_0$ fixes $H^+$ of a fixed point $p \in D$. So $D_1$ is a symmetric space.

**Definition 3.1.** We call the map $p$
\[
p : G/V \to G/K_0, \quad \{H^{p,q}\} \mapsto H^{n,0} + H^{n-2,2} + \cdots
\]
the natural projection of the classifying space.

The natural projection needs not to be holomorphic. In the next section, we will study the complex structures on the space $D$ and will explain why.

According to Griffiths, $D$ is a homogeneous complex manifold, i.e. The complex structure is invariant under the left transformation. There is a universal holomorphic bundle over $D$, namely we assign any point $p$ of $D$ the linear space
\[
0 \subset F^n \subset \cdots \subset F^1 \subset H
\]
or in other words, assign every point of $D$ the space $H = H_Z \otimes C$, with the Hodge decomposition

$$H = \sum H^{p,q}$$

It is well known that the holomorphic tangent bundle $T(D)$ can be realized by

$$T(D) \subset \oplus \text{Hom}(F^p, H/F^p) = \oplus_{r>0} \text{Hom}(H^{p,q}, H^{p-r,q+r})$$

such that the following compatible condition holds

$$\begin{array}{ccc}
F^p & \longrightarrow & F^{p-1} \\
\downarrow & & \downarrow \\
H/F^p & \longrightarrow & H/F^{p-1}
\end{array}$$

Thus we can define a subbundle $T_h(D)$ called the horizontal bundle of $D$, by

$$T_h(D) = \{ \xi \in T(D) | \xi F^p \subset F^{p-1} \}$$

The properties of the horizontal bundle or the horizontal distribution plays an important role in the theory of moduli space.

Let $g_R = f_0 + p_0$ be the Cartan decomposition of the Lie algebra $g_R$ into the compact part $f_0$ and noncompact part $p_0$. Suppose $f_0$ is the Lie algebra of $K_0$. Suppose $E$ is the fiber of $T_h(D)$ at the origin. By origin, we mean the point $eV$ on $G/V$ for the unit element of the group $G$.

**Lemma 3.5.** If $N = \dim H_R$ and $N \neq 4$, and $N \geq 2$, then $g_R$ is a simple Lie algebra.

**Proof:** Given a suitable basis of $H$, the quadratic form $Q$ can be represented as a matrix, also denoted as $Q$. Then

$$g_R = \{ g \in \mathfrak{gl}(n, R) | g^t Q g = Q \}$$

Let $N = \dim H_R$. Then the following facts are standard from linear algebra:
If \( n \) is odd, then \( N \) must be even, and \( \mathfrak{g}_R = \mathfrak{sp}(\frac{N}{2}, R) \);

If \( n \) is even, then \( \mathfrak{g}_R = \mathfrak{so}(p, q, R) \) for \( p + q = N \).

These algebra are simple Lie algebras from [13]. □

**Corollary 3.1.** The Killing form of \( \mathfrak{g}_R \) is

\[
B(X, Y) = c \text{tr}(XY)
\]

for some constant \( c \).

□

**Lemma 3.6.** If we identify \( T_0(G) \) with the Lie algebra \( \mathfrak{g}_R \). Then

\[ E \subset \mathfrak{p}_0 \]

Here \( E \) is the fiber of \( T_{\mathbb{R}}(D) \) at the original point.

**Proof:** Suppose

\[
\{0 \subset f^n \subset f^{n-1} \subset \cdots f^1 \subset H\} \quad \text{or} \quad \{h^{p,q}\}
\]

is the set of subspace representing the point \( eV \). Suppose \( X \in E \). Then we know \( X \in Hom(H, H) \) such that

\[
X : f^k \to f^{k-1}
\]

Let \( X = X_1 + X_2 \) be the Cartan decomposition with \( X_1 \in \mathfrak{f}_0 \), and \( X_2 \in \mathfrak{p}_0 \). Let

\[
h^+ = h^{n,0} + h^{n-2,2} + \cdots
\]

\[
h^- = h^{n-1,1} + h^{n-3,3} + \cdots
\]

be the subspaces of \( H \).

By definition \( X_1 \in \mathfrak{f}_0 \). So

\[
X_1 : h^+ \to h^+ \quad h^- \to h^-
\]
We know

\[ B(X_2, f_0) = 0 \]

So by the Corollary 3.1, for any \( X_1 \in f_0 \)

\[ tr(X_1X_2) = 0 \]

So we have

\[ X_2 : h^+ \to h^- \quad h^- \to h^+ \]

by Lemma 3.2. By definition, \( X \) maps \( f^k \) to \( f^{k-1} \). Thus \( X_1 \) maps \( f^k \) to \( f^{k-1} \). So \( X_1 \) leaves \( f^k \) invariant because \( X_1 \) sends \( h^+ \) to \( h^+ \), and \( h^- \) to \( h^- \).

So \( X_1 \in \mathfrak{v} \), the Lie algebra of \( V \). Thus \( X \) acting on the classifying space is the same as \( X_2 \). But \( X_2 \in \mathfrak{p} \). This completes the proof.

On the other hand, \( \forall h \in V, X \in E \), we have \( Ad(h)X \in E \). So there is a representation

\[ \rho : V \to Aut(E), \quad h \mapsto Ad(h) \]

Suppose \( T' \) is the homogeneous bundle

\[ T' = G \times_V E \]

whose local section can be represented as \( C^\infty \) functions

\[ f : G \to E \]

which is \( V \) equivariant

\[ f(ga) = Ad(a^{-1})f(g) \]

for \( a \in V, g \in G \). Our next lemma is

**Lemma 3.7.**

\[ T' = T_h(D) \]

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**Proof:** What we are going to prove is that both vector bundles will be coincided as subbundles of \( T(D) \).

Suppose \( \xi \in T_{gV}' \) for \( g \in G \) where \( T_{gV}' \) is the fiber of \( T' \) at \( gV \). Then \( \xi \) can be represented as

\[
\xi = (g, \xi_1) \quad \text{for} \quad \xi_1 \in E
\]

So the 1-jet in the \( \xi \) direction is \((g + \varepsilon g \xi_1) V\) for \( \varepsilon \) small. Such a point is

\[
(g + \varepsilon g \xi_1) \{f^p\} = (1 + \varepsilon g \xi_1 g^{-1}) \{F^p\}
\]

where \( \{F^p\} = g \{f^p\} \).

Suppose \( \xi_2 = g \xi_1 g^{-1} \), then

\[
\xi_2 \in (T_h)_{gV}(D)
\]

Thus \( \xi_2 \in (T_h)_{gV}(D) \) and

\[
T_{gV}' \subset (T_h)_{gV}(D)
\]

Thus

\[
T' \subset T_h(D)
\]

and \( T' \) is the subbundle of \( T_h(D) \). But they coincides at the origin. So they are equal.

**Corollary 3.2.** Suppose \( T_v(D) \) is the distribution of the tangent vectors of the fibers of the natural projection

\[
p : D \to G/K
\]

then

\[
T_v(D) \cap T_h(D) = \{0\}
\]
Proof:

\[ T_v(D) = G \times_V v_1 \]

where \( f = v + v_1 \) and \( v_1 \) is the orthonormal complement of the Lie algebra \( v \) of \( V \).

**Definition 3.2.** Let \( M \) be a complex manifold. Suppose \( M \subset D \) is a submanifold such that \( T(M) \subset T_h(D)|_M \). Then we say that \( M \) is a horizontal slice. To simplify the notations, we also say the pair \((M, f)\) or \( M \) or even \( F \) is a horizontal slice if

\[ f : M \to D \]

is an immersion and \( f(M) \) is a horizontal slice. In a word, a horizontal slice \( M \) of \( D \) is a complex integral submanifold of the distribution \( T_h(D) \).

**Corollary 3.3.** The map

\[ p : M \subset D \to G/K_0 \]

is an immersion.

### 3.2 The Invariant Complex Structure

Suppose \( D = G/V \) is the coset space of the subgroup \( V \) in \( G \). Let \( g = g_R = f_0 + p_0 = v_0 + v_1 + p_0 \) be the Cartan decomposition of the Lie algebra \( g \) of \( G \) into the compact part \( f_0 \) and noncompact part \( p_0 \), for which \( f_0 \) is the Lie algebra of \( K_0 \), and the decomposition of \( f_0 \) into \( v + v_1 \) where \( v_1 \) is the orthonormal complement of \( v \) in \( f_0 \) with respect to the Killing form of \( G \). It is well known that to give an invariant complex structure on \( G/V \) is equivalent to give a linear transformation \( J \) on \( v_1 + p_0 \) such that

\[ J^2 = -id_{(v_1 + p_0)} \]

\[ \rho(h)J = J\rho(h), \quad \forall h \in V \]
where

\[ \rho : V \to v_1 + p_0 \]

is the standard adjoint representation.

By the structure theory of the complex semisimple Lie algebra that the complexification of \( v_1 + p_0 \) can be written as the sum of the root spaces

\[ (v_1 + p_0)^c = \sum_{\alpha \in I} g_\alpha \]

for some index set \( I \). Suppose \( h \) is the Cartan subalgebra of \( g^c \). Let \( h_0 \subset m_0 \) be the maximal abelian algebra, then

\[ h_0 \subset v_0 \]

thus \( J \) is \( V \)-invariant implies that \( J \) is \( h_0 \) invariant. In particular

\[ [h, JX] = J[h, X]\]

for \( h \in h, X \in (v_1 + p_0)^c \). Now let \( X \in g_\alpha \), then

\[ [h, JX] = J[h, X] = J\alpha(h)X = \alpha(h)JX \]

thus

\[ JX \in g_\alpha \]

Since \( J^2 = -1 \), we know \( JX = \pm \sqrt{-1}X \). In particular, if \( J_1, J_2 \) are the two invariant complex structures, we have

\[ J_1J_2 = J_2J_1 \]

Suppose \( p \) is the projection of \( v_1 + p_0 \) to the second factor.
Definition 3.3. We call \( \omega \) is the fundamental form on \( D \) if \( \omega \) is \( G \) invariant and if on \( \mathfrak{v}_1 + \mathfrak{p}_0 \), the tangent space of \( D \) at \( T_e(D) \), \( \omega \) is defined as

\[
\omega(X, Y) = B(pX, pJY)
\]

where \( B \) is the Cartan-Killing form of the Lie algebra.

From the following proposition, we know \( \omega \) is well defined.

Proposition 3.1. We have

1. \( \omega(X, Y) = -\omega(Y, X) \), for \( X, Y \in \mathfrak{v}_1 + \mathfrak{p} \);  
2. \( \omega \) is \( \mathcal{V} \) invariant as a form on \( T_e(D) \).

Proof: We recall the root decomposition

\[
(\mathfrak{v}_1 + \mathfrak{p}_0)^c = \sum_{\alpha \in I} \mathfrak{g}_\alpha
\]

and the fact that

\[
Jg_\alpha \subset g_\alpha \quad \text{for} \quad \alpha \in I
\]

In particular, if \( X \in \mathfrak{v}_1 \), then \( JX \in \mathfrak{v}_1 \), and if \( X \in \mathfrak{p}_0 \), then \( JX \in \mathfrak{p}_0 \).

Now if \( X \in \mathfrak{v}_1 \), then \( pX = 0 \), \( pJX = 0 \). So

\[
\omega(X, Y) = 0 = -\omega(Y, X)
\]

\( \omega \) can be extended naturally to \( \mathfrak{v}_1^c + \mathfrak{p} \). So in order to verify the first claim, we need only to check the proposition for \( X \in \mathfrak{g}_\alpha \) and \( Y \in \mathfrak{g}_\beta \) for noncompact roots \( \alpha \) and \( \beta \).

Suppose

\[
JX = \sigma_X X, \\
JY = \sigma_Y Y
\]
then

$$\omega(X, Y) = \sigma_Y B(X, Y)$$

$$\omega(Y, X) = \sigma_X B(Y, X)$$

If $\alpha + \beta \neq 0$, then $B(X, Y) = 0$. So we need only assume that $\alpha + \beta = 0$. In this case, suppose that

$$JX = \sigma X$$

Then

$$JX = -\sigma X$$

because $J$ is a real operator and $\sigma$ is pure imaginary. But $\overline{X} \in \mathfrak{g}_{-\alpha}$. So

$$JY = -\sigma Y$$

So we have

$$\sigma_X = -\sigma_Y$$

and then

$$\omega(X, Y) = -\omega(Y, X)$$

Next, $\forall h \in V$, we have

$$\omega(Ad(h)X, Ad(h)Y) = \omega(X, Y)$$

because $Ad(h)$ commutes $p$ and $J$.

From the above theorem we know that we can define a $G$-invariant 2-form $\omega$ on $D$ by the left transformation.
3.3 The Pluriharmonicity

In this section, we are going to prove our main theorem in this chapter. We use the notations as in the previous sections.

Theorem 3.4. Let $g = v_0 + v_1 + p_0$ be the Cartan decomposition in the previous section. Suppose $U$ is an open set of $M$. Then if $X, Y, Z \in \Gamma(U, G \times V p_0)$ or $X, Y \in \Gamma(U, G \times V v_1)$ and $Z \in \Gamma(U, G \times V p_0)$, then

$$d\omega(X, Y, Z) = 0$$

Proof: We have

$$T(G/V) = T_v(G/V) \oplus G \times V p_0$$

where

$$T_v(G/V) = G \times V v_1$$

Suppose $\sigma_0 : G \to G$ is the involution, that is, $\sigma$ is a isomorphism of $G$ such that $\sigma_0^2 = 1$. We assume that $\sigma_0(X) = X$ for $X \in f_0$, $\sigma_0(X) = -X$ for $X \in p_0$. We also write $\sigma_0$ for the induced Lie algebra isomorphism.

$\sigma_0$ induced a $C^\infty$ map

$$G/V \to G/V$$

Let $\sigma_g$ be the map

$$\sigma_g = (L_g)\sigma_0(L_g^{-1})$$

on $G/V$ where $L_g$ is the left translation, then $\sigma_g(gV) = gV$.

Lemma 3.8. $\sigma_g(\omega) = \omega$ where $\omega$ is the differential form defined in the previous section.
**Proof:** By the definition of $\sigma_g$, we only need to prove that $\sigma_0(\omega) = \omega$.

First we observe that $\sigma_0(\omega)$ is also an invariant form. So we only check that $\sigma_0(\omega) = \omega$ at the original point.

We see that

$$\sigma_0 J = J\sigma_0, \quad p\sigma_0 = \sigma_0 p$$

So for any tangent vector $X, Y$,

$$(\sigma_0 \omega)(X, Y) = \omega(\sigma_0(X), \sigma_0(Y)) = B(p\sigma_0 X, pJ\sigma_0 Y)$$

$$= B(\sigma_0 p X, \sigma_0 p J Y) = B(p X, p J Y) = \omega(X, Y)$$

$\square$

Go back to the proof of theorem 3.4, if $X, Y, Z \in \Gamma(U, G \times_V p_0)$ or $X, Y \in \Gamma(U, G \times_V v_1)$ and $Z \in \Gamma(U, G \times_V p_0)$, then

$$d\omega(X, Y, Z) = -d\omega(\sigma_g X, \sigma_g Y, \sigma_g Z)$$

$$= -(\sigma_g d\omega)(X, Y, Z) = -d(\sigma_g \omega)(X, Y, Z)$$

$$= -d\omega(X, Y, Z)$$

So

$$d\omega(X, Y, Z) = 0$$

$\square$

Before proving our main theorem, we need a lemma to link the form $\omega$ with the invariant Riemannian metric on the space $G/K_0$.

We define an invariant Riemannian metric on $G/K_0$, once and for all, by

$$(X, Y)_{gV} = B((L_{g^{-1}})_* X, (L_{g^{-1}})_* Y)$$

where $L_{g^{-1}}$ is the left translation of the group $G$ by $g^{-1}$. 32
Lemma 3.9.
\[ \omega(X, Y) = (p_* X, p_* JY) \]

Proof: At the origin point, the lemma is trivially true by the definition. At a general point \( gV \) of \( D \), note that the left translation \( L_g^{-1} \) commutes with the projection \( p \), we have
\[
\omega_{gV}(X, Y) = \omega_0((L_{g^{-1}})_* X, (L_{g^{-1}})_* Y) = B(p(L_{g^{-1}})_* X, pJ(L_{g^{-1}})_* Y) \\
= B((L_{g^{-1}})_* p_* X, (L_{g^{-1}})_* p_* JY) = (p_* X, p_* JY)
\]

Definition 3.4. An immersion map
\[ p : M \to N \]
from a complex manifold \( M \) to a Riemannian manifold \( N \) is called pluriharmonic, if
\[ \nabla_{p_* X} p_* X + \nabla_{p_* JX} p_* JX + p_* J [X, JX] = 0 \]
for the Riemannian connection \( \nabla \) of \( N \).

Now we begin to prove our main theorem:

Theorem 3.5. Suppose \( \mathcal{M} \) is a horizontal slice of \( D \). That is \( \mathcal{M} \) is an immersed complex integral submanifold of the horizontal distribution, then
\[ p : \mathcal{M} \to D \to G/K_0 \]
is a pluriharmonic immersion for the invariant Riemannian connection \( \nabla \) on \( G/K_0 \).

Proof: We have already proved that the map is an immersion in the previous sections (Corollary 3.3). So it remains to prove the pluriharmonicity of the map. We say a complex submanifold \( R \) of \( D \) is integrable at a point \( q \), if
\[ T_q(R) \subset (G \times_V p_0)_q \]
and if at a neighborhood of \( q \), the map \( p \) is an immersion on this neighborhood. If \( U \) is such an open neighborhood of \( R \). Let \( X, Y, Z \in \Gamma(U, TR) \) and \( X_q, Y_q, Z_q \in (G \times V \mathfrak{p}_0)_q \). Then at \( q \)

\[
d\omega(X, Y, Z) = 0
\]

by theorem 3.4 And \( p_*X, p_*Y, p_*Z \) are well defined and \( C^\infty \). Then by Lemma 3.9

\[
0 = d\omega(X, Y, Z) = X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\
+ \omega([X, Y, Z]) - \omega([X, Z]) + \omega([X, Y])
\]

\[
\]

\[
- (p_*JX, [p_*Y, p_*Z]) + (p_*JY, [p_*X, p_*Z]) - (p_*JZ, [p_*X, p_*Y])
\]

\[
= (\nabla_{p_*X}p_*Y, p_*Z) + (\nabla_{p_*Y}p_*X, p_*Z) - (\nabla_{p_*Y}p_*X, p_*JZ)
\]

\[
- (p_*X, \nabla_{p_*Z}p_*Y) + (\nabla_{p_*Y}p_*X, p_*JY)
\]

\[
- (p_*JX, \nabla_{p_*Z}p_*Y) + (p_*JX, \nabla_{p_*Z}p_*Y) + (p_*JY, \nabla_{p_*Z}p_*Y)
\]

\[
- (p_*JY, \nabla_{p_*Z}p_*X) - (p_*JZ, \nabla_{p_*Z}p_*Y) + (p_*JZ, \nabla_{p_*Z}p_*X)
\]

\[
= (p_*Y, \nabla_{p_*X}p_*JZ) + (p_*JY, \nabla_{p_*X}p_*Z) - (p_*X, \nabla_{p_*Y}p_*JZ)
\]

\[
+ (p_*X, \nabla_{p_*Z}p_*Y) - (p_*JX, \nabla_{p_*Z}p_*Z) + (p_*JX, \nabla_{p_*Z}p_*Y)
\]

If we substitute \( X \) by \( JX \), \( Y \) by \( JY \) and \( Z \) by \( JZ \), we have

\[
0 = -(p_*JY, \nabla_{p_*JX}p_*Z) - (p_*Y, \nabla_{p_*JX}p_*Z) + (p_*JX, \nabla_{p_*JY}p_*Z)
\]

\[
- (p_*JX, \nabla_{p_*JZ}p_*Y) + (p_*X, \nabla_{p_*JY}p_*JZ) - (p_*X, \nabla_{p_*JZ}p_*JY)
\]

Substituting \( X \) by \( JX \), we have

\[
0 = (p_*Y, \nabla_{p_*JX}p_*Z) + (p_*JY, \nabla_{p_*JX}p_*Z) - (p_*JX, \nabla_{p_*JY}p_*JZ)
\]

\[
+ (p_*JX, \nabla_{p_*Z}p_*JY) + (p_*X, \nabla_{p_*Z}p_*Z) - (p_*X, \nabla_{p_*Z}p_*Y)
\]
Comparing the above two equations, we get

\[ 0 = -2(p_*JY, \nabla_{p_*JX}p_*Z) - 2(p_*Y, \nabla_{p_*JX}p_*JZ) + (p_*JX, [p_*JY, p_*Z]) \]
\[ - (p_*JX, [p_*JZ, p_*Y]) + (p_*X, [p_*JY, p_*JZ]) - (p_*X, [p_*Y, p_*Z]) \]

where the sum of the last four terms is equal to

\[ - (p_*X, p_*J[Y, Z]) + (p_*X, p_*J[Z, Y]) \]
\[ + (p_*X, p_*J[Y, JZ]) - (p_*X, p_*[Y, Z]) = 0 \]

because of the integrability condition of \( J \).

So we have

\[ (p_*Y, \nabla_{p_*JX}p_*JZ) + (p_*JY, \nabla_{p_*JX}p_*Z) = 0 \quad (3.3.1) \]

Let \( X = Z \), we have

\[ (p_*Y, \nabla_{p_*JX}p_*JX) + (p_*JY, \nabla_{p_*JX}p_*X) = 0 \]

Substituting \( X \) by \( JX \), we have

\[ (p_*Y, \nabla_{p_*X}p_*X) - (p_*JY, \nabla_{p_*X}p_*JX) = 0 \]

So

\[ (p_*Y, \nabla_{p_*X}p_*X) + (p_*Y, \nabla_{p_*JX}p_*JX) + (p_*JY, p_*[JX, X]) = 0 \]

Then the theorem follows from the fact that for any \( \tilde{X}, \tilde{Y}, \tilde{Z} \in T_{p(q)}(G/K) \), there is an integral submanifold \( R \) at \( q \) and \( X, Y, Z \in T_q(D) \) such that \( p_*X = \tilde{X}, p_*Y = \tilde{Y}, p_*Z = \tilde{Z} \).

**Corollary 3.4.** Use the same notation as above, We also have

\[ (p_*Z, \nabla_{p_*JX}p_*Y) + (p_*JZ, \nabla_{p_*JX}p_*Y) = 0 \]

**Proof:** This is the same as equation 3.3.1.
3.4 The Curvature of the VHS Metric

We use the notation in the previous sections. Now suppose $\mathcal{M}$ is a horizontal slice. i.e. $\mathcal{M}$ is an integral complex submanifold of the horizontal distribution. We also denote $\omega$ the restriction to $\mathcal{M}$ of the fundamental form (also noted $\omega$ in the previous sections).

Then we have

**Lemma 3.10.** On the horizontal slice $\mathcal{M}$

1. $d\omega = 0$;

2. $\omega$ is a 1-1 form;

3. $\omega(X, JX) < 0, X \neq 0$

**Proof:** The first assertion is easy from theorem 3.4. To prove the second assertion, let assume $X_\alpha, X_\beta \in T^{1,0}(G/V)$, where $\alpha, \beta$ are roots, then $\omega(X_\alpha, X_\beta) \neq 0$ if and only if $\beta = -\alpha$. In that case

$$X_\beta = -X_\alpha$$

contradicts to the fact that they are in $T^{1,0}(G/V)$. So $\omega$ is a 1-1 form on $G/V$. Thus a 1-1 form on $\mathcal{M}$. And we have

$$\omega(X, JX) = -(p_*X, p_*X) < 0$$

if $X \neq 0$.

Thus $\omega$ defines a Kähler metric whose underlying Riemannian metric is $( , )$.

**Definition 3.5.** The Kähler metric defined by $\omega$ is called the metric of the Variation of Hodge Structure, or VHS metric for short. By simplifying the notations, we also denote $\omega$ the VHS Kähler form.

Suppose $\Gamma \subset G$ is the monodromy group of $\mathcal{M}$, then we have
Theorem 3.6. \( \omega \) is a \( \Gamma \)-invariant Kähler metric.

Proof: By the above lemma, \( \omega \) is a Kähler metric. Since \( \omega \) is \( G \)-invariant, and since \( \Gamma \subset \text{Aut}(\mathcal{M}) \) is the monodromy group, then on \( \mathcal{M} \), \( \omega \) is \( \Gamma \)-invariant. \( \square \)

Theorem 3.7. With the notations as in the previous theorems, the holomorphic bisectional curvature and the Ricci curvature are nonpositive. Furthermore, the holomorphic sectional curvature and the Ricci curvature are negative away from zero by a constant number.

Proof: By the definition of \( \omega \), we see that the underlying Riemannian metric of \( \mathcal{M} \) is the metric of \( \mathcal{M} \) as the submanifold of \( G/K_0 \), or in other words, \( \mathcal{M} \) is a Riemannian submanifold of \( G/K_0 \).

\( \mathcal{M} \subset G/K_0 \) is an immersed submanifold as we have proved before. Suppose \( R, \tilde{R} \) are the curvature tensor of \( G/K_0 \) and \( \mathcal{M} \), respectively. Then we have the Gauss formula:

\[
\tilde{R}(X,Y,X,Y) + \tilde{R}(X,JY,X,JY) = R(X,Y,X,Y) + R(X,JY,X,JY) + (B(X,X), B(Y,Y) + B(JY,JY)) - |B(X,Y)|^2 - |B(X,JY)|^2
\]

here \( B(\cdot, \cdot) \) refers to the second fundamental form and \( X,Y \in T\mathcal{M} \). Now the pluriharmonicity of the map \( \mathcal{M} \rightarrow G/K_0 \) (theorem 3.5) implies that

\[
B(Y,Y) + B(JY,JY) = 0
\]

Thus the nonpositivity of the bisectional curvature follows from the fact

Lemma 3.11. \( G/K \) has nonpositive sectional curvature. Furthermore, we have

\[
R(X,Y,X,Y) = -||[X,Y]||^2
\]

Proof: Since \( G/K_0 \) is a symmetric space of noncompact type, the curvature formula follows from [13]. In particular, the curvature tensor is nonpositive. \( \square \)
Continuation of the Proof of the Theorem 3.7: Now we consider the holomorphic sectional curvature. By the above computation, we have

\[ \tilde{R}(X, JX, X, JX) \leq R(X, JX, X, JX) \]

That \( R(X, JX, X, JX) \) is negative away from zero is hinted by Griffiths and Schmidt [11]: If \([X, JX] \neq 0\) for all \(X \neq 0\) and \(X \in T_e \mathcal{M}\), the tangent space of \( \mathcal{M} \) at the original point, and if we can prove that

\[ ||[X, JX]|| \geq c||X||^2 \]

for some \( c > 0 \), then we have

\[ R(X, JX, X, JX) \leq -c^2||X||^4 \]

Thus the holomorphic sectional curvature is negative away from zero at the original point. Because the homogeneity of the metric, we know that it is negative everywhere.

Now we let

\[ X = \sum_i a_i g_i + \overline{a}_i g_{-i} \]

where \( Jg_i = \sqrt{-1}g_i \) and \( g_{-j} = \overline{g}_j \). Then

\[ [X, JX] = -2\sqrt{-1} \sum_i |a_i|^2 [g_i, g_{-i}] + \cdots \]

here \( \cdot \) refers to the terms which are not in the maximum torus. Since \( \sum_i |a_i|^2 [g_i, g_{-i}] \) belongs to the cone of positive roots, it will be never zero unless \( a_i \equiv 0 \). We proved that \( ||[X, JX]|| \geq c||X||^2 \).

So the holomorphic sectional curvature is negative away from zero. From the fact that the holomorphic bisectional curvature is nonpositive and the fact that the holomorphic sectional curvature is negative away from zero, we concluded that the Ricci curvature is negative away from zero.

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Corollary 3.5. If $\mathcal{M}$ is complete with the bounded sectional curvature, then $\mathcal{M}$ is quasi-projective.

Proof: it follows from a theorem of Yeung [29].

Remark 3.1. The above corollary is valid in a much weaker assumption. We will write out the proof in a subsequent paper.

3.5 On $\Gamma$-invariant Kähler Metrics

Suppose $D$ is the classifying space defined by the set of filtrations

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset H$$

satisfying the Hodge-Riemannian relations. Suppose $\Gamma$ is a cocompact lattice of $G$, the automorphism group of $D$. We say that $\Gamma$ is cocompact if $\Gamma \backslash G$ is a compact topological space. In this section, we restrict ourselves in the case that $n$ is an odd number, and consider the problem of the existence of Kähler metrics on $D$ which are $\Gamma$-invariant.

The motivation of this problem is from the recent work of Rajan [21]. In his paper, Rajan proved the infinitesimal rigidity of the complex structure for a large family of homogeneous Kähler manifolds. It would be very interesting to prove the same rigidity theorem for the classifying space. However, in general, $D$ is not a homogeneous Kähler manifold. Furthermore, by an interesting application of the same technique we have used in the previous sections, we proved that $D$ even does not admit a Kähler metric which is $\Gamma$-invariant.

To be more precise, we have

Theorem 3.8. Suppose $D$ is a classifying space. It is the set of filtrations

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset H$$
satisfying the two Hodge-Riemannian conditions. Suppose that \( n \) is an odd number and \( n > 1 \) and \( F^p \cap \overline{F^q} = H^{p,q} \neq 0, \forall p, q \). Suppose further that \( \Gamma \) is a cocompact lattice of \( D \). Then there are no Kähler metrics on \( D \) which are \( \Gamma \)-invariant.

Recall that \( \Gamma \) is cocompact if and only if \( \Gamma \setminus D \) is a compact space. The idea to prove the theorem is to consider the projection

\[ p : D \to G/K \]

where \( K \) is the maximum compact subgroup of \( G \) containing \( V \).

Recall that we have proved in lemma 3.5 that if \( n \) is odd, then \( G = \mathfrak{Sp}(N + 1, R) \) for some positive integer \( N \). In particular, \( G/K \) is a Hermitian symmetric space. It is the Siegal Space of the third kind.

We observed that \( p \) is not holomorphic nor anti-holomorphic. The reason is that if \( p \) is anti-holomorphic, then we can reverse the complex structure on \( D \) to make it holomorphic. By a general theorem in Murakami [23], we know that in that case, \( D \) will be a homogeneous Kähler manifold. But we have

**Lemma 3.12.** If \( n > 1 \) and \( n \) odd and \( H^{p,q} \neq 0, \forall p, q \), then \( D \) is not a homogeneous Kähler manifold.

**Proof:** By a homogeneous Kähler manifold we mean a homogeneous space whose invariant Riemannian metric is Kähler. The case \( n = 3 \) and \( \dim F^1 = 1 \) is proved in [17]. In general let’s consider the following map from the classifying space \( D \) to its dual space \( \tilde{D} \): for a point \( \{F^p\} = \{H^{p,q}\} \in D \), define

\[ F^p_1 = H^{n,0} + H^{1,n-1} + H^{n-2,2} + \cdots + H^{r,s} \]

here

\( (r, s) = \begin{cases} 
(p, n-p) & \text{if } p \text{ odd} \\
(n-p, p) & \text{if } p \text{ even}
\end{cases} \)
Obviously, such a map is not holomorphic. On the other hand, the map
\[ \{ F_1^p \} \mapsto H^+ \]
where \( H^+ = H^{n,0} + H^{n-2,2} + \cdots = F_1^{n+1} \) is always holomorphic. By the discussion in the previous sections we know that \( p \) is not holomorphic nor anti-holomorphic and \( D \) is not a homogeneous Kähler manifold.

\[ \square \]

**Lemma 3.13.** If \( D \) admits a \( \Gamma \)-invariant Kähler metric, then there is a \( \Gamma \)-equivariant pluriharmonic map \( f : D \to G/K \) which is surjective at some point \( x_0 \in D \).

**Proof:** Suppose \( \tilde{\omega} \) is the Kähler form on \( D \) which is \( \Gamma \)-invariant. According to Eells-Sampson [7], Jost-Yau [14], and Loubrie [15], there is a \( \Gamma \)-equivariant harmonic map \( f : D \to G/K \) such that \( f \) on \( \Gamma \setminus D \) is homotopic to \( p \). By topology, we know \( p, f \) induce the same map between the cohomology groups:

\[ p^*, f^* : H^{2N}(\Gamma \setminus G/K, C) \to H^{2N_1}(\Gamma \setminus D, C) \]

Here \( 2N \) is the real dimension of \( \Gamma \setminus G/K \). \( 2N_1 \) is the real dimension of \( \Gamma \setminus D \). In particular, if \( \eta \) is the volume form of \( \Gamma \setminus G/K \) then we have

\[ \int_{\Gamma \setminus D} f^* \eta \wedge \tilde{\omega}^{N_1-N} = \int_{\Gamma \setminus D} p^* \eta \wedge \tilde{\omega}^{N_1-N} \]

On the other hand

\[ \int_{\Gamma \setminus D} p^* \eta \wedge \tilde{\omega}^{N_1-N} = \int_{\Gamma \setminus G/K} \eta \int_{\Gamma \setminus D} \tilde{\omega}^{N_1-N} > 0 \]

So

\[ \int_{\Gamma \setminus D} f^* \eta \wedge \tilde{\omega}^{N_1-N} \neq 0 \]

and in particular, \( f^* \neq 0 \) at some point \( x_0 \).
By the rigidity theorem of Siu [22], $f$ is a holomorphic or anti-holomorphic map. Since $D$ is a homogeneous manifold, then there is an invariant Riemannian metric $g$ on $D$. Such a metric turns out to be Hermitian. We have

**Lemma 3.14.** The pluriharmonic map $f$ is a harmonic map with respect to $g$.

**Proof:** We assume that $f$ is holomorphic without losing generality. First note that each fiber of $p : D \to G/K$ is a compact Kähler submanifold of $D$. Thus according to Liouville’s theorem $f$ is a constant along each fiber.

Now suppose that $(x^1, \cdots, x^N)$ is the local coordinate of $G/K$ and $(z^1, \cdots, z^N_1)$ is the corresponding holomorphic local coordinate such that $f = (f^1, \cdots, f^N)$ with respect to these coordinates. $f$ is pluriharmonic, that is

$$\frac{\partial^2 f^\alpha}{\partial z^i \partial \bar{z}^j} + \Gamma^\alpha_{\beta\gamma}(f) \frac{\partial f^\beta}{\partial z^i} \frac{\partial f^\gamma}{\partial \bar{z}^j} = 0, \forall i, j$$

Here $\Gamma^\alpha_{\beta\gamma}$ is the connection coefficients of the invariant Riemannian metric of the symmetric space $G/K$. In particular, we suppose at some point $f(x)$, $\Gamma^\alpha_{\beta\gamma} = 0$. Then the pluriharmonicity at $x$ implies

$$\frac{\partial^2 f^\alpha}{\partial z^i \partial \bar{z}^j}(x) = 0$$

for all $i, j$, and $\alpha$. Now suppose $\Delta$ is the Laplacian of the metric $g$. Then we are going to prove that

$$\Delta f^\alpha(x) = 0$$
Now

\[- \Delta f^\alpha = \delta df^\alpha = \delta \partial f^\alpha + \delta \overline{\partial} f^\alpha \]

\[= - * d * \frac{\partial f^\alpha}{\partial z^i} dz^i - * d * \frac{\partial f^\alpha}{\partial \overline{z}_i} d\overline{z}_i \]

\[= - * d \frac{\partial^2 f^\alpha}{\partial z^i \partial \overline{z}_j} dz^j \wedge * dz^i - \frac{\partial f^\alpha}{\partial \overline{z}_i} * d * dz^i \]

Here we use the fact that \( f \) is a pluriharmonic map.

We have

\[d z^k \wedge * d z^j = 0\]

for all \( k, j \) by the type considerations.

**Lemma 3.15.**

\[\sum_j \frac{\partial f^\alpha}{\partial z^j} * d * d z^j = \sum_j \frac{\partial f^\alpha}{\partial \overline{z}_j} \Delta z^j = 0\]

**Proof:** Suppose that \( z^j = x^j + \sqrt{-1} x^{j+n} \). Here \( (y^1, \ldots, y^{2N_1}) \) is the real local coordinate. Suppose

\[g = g_{i,j} dz^i \otimes d\overline{z}_j\]

is the Hermitian metric defined by \( \omega \). The corresponding Riemannian metric is then

\[G = G_{a,b} dy^a dy^b \]

\[= 2(Re g_{i,j} dy^i dy^j + Re g_{i,j} dy^{i+n} dy^{j+n} \]

\[- Im g_{i,j} dy^{i+n} dy^j + Im g_{i,j} dy^i dy^{j+n})\]
Here we assume the sum over \( i, j, k, \ldots \) are from 1 to \( N_1 \) and the sum over \( a, b, \ldots \) are from 1 to \( 2N_1 \). Thus

\[
\Delta z^k = \frac{1}{\sqrt{\det G}} \frac{\partial}{\partial x^a}(\sqrt{\det G} g^{ab} \frac{\partial}{\partial x^b} z^k)
\]

\[
= \frac{1}{\det g} \frac{\partial}{\partial x^a}(\det g (G^{ak} + \sqrt{-1} G^{a(k+n)}))
\]

Now

\[
(G^{ak} + \sqrt{-1} G^{a(k+n)}) g_{kj}
\]

\[
= \frac{1}{2} (G^{ak} + \sqrt{-1} G^{a(k+n)}) (G_{kj} + \sqrt{-1} G_{k,j+n}) = \delta_{aj} + \sqrt{-1} \delta_{a(j+n)}
\]

Thus

\[
G^{ak} + \sqrt{-1} G^{a(k+n)} = \begin{cases} 
  g^{k\pi} & a \leq N_1 \\
  -\sqrt{-1} g^{a,a-N_1} & a > n
\end{cases}
\]

So finally we have

\[
\Delta z^k = \frac{1}{\det g} \frac{\partial}{\partial x^l} (\det g g^{kl}) + \frac{1}{\det g} \frac{\partial}{\partial x^{l+N_1}} (\det g \sqrt{-1} g^{kl})
\]

\[
= 2 \frac{1}{\det g} \frac{\partial}{\partial x^l} (\det g g^{kl}) = 2 g^{kl} g^{\overline{j} \overline{l}} (d\tilde{\omega})_{ijl}
\]

So we have

\[
\sum_j \frac{\partial f^\alpha}{\partial z^j} \ast d \ast dz^j = C \frac{\partial f^\alpha}{\partial z^j} g^{j\overline{k}} g^{r \overline{s}} (d\omega)_{r\overline{s}k}
\]

where \( C \) is a constant. Thus we have to prove

\[
C \frac{\partial f^\alpha}{\partial z^j} g^{j\overline{k}} g^{r \overline{s}} (d\omega)_{r\overline{s}k} = 0
\]

Claim:

\[
C \frac{\partial f^\alpha}{\partial z^j} g^{j\overline{k}} g^{r \overline{s}} (d\omega)_{r\overline{s}k} = C < df^\alpha \wedge \omega, d\omega >
\]
Let $x \in D$. Then we have the symmetry $\sigma_x$ defined in the previous section. Now since $f$ is constant along each fiber, there is an $\tilde{f}$ on $G/K$ such that

$$ f = \tilde{f} \circ p $$

We have $\sigma_x p = p\sigma_x$. So we have

$$ \sigma_x df^\alpha = \sigma_x \tilde{f}^\alpha p = p\sigma_x \tilde{f}^\alpha = -p d\tilde{f}^\alpha = -df^\alpha $$

On the other hand, we have

$$ \sigma_x \omega = \omega $$

Thus

$$ < df^\alpha \wedge \omega, d\omega > = \sigma_x < df^\alpha \wedge \omega, d\omega > = - < df^\alpha \wedge \omega, d\omega > $$

The lemma is proved.

**Continuation of the Proof:** Now we know $f$ is a harmonic map with respect to the metric $g$. However, we know that there is a complex structure on $\Gamma \setminus D$ such that $g$ is holomorphic. In particular, $p$ is harmonic with respect to $g$. Since $p$ is a surjective map, by the uniqueness theorem of the Harmonic map, we know $p = f$. But this is a contradiction, because $f$ is a holomorphic map but $p$ is not. So we have proved that there are no Kähler metrics on $D$ which are $\Gamma$-invariant.
Let $\mathcal{M}$ be the moduli space of a polarized Calabi-Yau manifold $(X, \omega)$. There is a natural map $\mathcal{M} \to D$ which is an immersion of $\mathcal{M}$ to the classifying space such that $\mathcal{M}$ is a horizontal slice (see definition 3.2). The holomorphic tangent space of $\mathcal{M}$ at $X$ is $H^1(X, \Theta)\omega$ defined by

$$H^1(X, \Theta)\omega = \{ \varphi \in H^1(X, \Theta) | \varphi \omega = 0 \}$$

If dim $X = 3$ and if $X$ is simply connected, then

$$H^1(X, \Theta)\omega = H^1(X, \Theta)$$

So in the case of Calabi-Yau threefold, the dimension of Kodaira-Spencer space and the dimension of the polarized moduli space are the same.

In this chapter, we focus our attention on Calabi-Yau threefolds and their classifying spaces. First we write out explicitly the invariant complex structure on the classifying space and the projection $D \to D_1$. Using this, we reproved the pluriharmonicity and the Ricci-negativeness in the last chapter. This is done in the second section. In the third section, we generalized the definition of the Weil-Petersson metric to the case of horizontal slices and proved a formula between the Weil-Petersson metric and the
metric of Variation of Hodge Structure (VHS) (definition 3.5). Professor Albrecht Klemm was kindly pointed out that the physicist de Wit also obtained a similar formula. The last several sections are related to the Weil-Petersson metric and the VHS metric. We proved that the completeness of the Weil-Petersson metric is equivalent to the boundness of the cubic forms by a gradient estimate; we also write out explicitly the curvature of the VHS metric and got the optimal estimate of the upper bound of the Ricci and holomorphic sectional curvature, and proved that the boundness of the (Riemannian) sectional curvature is equivalent to the boundness of the Ricci curvature. From this we see that if the Ricci curvature is bounded, then a complete horizontal slice, is quasi-projective. Finally, in the last section, we got a asymptotic estimate of the Weil-Petersson metric near the degeneration of a Calabi-Yau threefold. C-L. Wang independently got a slight sharper estimate using Nilpotent Orbit Theorem.

4.1 The Classifying Space of the Calabi-Yau Three-folds

In this section, we will write out explicitly the complex structure defined on \( D \), the classifying space. We will prove that, such a complex structure is not canonical in the sense that there is no homogeneous Kähler metric on such a complex structure. We will also write out explicitly the projection of the classifying space to the base symmetric space via local coordinates. These formula we obtained is fundamental to the study of moduli space of Calabi-Yau threefold, as we will see in the subsequent sections.

We assume that the Calabi-Yau threefolds we studied in this section are all simply connected, unless otherwise stated.

Suppose that \( Q \) is the quadratic form on \( H = H^3(X, \mathbb{C}) \) such that

\[
Q(\omega, \eta) = -\int_X \omega \wedge \eta
\]

where \( \omega, \eta \in H^3(X, \mathbb{C}) \). \( Q \) is skew-symmetric and nondegenerated.
Let \( n = \dim H^1(X, \Theta) \). The dual classifying space \( \hat{D} \) is the set of subspaces of \( H \)
\[
0 \subset F^3 \subset F^2 \subset F^1 \subset H = \mathbb{C}^{2n+2}
\]
such that
\[
Q(F^2, F^2) = 0, Q(F^1, F^3) = 0
\]
with \( \dim F^3 = 1, \dim F^2 = n + 1, \dim F^1 = 2n + 1 \).

\((F^3, F^2, F^1, Q)\) determines a set of subspaces \( \{H^{p,q}, p + q = 3\} \). Here \( H^{3,0} = F^3 \), \( H^{2,1} \) is the orthonormal complement of \( H^{3,0} \), and so on.

The classifying space \( D \) is an open subset of \( \hat{D} \) such that
\[
\begin{cases}
\iota Q(x, \bar{x}) < 0 & 0 \neq x \in H^{3,0} \\
\iota Q(x, \bar{x}) > 0 & 0 \neq x \in H^{2,1}
\end{cases}
\]
here and in what follows, \( \iota \) is \( \sqrt{-1} \), unless otherwise stated.

**Proposition 4.1.** There is a basis \( e_1, \ldots, e_{2n+2} \) of \( H \) under which \( Q \) can be represented as
\[
Q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
And if we let
\[
f^3 = \text{span}\{e_1 - \sqrt{-1}e_{n+2}\}
\]
\[
f^2 = \text{span}\{e_1 - \sqrt{-1}e_{n+2}, e_2 + \sqrt{-1}e_{n+3}, \ldots, e_{n+1} + \sqrt{-1}e_{2n+2}\}
\]
and \( f^1 \) is the hyperplane perpendicular to \( f^3 \) with respect to \( Q \), then
\[
\{0 \subset f^3 \subset f^2 \subset f^1\} \in D
\]
By the proposition, we know $G = \mathfrak{Sp}(n+1,\mathbb{R})$ and $G_C = \mathfrak{Sp}(n+1,\mathbb{C})$, the real and complex symplectic group, respectively (cf. lemma 3.5).

$G_C$ acts on $\tilde{D}$ transitively, with the stabilization subgroup $B$. Thus $G_c/B = \tilde{D}$.

The $G$-orbit of $\{f^1, f^2, f^3\}$ is an open set $D$ of $\tilde{D}$, with the stabilization subgroup $V = G \cap B$ which is compact, and $D = G/V$.

It is known that $B$ is a parabolic subgroup. So $G_C/B$ is a Kähler $C$-Space. Since $G$ acting on $G/G \cap B$ holomorphically, it is a homogeneous complex manifold.

**Proposition 4.2.** Suppose $K$ is the maximal connected compact subgroup of $G$ containing $V$. Then there is an invariant complex structure $J_1$ on $G/V$ such that $G/V$ becomes a homogeneous Kähler manifold and the projection $p : G/V \rightarrow G/K$ is holomorphic.

**Proof:** See [20].

Unfortunately, the complex structure on $D$ is not the complex structure mentioned above. Of course $D$ is a Kähler manifold, because it is an open subset of the Kähler space $\tilde{D}$. But it needs not to be a homogeneous Kähler manifold. Later in this chapter, we are going to see that $D$ is not a homogeneous Kähler manifold. We denote $D'$ the homogeneous Kähler manifold in the above proposition. The underlying differentiable structures of $D$ and $D'$ are the same.

Now we write out the invariant complex structure explicitly. In order to do this, we need only to write out the action $J$ on the space $T_e(D)$. By the invariance of $J$, we know the action of $J$ on any tangent space.

$V$ is a compact subgroup. Let $\mathfrak{v}$ be its Lie algebra. Let $\mathfrak{b}$ be the Lie algebra of $B$. Write

$$\mathfrak{b} = \mathfrak{v}^c \oplus \mathfrak{n}_-$$

where $\mathfrak{n}_-$ is a linear subspace and

$$\mathfrak{g}_c = \mathfrak{v}^c \oplus \mathfrak{n}_- \oplus \tau(\mathfrak{n}_-)$$
where $\tau$ is the conjugate of the $\mathfrak{g}_c$ with respect to its compact real form. We will write out these subspaces concretely.

We use a $(2n + 2) \times (2n + 2)$ matrix to represent a point in $\tilde{D}$. Namely, the vector space $F^3$ is spanned by the first column, $F^2$ is spanned by the first $(n + 1)$ columns and $F^1$ is spanned by the first $(n + 1)$ columns and the last $n$ columns. If the two matrices $A, B$ represent the same point in $\tilde{D}$, we write

$$A \sim B$$

Since $F^1$ is always determined by $F^3$, some time we use the first $(n + 1)$ columns of the above matrices to represent the points in $\tilde{D}$. The notation $\sim$ should carry the same meaning.

Suppose $X \in \mathfrak{g}_c \in \mathfrak{sp}(n + 1, \mathbb{C})$, then

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad B^t = B, \quad C^t = C$$

for $A, B, C \in \mathfrak{gl}(n + 1, \mathbb{C})$. Let

$$\begin{pmatrix} 1 & 1 \\ iK & -iK \end{pmatrix}$$

represent the original point $\{f^3 \subset f^2 \subset f^1\}$ where

$$K = \begin{pmatrix} -1 & 1 \\ & 1 \\ & & \ddots \\ & & & 1 \end{pmatrix}$$

Sometimes we also denote $e, eV, eB$ as the original point, depending on the context.

If $X \in \mathfrak{b}$, and if $\varepsilon > 0$ is small then

$$(1 + \varepsilon X) \begin{pmatrix} 1 & 1 \\ iK & -iK \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ iK & -iK \end{pmatrix}$$
We suppose that

\[
X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} = \begin{pmatrix} a & \alpha_1 & b & \beta^t \\ \alpha_2 & A_1 & \beta & B_1 \\ c & \gamma^t & -a & -\alpha_2^t \\ \gamma & C_1 & -\alpha_1 & -A_1^t \end{pmatrix} = \begin{pmatrix} B_1^t = B_1 \\ C_1^t = C_1 \end{pmatrix}
\]

for \( A_1, B_1, C_1 \in \mathfrak{gl}(n, \mathbb{C}), \alpha_1, \alpha_2, \beta, \gamma \in \mathbb{C}^n, a, b, c \in \mathbb{C} \). Then

\[
X \begin{pmatrix} 1 & 1 \\ iK & -iK \end{pmatrix} = \begin{pmatrix} a - ib & \alpha_1^t + i\beta^t & a + ib & \alpha_1^t - i\beta^t \\ \alpha_2 - i\beta & A_1 + iB_1 & \alpha_2 + i\beta & A_1 - iB_1 \\ c + ia & \gamma^t - i\alpha_2^t & c - ia & \gamma^t + i\alpha_2^t \\ \gamma + i\alpha_1 & C_1 - iA_1^t & \gamma - i\alpha_1 & C_1 + iA_1^t \end{pmatrix}
\]

Since \( F^1 \) is perpendicular to \( F^3 \), \( F^3, F^2 \) carry all the information of a point. So in what follows we need only to compute the first \((n + 1)\) columns of the matrix that represents a point. Now

\[
(1 + \varepsilon X) \begin{pmatrix} 1 \\ iK \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon(a - ib) & \varepsilon(\alpha_1^t + i\beta^t) \\ \varepsilon(\alpha_2 - i\beta) & 1 + \varepsilon(A_1 + iB_1) \\ -i + \varepsilon(c + ia) & \varepsilon(\gamma^t - i\alpha_2^t) \\ \varepsilon(\gamma + i\alpha_1) & i + \varepsilon(C_1 - iA_1^t) \end{pmatrix} + O(\varepsilon^2)
\]

\[
\sim \begin{pmatrix} 1 \\ \varepsilon(\alpha_2 - i\beta) & 1 + \varepsilon(A_1 + iB_1) \\ -i + \varepsilon(2ia + b + c) & \varepsilon(\gamma^t - i\alpha_2^t) \\ \varepsilon(\gamma + i\alpha_1) & i + \varepsilon(C_1 - iA_1^t) \end{pmatrix} + O(\varepsilon^2)
\]

\[
\sim \begin{pmatrix} 1 \\ \varepsilon(\alpha_2 - i\beta) \\ -i + \varepsilon(2ia + b + c) \\ \varepsilon(\gamma + i\alpha_1) \end{pmatrix} + O(\varepsilon^2)
\]

(4.4.1)
Thus if $X \in b$, then

\[
\begin{align*}
\alpha_2 - i\beta &= 0 \\
\gamma + i\alpha_1 &= 0 \\
2ia + b + c &= 0 \\
C_1 + B_1 - i(A_1 + A_1^t) &= 0
\end{align*}
\]

On the other hand, $v = g \cap b$. So if $X$ is a real matrix, then

\[
\alpha_1 = \alpha_2 = \beta = \gamma = 0, C_1 = -B_1, A_1 + A_1^t = 0, a = 0, b = -c
\]

In another word

\[
X \in v \Rightarrow X = \begin{pmatrix}
0 & 0 & b & 0 \\
0 & A_1 & 0 & B_1 \\
-b & 0 & 0 & 0 \\
0 & -B_1 & 0 & A_1
\end{pmatrix} \quad B_1^t = B_1, \quad A_1^t = -A_1
\]

We have the vector space decomposition

\[
b = v^c + n_-
\]

thus

\[
n_- = \left\{ \begin{pmatrix}
a & i\gamma^t & -ia & \beta^t \\
i\beta & A_1 & \beta & iA_1 \\
-ia & \gamma^t & -a & -i\beta^t \\
\gamma & iA_1 & -i\gamma & -A_1
\end{pmatrix} \right\}, \quad A_1^t = A_1
\]

and

\[
\tau(n_-) = \left\{ \begin{pmatrix}
a & -i\beta^t & ia & \gamma^t \\
-\gamma & A_1 & \gamma & -iA_1 \\
-ia & \beta^t & -a & i\gamma^t \\
\beta & -iA_1 & i\beta & -A_1
\end{pmatrix} \right\} \quad A_1^t = A_1
\]
\( \tau(n_-) \) is the holomorphic tangent space at the origin of \( \tilde{D} \). So \( J \) acting on it is equal to \( \sqrt{-1} \). Now suppose \( Y \in \tau(n_-) \), and

\[
Y = \begin{pmatrix}
a & -i\beta^t & ia & \gamma^t \\
-i\gamma & A_1 & \gamma & -iA_1 \\
i\alpha & \beta^t & -a & i\gamma^t \\
\beta & -iA_1 & i\beta & -A_1 \\
\end{pmatrix}
\]

(4.4.2)

then by equation (4.4.1)

\[
(1 + \varepsilon Y) \begin{pmatrix} 1 \\ iK \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon(-2i\gamma) & 1 \\ -i + \varepsilon(4\alpha a) & \varepsilon(2(\beta - \gamma)^t) \\ \varepsilon 2\beta & i + \varepsilon(-4iA_1) \end{pmatrix}
\]

(4.4.3)

Suppose that

\[
g = v + v_1 + p_0
\]

We identify \( v_1 + p_0 \) to the tangent space of \( D \) at the origin. Let \( J \) be the invariant complex structure of \( D \). Then in particular \( J \) is a real linear transformation of \( v_1 + p \).

Our next theorem gives a concrete representation of such linear transformation.

**Theorem 4.1.** Suppose that \( X \in p \)

\[
X = \begin{pmatrix}
a & \alpha^t & b & \beta^t \\
\alpha & A_1 & \beta & B_1 \\
b & \beta^t & -a & -\alpha^t \\
\beta & B_1 & -\alpha & -A_1 \\
\end{pmatrix}
\]

Then

\[
A_1^t = A_1 \quad B_1^t = B_1
\]
\[ JX = \begin{pmatrix} b & \beta t & -a & -\alpha t \\ \beta & -B_1 & -\alpha & A_1 \\ -a & -\alpha t & -b & -\beta t \\ -\alpha & A_1 & -\beta & B_1 \end{pmatrix} \]

if \( X \in v_1 \)

\[ X = \begin{pmatrix} \alpha t & \beta t \\ -\alpha & \beta \\ -\beta t & \alpha t \\ -\beta & -\alpha \end{pmatrix} \]

Then

\[ JX = \begin{pmatrix} -\beta t & \alpha t \\ \beta & \alpha \\ -\alpha t & -\beta t \\ -\alpha & \beta \end{pmatrix} \]

Proof: Recall that the inclusion

\[ G/V \to G_{c}/B \]

is holomorphic. Now at the origin, \( v_1 + p_0 \) is the tangent space of \( G/V \) whose complex structure we are going seek. and \( \text{Re}\tau(n_-) \) is the tangent space of \( G_{c}/B \). Its complex structure is known.  

We see from

\[ G/V \to G_{c}/B \]

we have

\[ v_1 + p_0 \to \text{Re}\tau(n_-) \to \tau(n_-) \]

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where
\[ \nu_1 + p_0 \rightarrow \tau(n_-) \]
can be identified as follows:

If \( X \in p_0 \), then by equation (4.4.1)
\[
(1 + \varepsilon X) \begin{pmatrix} 1 \\ iK \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon(\alpha - i\beta) & 1 \\ -i + \varepsilon(2ia + 2b) & 0 \\ \varepsilon(\beta + i\alpha) & i + \varepsilon(2B_1 - 2A_1i) \end{pmatrix}
\] (4.4.4)

If \( X \) corresponds to \( Y \) where \( Y \) is defined by the equation (4.4.2), then comparing equation (4.4.2) to equation (4.4.3), we have
\[
a \triangleright \frac{1}{2}(a - ib) \quad \text{entries of } Y
\]
\[
\gamma \triangleright \frac{1}{2}(ia + \beta) \quad \text{entries of } X
\]
\[
A_1 \triangleright \frac{1}{2}(A_1 + iB_1) \quad \text{entries of } X
\]
where \( \triangleright \) means that the correspondence of entries of \( X \) to those of \( Y \) via the identification. Thus
\[
aA \triangleright \frac{1}{2}(b + ia) \quad \text{entries of } Y
\]
\[
\gamma \triangleright \frac{1}{2}(i\beta - \alpha) \quad \text{entries of } X
\]
\[
\iota A_1 \triangleright \frac{1}{2}(B_1 + iA_1) \quad \text{entries of } X
\]
Since \( JY = \sqrt{-1}Y \), we see for the entries of \( X \), \( J \) maps
\[
\alpha \rightarrow \beta \quad a \rightarrow b \quad A_1 \rightarrow -B_1
\]
\[
\beta \rightarrow -\alpha \quad b \rightarrow -a \quad B_1 \rightarrow A_1
\]
Thus
\[
JX = \begin{pmatrix} b & \beta^t & -a & -\alpha^t \\ \beta & -B_1 & -\alpha & A_1 \\ -a & -\alpha^t & -b & -\beta^t \\ -\alpha & A_1 & -\beta & B_1 \end{pmatrix}
\]
If $X \in v_1$

$$(1 + \varepsilon X) \begin{pmatrix} 1 \\ iK \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\varepsilon(\alpha + i\beta) & 1 \\ -i & 2\varepsilon i(\alpha + i\beta) \\ i \\ \varepsilon i(\alpha + i\beta) & \ddots \end{pmatrix}$$

Using exactly the same method, we have

$$JX = \begin{pmatrix} -\beta^t & \alpha^t \\ \beta & \alpha \\ -\alpha^t & -\beta^t \\ -\alpha & \beta \end{pmatrix}$$

\[ \square \]

**Corollary 4.1.** The complex structure $J$ on $v_1 + p_0$ has the following properties:

1. If $X \in p_0, JX \in p_0$; $X \in v_1, JX \in v_1$;

2. $J$ is $V$-invariant;

3. $J$ is not $K_0$ invariant.

**Proof:** A straightforward computation.

Using the same method, we can compute the complex structure $J_1$ on $v_1 + p_0$ corresponding to the complex manifold $D'$ defined in the beginning of this section.

**Theorem 4.2.** If $X \in p_0$, where

$$X = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$
then

\[ J_1X = \begin{pmatrix} B & -A \\ -A & -B \end{pmatrix} = X \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

If \( X \in v_1 \),

\[ X = \begin{pmatrix} \alpha t & \beta \\ -\alpha & \beta \\ -\beta & \alpha \end{pmatrix} \]

then

\[ J_1X = \begin{pmatrix} -\beta & \alpha t \\ \beta & \alpha \\ -\alpha t & -\beta \end{pmatrix} \]

**Corollary 4.2.** \( D \) and \( D' \) are not holomorphically equivalent.

**Proof:** By the straightforward computation we see that \( J_1 \) is \( K_0 \) invariant on \( p \) but \( J \) is not. \( \Box \)

Now we are going to write out the projection explicitly

\[ \frac{G}{V} \to \frac{G}{K} \]

By definition (3.1), the projection send

\[ 0 \subset F^3 \subset F^2 \subset F^1 \subset H \]

to

\[ F^3 \oplus H^{1,2} \]
Suppose now near the original point, $F^3$ can be represented by

$$(1, z^t, a, \alpha^t)$$

where $z, \alpha \in \mathbb{C}^n, a \in \mathbb{C}$. And suppose $F^2$ can be represented by

$$\begin{pmatrix} 1 & z^t & a & \alpha^t \\ 0 & 1 & \beta & A \end{pmatrix}$$

for $\beta \in \mathbb{C}^n, A \in \mathfrak{gl}(n, \mathbb{C})$. Then by the equation

$$Q(F^2, F^2) = 0$$

we know that

$$\beta = \alpha - Az, \quad A^t = A$$

So locally, we can represented $F^2$ by the matrix

$$\begin{pmatrix} 1 & z^t & a & \alpha^t \\ 0 & 1 & \alpha - Az & A \end{pmatrix} \quad A^t = A \quad (4.4.5)$$

Let $\Omega$ be a local section of $F^3$, $\Omega = (1, z^t, a, \alpha^t)$, and let $(\Omega, \Theta)$ be a local section of $F^2$ with $\Theta = (0, 1, \alpha - Az, A)$. Let

$$m = Q(\Omega, \Theta) = -a + \overline{a} - \alpha^t \overline{z} + \overline{\alpha^t} z$$

$$\xi = Q(\Omega, \Theta) = -\alpha + \overline{\alpha} - \overline{A}(\overline{z} - z)$$

where $m \in \mathbb{C}, \xi \in \mathbb{C}^n$. It is checked that

$$\Omega = (1, z^t, a, \alpha^t)$$

$$\Theta = (0, 1, \alpha - Az, A)$$

We know that

$$Q(\Omega, \Theta - \frac{\xi}{m}) = 0$$
So $\Theta - \frac{\xi}{m} \Pi \in F^1$ and the row vectors of it spanned $H^{1,2}$.

The canonical map can be locally represented as

\[
\begin{pmatrix}
1 & z^t & a & \alpha^t \\
0 & 1 & \alpha - Az & A
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & z^t & a & \alpha^t \\
-\frac{\xi}{m} & 1 - \frac{\xi z^t}{m} & \frac{m}{\alpha - Az} - \frac{\pi\xi}{m} & \frac{\xi}{m} \alpha^t
\end{pmatrix}
\]

Now let

\[
\mu = \frac{1}{m - (z^t - z^t)\xi}
\]

Then as a matrix

\[
1 + \mu \xi (z^t - z^t) = (1 - \frac{\xi}{m} (z^t - z^t))^{-1}
\]

Under this notation, we have

\[
\begin{pmatrix}
1 & z^t \\
-\frac{\xi}{m} & 1 - \frac{\xi z^t}{m}
\end{pmatrix}^{-1} = \begin{pmatrix}
1 - z^t B \frac{\xi}{m} & -z^t B \\
B \frac{\xi}{m} & B
\end{pmatrix}
\]

where $B = 1 + \mu \xi (z^t - z^t)$

\[
\begin{pmatrix}
1 - z^t B \frac{\xi}{m} & -z^t B \\
B \frac{\xi}{m} & B
\end{pmatrix} \begin{pmatrix}
a & \alpha^t \\
\frac{m}{\alpha - Az} - \frac{\xi}{m} & \frac{\xi}{m} \alpha^t
\end{pmatrix} = \begin{pmatrix}
D_1 & D_2^t \\
D_3 & D_4
\end{pmatrix}
\]

for $D_1 \in \mathbb{C}, D_2, D_3 \in \mathbb{C}^n, D_4 \in \mathfrak{gl}(n, \mathbb{C})$. Then it can be computed

\[
\begin{align*}
D_1 &= a - z^t (\alpha - A\overline{z}) + \mu (z^t \xi)^2 \\
D_2 &= D_3 = (a - \overline{a}) \mu \xi + B(\overline{\alpha - Az}) \\
D_4 &= \overline{A} + \mu \xi \overline{\alpha^t}
\end{align*}
\]

(4.4.6)

So we have the following proposition
Proposition 4.3. Under the notation as above, the map

\[ p : G/V \to G/K \]

under the local coordinate described as above is

\[
\begin{pmatrix}
1 & z^t & a & a^t \\
1 & \alpha - Az & A
\end{pmatrix}
\to
\begin{pmatrix}
D_1 & D_2 \\
D_3 & D_4
\end{pmatrix}
\]

where the \( D_i \)'s are defined as in equation (4.4.6).

4.2 Pluriharmonicity and Ricci Negativeness

In this section, we reprove the results in Chapter 3 in the case of Calabi-Yau threefold.

Recall that an immersion \( f \) from a complex manifold \( M \) to a Riemannian manifold \( N \) is called pluriharmonic if for any \( X \in TM \), we have

\[
\nabla_{f^*X} f^*X + \nabla_{f^*JX} f^*JX + f^*J[X,JX] = 0
\]

where \( J \) is the complex structure.

Lemma 4.1. Let \((z^1, \cdots, z^n)\) be the local holomorphic coordinate of \( M \) at some point \( p \). Suppose that

\[ f : M \to N \]

is an immersion to a Riemannian manifold \( N \). And suppose that

\[ \nabla_{f^*\frac{\partial}{\partial z^i}} f^* \frac{\partial}{\partial \overline{z}^j} = 0 \quad (4.4.7) \]

for any \( i, j \). Then \( f \) is a pluriharmonic map.

Proof: Let

\[ X = a^i \frac{\partial}{\partial z^i} + b^j \frac{\partial}{\partial \overline{z}^j}, \quad JX = \sqrt{-1} a^i \frac{\partial}{\partial z^i} - \sqrt{-1} b^j \frac{\partial}{\partial \overline{z}^j} \]

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Let
\[ f_\ast X = (a^i \circ f^{-1}) \frac{\partial x^\alpha}{\partial z^i} \frac{\partial}{\partial x^\alpha} + (b^j \circ f^{-1}) \frac{\partial x^\alpha}{\partial z^j} \frac{\partial}{\partial x^\alpha} \]
\[ = (a^i \circ f^{-1}) f_\ast \frac{\partial}{\partial z^i} + (b^j \circ f^{-1}) f_\ast \frac{\partial}{\partial z^j} \]
\[ f_\ast JX = \sqrt{-1} (a^i \circ f^{-1}) \frac{\partial x^\alpha}{\partial z^i} \frac{\partial}{\partial x^\alpha} - \sqrt{-1} (b^j \circ f^{-1}) \frac{\partial x^\alpha}{\partial z^j} \frac{\partial}{\partial x^\alpha} \]
\[ = \sqrt{-1} (a^i \circ f^{-1}) f_\ast \frac{\partial}{\partial z^i} - \sqrt{-1} (b^j \circ f^{-1}) f_\ast \frac{\partial}{\partial z^j} \]
where \((x^1, \ldots, x^n)\) is a local coordinate of \(N\) at \(f(p)\). Thus using equation 4.4.7,
\[ \nabla_{f_\ast X} f_\ast X \]
\[ = \nabla_{(a^i \circ f^{-1}) f_\ast} \frac{\partial}{\partial z^i} + (b^j \circ f^{-1}) f_\ast \frac{\partial}{\partial z^j} \]
\[ = (a^i \circ f^{-1}) \frac{\partial a^k}{\partial z^i} f_\ast \frac{\partial}{\partial z^k} + (a^i \circ f^{-1}) \frac{\partial b^j}{\partial z^i} f_\ast \frac{\partial}{\partial z^j} \]
\[ + (a^i \circ f^{-1}) (a^k \circ f^{-1}) \nabla_{f_\ast} \frac{\partial}{\partial z^k} f_\ast \frac{\partial}{\partial z^i} + (b^j \circ f^{-1}) \frac{\partial a^k}{\partial z^j} f_\ast \frac{\partial}{\partial z^k} \]
\[ + (b^j \circ f^{-1}) \frac{\partial b^j}{\partial z^j} f_\ast \frac{\partial}{\partial z^j} + (b^j \circ f^{-1}) (b^j \circ f^{-1}) \nabla_{f_\ast} \frac{\partial}{\partial z^j} f_\ast \frac{\partial}{\partial z^i} \]
Using the same method, we have
\[ \nabla_{f_\ast JX} JX f_\ast JX \]
\[ = -(a^i \circ f^{-1}) \frac{\partial a^k}{\partial z^i} f_\ast \frac{\partial}{\partial z^k} + (a^i \circ f^{-1}) \frac{\partial b^j}{\partial z^i} f_\ast \frac{\partial}{\partial z^j} \]
\[ + (b^j \circ f^{-1}) \frac{\partial a^k}{\partial z^j} f_\ast \frac{\partial}{\partial z^k} - (b^j \circ f^{-1}) \frac{\partial b^j}{\partial z^j} f_\ast \frac{\partial}{\partial z^j} \]
\[ - (b^j \circ f^{-1}) (b^j \circ f^{-1}) \nabla_{f_\ast} \frac{\partial}{\partial z^j} f_\ast \frac{\partial}{\partial z^j} \]
\[ - (a^i \circ f^{-1}) (a^k \circ f^{-1}) \nabla_{f_\ast} \frac{\partial}{\partial z^k} f_\ast \frac{\partial}{\partial z^i} \]
Thus
\[ \nabla_{f_\ast X} f_\ast X + \nabla_{f_\ast JX} JX f_\ast JX = 2(a^i \circ f^{-1}) \frac{\partial b^j}{\partial z^i} f_\ast \frac{\partial}{\partial z^j} + 2(b^j \circ f^{-1}) \frac{\partial a^k}{\partial z^j} f_\ast \frac{\partial}{\partial z^k} \]
Since we have
\[ [X, JX] = -2\sqrt{-1} \left( a^i \frac{\partial b^j}{\partial z^i} \frac{\partial}{\partial z^j} - b^j \frac{\partial a^i}{\partial z^j} \frac{\partial}{\partial z^i} \right) \]
\[ J[X, JX] = -2 \left( a^i \frac{\partial b^j}{\partial z^i} \frac{\partial}{\partial z^j} + b^j \frac{\partial a^i}{\partial z^j} \frac{\partial}{\partial z^i} \right) \]
So we have
\[ \nabla_{f_*X} f_*X + \nabla_{f_*JX} f_*JX + f_*J[X, JX] = 0 \]

\[ \square \]

**Lemma 4.2.** Let 

\[ f : M \to G/V \]

be a horizontal slice. Then

\[ \tilde{f}(p) = af(p) \]

\[ \forall a \in G \text{ is also a horizontal slice}. \]

**Proof:** We need only to prove that \( T_h(D) \), the horizontal distribution, is a homogeneous vector bundle. This can be seen as follows:

Let \( X \in T_h(D) \) at \( p \). Let \( \sigma(t) \) be its integral curve. Then \( \forall g \in G, g\sigma(t) \) is the integral curve of \( (dL_g)X \). We have

\[ \frac{d}{dt}|_{t=0} F^k(g\sigma(t)) \subset F^{k-1}(g\sigma(0)) \]

So \( dL_g X \in T_h(D) \).

\[ \square \]

Now suppose

\[ f(p) = aV \]

If \( f \) is a horizontal slice, then so is

\[ \tilde{f}(p) = a^{-1}f(p) \]

If at \( eV \), the original point of \( G/V \), \( \tilde{f} \) is pluriharmonic, that is, if

\[ \nabla_{(p\tilde{f})_*X} (p\tilde{f})_*X + \nabla_{(p\tilde{f})_*JX} (p\tilde{f})_*JX + (p\tilde{f})_*J[X, JX] = 0 \]
Then since

\[ pL_{a-1} = L_{a-1}p \]

and the connection is left invariant, we know

\[
(L_{a-1})_*(\nabla_{(pf)_*X}(pf)_*X + \nabla_{(pf)_*JX}(pf)_*JX + (pf)_*J[X,JX]) = 0
\]

By the above observation we need only to prove the pluriharmonicity of \( f \) at the origin.

Now we suppose the horizontal slice

\[ f : \mathcal{M} \rightarrow G/V \]

satisfies \( f(0) = eV \). Without losing the generality, we assume locally the map

\[ \mathcal{M} \rightarrow F^3 \quad (t_1, \ldots, t_n) \mapsto (1, z^t, a, \alpha^t) \]

is an immersion. According to Bryant-Griffiths [4], this is the horizontal slice of the maximal dimension. Let

\[ u = a + z^t \alpha \]

Then \( u \) is a holomorphic function on a neighborhood of \( eV \) in \( \mathcal{M} \). We know that at the point \( eV \), \( \frac{\partial z^t}{\partial \mathcal{M}} \neq 0 \). So using the implicit function theorem, we can assume that

\[ z_i = \frac{1}{\sqrt{2}} t_i \]

Then in order that

\[ Q((1, z^t, a, \alpha^t), (0, (dz)^t, da, (d\alpha)^t)) = 0 \]

We must have

\[ z_i = \frac{1}{\sqrt{2}} t_i, \quad a = u - \frac{1}{2} t_i u_i, \quad \alpha_i = \frac{1}{\sqrt{2}} u_i \]
In particular, at the original point, we have

\[ u = \sqrt{-1}, \nabla u = 0, \nabla^2 u = -\sqrt{-1}I \]

using this and the definition of \( D_i \), we know that at the original point

\[
\begin{align*}
\frac{\partial D_1}{\partial k} &= 0, \frac{\partial D_1}{\partial k} = 0 \\
\frac{\partial (D_3)_r}{\partial k} &= -\sqrt{2i}S_{rk}, \frac{\partial (D_3)_r}{\partial k} = 0 \\
\frac{\partial D_4}{\partial k} &= 0, \frac{\partial (D_4)_{rs}}{\partial k} = \pi_{rs} \\
\frac{\partial^2 D_1}{\partial k \partial t} &= 0, \frac{\partial^2 D_4}{\partial k \partial t} = 0, \frac{\partial^2 (D_3)_r}{\partial k \partial t} = -\frac{1}{\sqrt{2}} \pi_{klr}
\end{align*}
\] (4.4.8)

We suppose that

\[ (t_1, \ldots, t_n) \mapsto (A_{kl}), \quad 1 \leq k, l \leq n + 1 \]

Then we have

\[
\begin{align*}
D_1 &= A_{11} \\
(D_3)_r &= A_{r+1,1} \quad 1 \leq r \leq n \\
(D_4)_{rs} &= A_{r+1,s+1} \quad 1 \leq r, s \leq n
\end{align*}
\]

It is easy to see that if

\[
\frac{\partial^2 A_{rs}}{\partial i \partial j} + \sum_{m,n,p} \Gamma_{mn,pq}^{rs} \frac{\partial A_{mn}}{\partial i} \frac{\partial A_{pq}}{\partial j} = 0
\] (4.4.9)

Then

\[ \nabla^2 \psi_{ij} \frac{\partial f}{\partial \psi_{ij}} = 0 \]

This is valid if we have

1. \( \Gamma_{1,(i+1),pq}^{11} = 0, 2 \leq p \leq q \leq n + 1, 1 \leq i \leq n \)

2. \( \Gamma_{1,(i+1),pq}^{rs} = 0, 2 \leq r \leq s, 2 \leq p \leq q \leq n + 1, 1 \leq i \leq n \)
3. \[ u_{i(s-1)} + \sum_{2 \leq p \leq q} 2\sqrt{-1} \Gamma_{1(i+1),pq}^{1s} u_{(p-1)(q-1)j} = 0, 2 \leq s \]

Let \( G = \text{Im} \, A \). We know at the original point

\[ G_{ij} = \delta_{ij} \]

The invariant Kähler metric on \( G/K \) is

\[ \omega = \partial \overline{\partial} \log \det G \]

It is easy to compute that

\[
G_{i,j\bar{k}l} = \begin{cases} 
\frac{1}{2} (G^{ik}G^{jl} + G^{il}G^{jk}) & i < j, k < l \\
\frac{1}{2} G^{ik}G^{il} & i = j, k < l \\
\frac{1}{2} G^{ik}G^{jk} & i < j, k = l \\
\frac{1}{4} (G^{ik})^2 & i = j, k = l
\end{cases}
\] (4.4.10)

At the origin, \( G_{ij} = \delta_{ij} \). From this, it is easy to verify the above equations.

Thus we have proved

**Theorem 4.3.** With the notations as above. If \( \mathcal{M} \to G/V \) is a horizontal slice where \( G/V \) is a classifying space of a Calabi-Yau threefold, then \( \mathcal{M} \to G/K \) is a pluriharmonic map.

**Remark 4.1.** The projection \( p \) is in general, not a pluriharmonic map. For example, in our case, we see at the origin point

\[
\frac{\partial D_i}{\partial \bar{z}_1} = 0
\]

for \( i=1,2,4 \). But

\[
\frac{\partial^2 (d_4)_{11}}{\partial z_1 \partial \bar{z}_1} = -\sqrt{-1} \neq 0
\]

So it does not satisfy the equation

\[
\frac{\partial^2 f^\alpha}{\partial z_1 \partial \bar{z}_1} + \Gamma^\alpha_{\beta \gamma} \frac{\partial f^\beta}{\partial z_1} \frac{\partial f^\gamma}{\partial \bar{z}_1} = 0
\]
Now suppose that $\mathcal{M}$ is a horizontal slice. We denote $\omega$ the Kähler form of its VHS metric (see definition 3.5).

**Theorem 4.4.** The Ricci curvature of $\omega$ is nonpositive.

**Proof:** Let $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ be a local frame. Suppose $\nabla$ is the connection of $G/K_0$. Suppose $R$ and $\tilde{R}$ is the invariant Ricci curvature tensor of $\mathcal{M}$ and $\tilde{R}$ is the curvature tensor of $G/K$, then by the Gauss formula

$$R(p_*X, p_*X) = \tilde{R}(p_*X, p_*e_i, p_*X, p_*e_i) + \tilde{R}(p_*X, p_*Je_i, p_*X, p_*Je_i)$$

By theorem 4.3, we see that the last term is zero. So we concluded that

$$R(p_*X, p_*X) \leq \tilde{R}(p_*X, p_*e_i, p_*X, p_*e_i) + \tilde{R}(p_*X, p_*Je_i, p_*X, p_*Je_i)$$

(4.4.11)

So the theorem follows from lemma 3.11.

**Theorem 4.5.** Let $\mathcal{M}$ be the horizontal slice of the Calabi-Yau classifying space, then its Ricci curvature of $\omega$ satisfies

$$\text{Ricci}(\omega) \leq \alpha \omega < 0$$

for some constant $\alpha < 0$.

**Proof:** Without losing generality, we assume $X = e_1$. Thus by equation 4.4.11 and Lemma 3.11, we have

$$R(p_*X, p_*X) \leq \tilde{R}(p_*e_1, p_*Je_1, p_*e_1, p_*Je_1)$$

By Helgason [13], or lemma 3.11 we know that if $e_1 = (g, E_1)$ for $g \in G$, $E_1 \in \mathfrak{p}_0 \subset \mathfrak{g}$, then

$$\tilde{R}(p_*e_1, p_*Je_1, p_*e_1, p_*Je_1) = -||[E_1, JE_1]||^2$$

66
By Theorem 4.1, we assume

\[
E_1 = \begin{pmatrix}
  a & \alpha^t & b & \beta^t \\
  \alpha & A_1 & \beta & B_1 \\
  b & \beta^t & -a & -\alpha^t \\
  \beta & -B_1 & -\alpha & -A_1
\end{pmatrix}
\]

\[A_1^t = A_1, \quad B_1^t = B_1\]

and then

\[
JE_1 = \begin{pmatrix}
  b & \beta^t & -a & -\alpha^t \\
  \beta & -B_1 & -\alpha & A_1 \\
  -a & -\alpha^t & -b & -\beta^t \\
  -\alpha & A_1 & -\beta & B_1
\end{pmatrix}
\]

So

\[
[E_1, JE_1] = \begin{pmatrix}
  0 & a\beta^t - ba^t \\
  b\alpha - a\beta & \alpha\beta^t - \beta\alpha^t - A_1B_1 + B_1A_1 \\
  |a|^2 + |b|^2 + |\alpha|^2 + |\beta|^2 & a\alpha^t + b\beta^t \\
  b\beta + a\alpha & \beta\beta^t + \alpha\alpha^t - A_1^2 - B_1^2 \\
  -(|a|^2 + |b|^2 + |\alpha|^2 + |\beta|^2) & -a\alpha^t - b\beta^t \\
  -a\alpha - b\beta & -\alpha\alpha^t - \beta\beta^t + A_1^2 + B_1^2 \\
  0 & -b\alpha^t + a\beta^t \\
  -a\beta + ba\alpha & -\beta\alpha^t + \alpha\beta^t + B_1A_1 - A_1B_1
\end{pmatrix}
\]

Thus

\[
||[E_1, JE_1]||^2 \geq (|a|^2 + |b|^2 + |\alpha|^2 + |\beta|^2)^2 + tr((a\alpha^t + \beta\beta^t - A_1^2 - B_1^2)^2)
\]

Let

\[
P = a\alpha^t + \beta\beta^t \quad Q = A_1^2 + B_1^2
\]
Then
\[ tr(P - Q)^2 = tr(P^2 + Q^2 - 2PQ) \]
\[ \geq tr(P^2) + tr(Q^2) - 2tr(P^2) - \frac{1}{2}tr(Q^2) \tag{4.4.12} \]
\[ \geq \frac{1}{2}tr(Q^2) - tr(P^2) \]

\[ tr(P^2) \leq 4(|\alpha|^2 + |\beta|^2)^2 \tag{4.4.13} \]

\[ ||[E_1, JE_1]||^2 \geq (|\alpha|^2 + |\beta|^2)^2 + (|a|^2 + |b|^2)^2 + \frac{1}{10} tr(P - Q)^2 \]
\[ \geq \frac{1}{20}tr Q^2 + \frac{3}{5}(|\alpha|^2 + |\beta|^2) + |a|^2 + |b|^2 \tag{4.4.14} \]

\[ tr(A_1^2B_1^2) = tr(A_1B_1A_1) \geq 0 \tag{4.4.15} \]

Now by equations (4.4.12), (4.4.13), (4.4.14), (4.4.15)

\[ ||[E_1, JE_1]||^2 \geq tr(A_1^4 + B_1^4) + |a|^2 + |b|^2 + |\alpha|^2 + |\beta|^2 \]
\[ \geq c tr(A_1^2 + B_1^2) + a^2 + b^2 + |\alpha|^2 + |\beta|^2 \]
\[ \geq c(E_1, E_1)^2 \]
for some constant \( c \).

### 4.3 The Weil-Petersson Metric and the VHS Metric

In this section, we generalized the notation of Weil-Petersson metric to some horizontal slices and proved a relation between the Weil-Petersson metric and the VHS metric in the case of dimension 3.

Let \( D \) be the classifying space of some moduli space \( \mathcal{M} \) of a polarized Calabi-Yau manifold. There is a line bundle \( F^n \) over \( \mathcal{M} \). Let \( \Omega \) be a local section of \( F^n \), then G.
Tian [24] proved that
\[ \omega_{WP} = -\frac{\sqrt{-1}}{2} \partial \overline{\partial} \log Q(\Omega, \overline{\Omega}) \]
where \( Q \) is the quadratic form represent the polarization. In view of this we make the following definition

**Definition 4.1.** Let \( \mathcal{M} \to D \) be a horizontal slice. Let \( \Omega \) be a local section of \( F^n \to \mathcal{M} \). Then the Weil-Petersson metric \( \omega \) is defined as
\[ \omega = -\frac{\sqrt{-1}}{2} \partial \overline{\partial} \log Q(\Omega, \overline{\Omega}) \]

The following proposition \( \omega \) guaranteed that the Weil-Petersson metric is well defined.

**Proposition 4.4.** Let \( \omega = \frac{\sqrt{-1}}{2} g_{\alpha \overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta \) in local coordinate. Then \( g_{\alpha \overline{\beta}} > 0 \) is positive definite.

**Proof:** Let \( K_\alpha = -\partial_\alpha \log Q(\Omega, \overline{\Omega}) \). Then
\[ g_{\alpha \overline{\beta}} = -\frac{Q(\partial_\alpha \Omega - K_\alpha \Omega, \partial_\beta \Omega - K_\beta \Omega)}{Q(\Omega, \overline{\Omega})} \]
But \( \partial_\alpha \Omega - K_\alpha \Omega \in H^{n-1,1} \). The proposition follows from the second Hodge-Riemannian Relation.

**Definition 4.2.** The cubic form \( F = F_{ijk} \) is a section of the bundle \( \text{Sym}^3(T^* \mathcal{M}) \otimes (F^n)^\otimes 2 \) defined by
\[ F_{ijk} = Q(\Omega, \partial_i \partial_j \partial_k \Omega) \]
in local coordinates \( (z^1, \ldots, z^n) \).

Suppose \( D = G/V \) for a noncompact semisimple Lie group \( G \) and a compact subgroup \( V \). In lemma 3.5, it is proved that actually \( G = \mathfrak{Sp}(n + 1, R) \). It is not hard to see that \( V = \mathfrak{U}(1) \times \mathfrak{U}(n) \).

We are going to prove
Theorem 4.6. Suppose $\omega$ is the Kähler form of the Weil-Petersson metric. Let

$$\omega_1 = (n + 3)\omega + \text{Ric}(\omega)$$

then $\omega_1$ is a constant multiple of the VHS metric.

Proof: By the curvature formula of Strominger [23], we have

$$R_{i\bar{j}} = -(n + 1)g_{i\bar{j}} + e^{2K} F_{i \bar{p} q} F_{j mn} \overline{g^{p m}} \overline{g^{q n}}$$

where we suppose $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ in the local coordinates $(z^1, \cdots, z^n)$ and $K$ is the local function: $K = -\log Q(\Omega, \overline{\Omega})$.

Let $\omega_1 = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then we need only to prove

$$h_{i\bar{j}} = 2g_{i\bar{j}} + e^{2K} F_{i \bar{p} q} F_{j mn} \overline{g^{p m}} \overline{g^{q n}}$$

is a multiple of VHS metric.

Consider the projection

$$p : D = G/V \to G/K$$

where $K$ is the maximal connected compact subgroup of $G$ containing $V$.

We have

Lemma 4.3. $p$ is an isometry between the Riemannian submanifold $\mathcal{M}$ of $D$ and the Riemannian submanifold $p(\mathcal{M})$ of $G/K$.

Proof: Note that $\mathcal{M}$ is a horizontal slice of $D$. The lemma follows from the definition of the invariant Hermitian metric on both manifold.

Recall the result of Bryant and Griffiths [4]. Their results can be briefly written as follows:

Suppose

$$\pi' : D \to CP^{2n+1}$$
is the projection of $D$ to $CP^{2n+1}$ by sending $(F^3, F^2, F^1)$ to $F^3$. Then

$$\pi': \mathcal{M} \rightarrow D \rightarrow CP^{2n+1}$$

is an immersion.

If furthermore we assume that $eV \in \mathcal{M}$, i.e. the moduli space passes the original point of $D$, where the original point is defined as $\{f^3, f^2, f^1\} \in D$ in proposition 4.1. Then there is a holomorphic function $u$ defined on a neighborhood of the original point of $\mathbb{C}^n$ such that at a neighborhood of $eV$

$$F^3 = (1, \frac{1}{\sqrt{2}} z^1, \cdots, \frac{1}{\sqrt{2}} z^n, u - \sum_i \frac{1}{2} z^i u_i, \frac{1}{\sqrt{2}} u_1, \cdots, \frac{1}{\sqrt{2}} u_n)$$

and $F^2 = \nabla F^3$, $F^1 \perp F^3$ via $Q$.

In order to prove the theorem, we need only to prove it at any point. Since $D$ is a homogeneous manifold we can assume the point is the original point $eV$ of $D$.

First, at the original point, we have

$$\sqrt{-1}Q(\Omega, \bar{\Omega}) = 2$$

Now suppose that $(z^1, \cdots, z^n)$ is the local coordinate. Then by Tian’s theorem

$$g_{ij} = -\frac{Q(D_i \Omega, D_j \Omega)}{Q(\Omega, \bar{\Omega})}$$

where $D_i \Omega = \partial_i \Omega - K_i \Omega$. We get $g_{ij} = \frac{1}{2} \delta_{ij}$

Now the cubic form $F_{ijk}$ at $eV$ is

$$F_{ijk} = -\frac{1}{2} \frac{\partial^3 u}{\partial z^i \partial z^j \partial z^k}(0) = -\frac{1}{2} u_{ijk}(0)$$

Thus

$$2g_{ij} + e^{2K} F_{imn} \overline{F_{j pq}} \overline{g}^{m \overline{p}} g^{np} = \delta_{ij} + \frac{1}{4} u_{imn}(0) \overline{u}_{j mn}(0)$$

Next we compute $h_{ij}$ via the function $u$:
As a bounded domain, \( G/K \) can be realized as the Siegel Upper plane as follows:

\( G/K \) is the set of all the symmetric \((n + 1) \times (n + 1)\) matrix \( Z \) satisfying \( \text{Im} Z > 0 \) here \( \text{Im} Z > 0 \) means \( \text{Im} Z \) is a positive Hermitian matrix.

The Hermitian metric \( \partial \overline{\partial} \log \det \text{Im} Z \) is a multiple of the invariant VHS metric. At the point \( iI \), the metric matrix is an identity matrix. A straightforward computation using definition of the VHS metric gives

\[ h_{ij} = \delta_{ij} + \frac{1}{4} u_{imn} \overline{u}_{jmn} \]

### 4.4 On the Complete Weil-Petersson Metric

In this section, we give a necessary and sufficient condition for the Weil-Petersson metric to be complete. We use the gradient estimate to prove our result. At this moment we have not found any example for the theorem. But we hope that a local version of the theorem will be useful in study the moduli space near infinity.

**Theorem 4.7.** Suppose \( \mathcal{M} \) is the horizontal slice defined above. If the Weil Petersson metric on \( \mathcal{M} \) is complete, then the normal of the cubic form with respect to the Weil-Petersson metric is bounded. On the other hand, if the cubic form with respect to the Weil-Petersson metric is bounded, then the Weil-Petersson completeness is equivalent to the VHS completeness.

**Proof:** Let \( \omega_1 \) be the VHS metric, by the formula

\[ \omega_1 = 2 \omega + e^{2K} F_{imn} \overline{F}_{jpq} g^{mp} \overline{g}^{nq} dz_i \wedge \overline{dz_j} \]

we know that if the cubic form is bounded, then the VHS metric and the Weil-Petersson metric are mutually equivalent:

First

\[ \omega_1 \geq 2 \omega \]
On the other hand, suppose we are working on the normal coordinates and suppose that $K = 0$, then we have

$$|\sum F_{imn}F_{jmn}\alpha_i\alpha_j| = \sum_{mn} |\sum_i F_{imn}\alpha_i|^2$$

$$\leq (\sum_{mn} \sum_i |F_{imn}|^2) \sum_i |\alpha_i|^2 \leq M \sum_i |\alpha_i|^2$$

Thus

$$\omega_1 \leq (2 + M)\omega$$

Since $\omega_1$ is the metric induced from a complete metric of $D$, we know $\omega_1$ completeness is equivalent to $\omega$ completeness.

On the other hand, suppose

$$f = |F|^2 = e^{2K} F_{ijk} \overline{F_{abc}} g^{\overline{ia}} g^{\overline{jb}} g^{k\overline{c}}$$

where $K = -\log\sqrt{-1}Q(\Omega, \overline{\Omega})$. We compute under normal coordinates.

$$f_\alpha = 2e^{2K} K_\pi F_{ijk} \overline{F_{abc}} g^{\overline{ia}} g^{\overline{jb}} g^{k\overline{c}}$$

$$+ e^{2K} F_{ijk} \overline{\partial_\alpha F_{abc}} g^{\overline{ia}} g^{\overline{jb}} g^{k\overline{c}} + e^{2K} F_{ijk} \overline{F_{abc}} \partial_\alpha (g^{\overline{ia}} g^{\overline{jb}} g^{k\overline{c}})$$

Thus

$$\Delta f = e^{2K} |\partial_\alpha F_{ijk} + 2K_\alpha F_{ijk}|^2 + 2n|F|^2 + e^{2K} F_{ijk} \overline{F_{abc}} \partial_\alpha \overline{\partial_\alpha} (g^{\overline{ia}} g^{\overline{jb}} g^{k\overline{c}})$$

We see that

$$e^{2K} F_{ijk} \overline{F_{abc}} \partial_\alpha \overline{\partial_\alpha} (g^{\overline{ia}} g^{\overline{jb}} g^{k\overline{c}}) = 3e^{2K} F_{ijk} \overline{F_{ajk}} R_{a\overline{a}}$$

Thus

$$\Delta f \geq 2n|F|^2 + 3e^{2K} F_{ijk} \overline{F_{ajk}} R_{a\overline{a}}$$

Here $n$ is the complex dimension of the Moduli space. It is known that

$$R_{a\overline{a}} = -(n + 1)\delta_{a\overline{a}} + e^{2K} F_{amn} \overline{F_{imn}}$$

(4.4.16)
Thus

\[ \Delta f \geq 2n|F|^2 - 3(n + 1)|F|^2 + 3e^{4K} \sum_{a,i} \left( \sum_{j,k} F_{i,jk} F_{a,jk} \right)^2 \]

\[ \geq -(n + 3)|F|^2 + 3e^{4K} \left( \sum_{i,j,k} |F_{ijk}|^2 \right)^2 \]

\[ \geq -\frac{3}{n} f^2 - (n + 3)f \]

We now recall a version of the maximum principle from [24].

**Proposition 4.5.** Suppose that \((M, g)\) is a complete Kähler manifold. If the Ricci curvature of \(g\) is bounded from below and \(\varphi\) is a nonnegative function satisfying

\[ \Delta \varphi \geq c_1 \varphi^\alpha - c_2 \varphi - c_3 \]

where \(\alpha > 1, c_1 > 0, c_2, c_3 \geq 0\) are constants. then

\[ \sup \varphi \leq \text{Max}(1, \left(\frac{c_2 + c_3}{c_1}\right)^{\frac{1}{\alpha}}) \]

Now by equation 4.4.16 we know that the Ricci curvature is bounded from below. Thus

\[ f \leq \sqrt{\frac{n(n + 3)}{3}} \]

**Remark 4.2.** We can also get similar estimates on moduli spaces with noncomplete Weil-Petersson metric. In that case, a different version of Maximum principle should be set up.

### 4.5 The Curvature Computation

In [11], the authors estimated the holomorphic sectional curvature in the horizontal direction for the whole classifying space. In this section we give a more precise computation
of the holomorphic sectional curvature, bisectional curvature and the Ricci curvature on the horizontal slice. We give an optimal constant of the bound of the curvatures.

We are going to estimate the Ricci curvature and the holomorphic sectional curvature of the VHS metric.

Suppose \((g_{ij})\) is the Weil-Petersson metric, \((F_{ijk})\) is the cubic form, and \(K = -\log \sqrt{-1Q(\Omega, \overline{\Omega})}\) as in the previous sections. Then the VHS metric \((h_{ij})\) can be locally represented as:

\[
h_{ij} = 2g_{ij} + \sum_{rspbq} e^{2K} F_{irs} \overline{F_{jqs}} g_{rp} g_{sq}
\]

As we have proved, \((h_{ij})\) is a Kähler metric. So we have the curvature formula for the curvature tensor \(R_{ijkl}\) of \((h_{ij})\).

\[
\tilde{R}_{ijkl} = \frac{\partial^2 h_{ij}}{\partial z^k \partial \overline{z}^l} - h_{nm} \frac{\partial h_{im}}{\partial z^k} \frac{\partial h_{mj}}{\partial \overline{z}^l}
\]

Now we suppose that at point \(p\), the local coordinate for the Weil-Petersson metric is normal, i.e., at point \(p\), \(g_{ij} = \delta_{ij}\) and \(dg_{ij} = 0\). In particular, at point \(p\), the curvature tensor \(R_{ijkl}\) of \((g_{ij})\) is

\[
R_{ijkl} = \frac{\partial^2 g_{ij}}{\partial z^k \partial \overline{z}^l}
\]

We have

\[
\frac{\partial h_{im}}{\partial z^k} = e^{2K} \sum_{rs} F_{irs,k} \overline{F_{mrs}}
\]

where

\[
F_{irs,k} = \partial_k F_{irs} + 2K_k F_{irs}
\]

is the covariant derivative we defined before. Furthermore

\[
\frac{\partial^2 h_{ij}}{\partial z^k \partial \overline{z}^l} = 2R_{ijkl} - 2e^{2K} \sum_{aq} R_{aqkl} F_{irs} \overline{F_{jrq}}
\]

\[
+ 2e^{2K} \delta_{kl} \sum_{rs} F_{irs} \overline{F_{jrs}} + e^{2K} \sum_{rs} F_{irs,k} \overline{F_{jrs,l}}
\]
Thus we have the following

**Proposition 4.6.** If $K = 0$ at the point $p$, then at $p$,

\[
\tilde{R}_{ijkl} = 2R_{ijkl} + 2\delta_{kl} \sum_{rs} F_{irs} F_{jrs} - 2 \sum_{sq} R_{qijkl} F_{irs} F_{jrq} \\
+ \sum_{rs} F_{irs,kl} F_{jrs,l} - \sum_{mn} (\sum_{rs} F_{irs,k} F_{mrs}) (\sum_{rs} F_{jrs,l} F_{nrs}) h^{nm}
\]

(4.4.17)

Based on the above proposition, we get

**Theorem 4.8.** Let $c(n) = ((\sqrt{n} + 1)^2 + 1)$, then

\[
\text{Ric}(\omega_1) \leq -\frac{1}{c(n)} \omega \\
R \leq -\frac{1}{c(n)}
\]

where $R$ is the superium of the holomorphic sectional curvature. The constant here is optimal. Furthermore, the bisectional curvature is nonpositive.

**Proof:** Fixing $i$, let

\[
A_m = \sum_{rsk} F_{irs,k} a_k F_{mrs}
\]

for a vector $a = (a_1, \cdots, a_n)$. Then it is easy to see that

\[
\sum_{pq} |\sum_k F_{ipq,k} a_k|^2 - \sum_{mn} (\sum_{rsk} F_{irs,k} a_k F_{mrs})(\sum_{rsk} F_{irs,k} a_k F_{mrs}) h^{nm} \\
= \sum_{pq} |\sum_k F_{ipq,k} a_k|^2 - \sum_{mn} h^{nm} A_m F_{npq}^2 + 2 \sum_a |\sum_m h^{am} A_m|^2
\]

(4.4.18)

Here we use the fact that $h_{ij} = 2\delta_{ij} + F_{imm} F_{jmn}$

Define a generic vector $a^k = \delta_{ik}, k = 1, \cdots, n$. Using equation 4.4.18, we have

\[
\sum_{pq} |F_{ipq,i}|^2 - \sum_{mn} h^{nm} (\sum_{rs} F_{irs,i} F_{mrs})(\sum_{rs} F_{irs,i} F_{nrs}) \geq 0
\]

Now using proposition 4.6, we get

\[
\tilde{R}_{ijkl} a^i a^j a^k a^l = \tilde{R}_{iija} - 2 R_{iija} + 2 \sum_{rs} |F_{irs}|^2 - 2 \sum_{sq} r_{qija} F_{irs} F_{irq} \\
\geq 4 - 4 \sum_r |F_{iri}|^2 + 2 \sum_{rp} |\sum_q F_{qip} F_{irq}|^2
\]
Let
\[ x = \sum_r |F_{iir}|^2 \]

Then we have
\[
\sum_{rp} \sum_q |F_{qip}|^2 |F_{irq}|^2 \geq \sum_{q} |F_{iq}^2|^2 = x^2
\]
\[
\sum_{rp} \sum_q |F_{qip}|^2 |F_{irq}|^2 \geq \sum_r \sum_q |F_{qir}|^2 \geq \frac{1}{n} (h_{ii} - 2)^2
\]

So for \( a, b > 0, a + b = 1 \), we have
\[
\frac{1}{2} \tilde{R}_{i\bar{i}i} \geq 2 - 2x + ax^2 + \frac{b}{n} (h_{ii} - 2)^2 \\
\geq 2 - \frac{1}{a} + \frac{b}{n} (h_{ii} - 2)^2
\]

On the other hand, we have
\[ h_{ij} a^i a^j = h_{ii} \]

Let \( a = \frac{2 + \sqrt{n}}{2 + 2\sqrt{n}} \) and \( b = 1 - a \), we have
\[ \tilde{R}_{i\bar{i}i} \geq \frac{1}{(\sqrt{n} + 1)^2 + 1} h_{ii}^2 = |h_{ij} a^i a^j|^2 \]

It is a straightforward computation that the constant here is optimal. So we proved the assertion for the holomorphic sectional curvature because \( (a^k) \) can be an arbitrary vector.

Now we turn to the bisectional curvature. For any \( (a^1, \cdots, a^n) \), using the same inequalities before, we have
\[
\tilde{R}_{i\bar{i}k} a^k a^l \geq 2 \sum_k |a^k|^2 + 2 |a^l|^2 - 4 \sum_r \sum_k |F_{i rk} a^k|^2 \\
+ 2 \sum_{mr} \sum_{qk} |F_{qkm} a^k|^2 + 2 \sum_r \sum_{qk} |F_{iqk} a^k - a^r|^2 \geq 0
\]
This proves the nonpositivity of the bisectional curvature.

Now we consider the Ricci curvature. Suppose that $\xi$ is a unit vector. Then by the definition of the Ricci curvature and above results, we have

$$-\text{Ric}(\xi, \xi) \geq \tilde{R}(\xi, \xi, \xi, \xi)$$

Thus completes the proof of the theorem.

### 4.6 The Boundness of the Sectional Curvature

In this section, we prove that the boundness of the Ricci curvature implies the boundness of the Riemannian sectional curvature.

**Theorem 4.9.** Suppose $p \in M$ is a fixed point such that the Ricci curvature has a bound $C_p$ at $p$. That is

$$|\text{Ric}(\omega_1)_p| \leq C_p \omega_1$$

Then the Riemannian sectional curvature has a bound

$$|R(X, Y, X, Y)| \leq (2 + C_p)||X||^2||Y||^2$$

where $X, Y \in T_pM$ and $X \perp Y$.

We begin by restate the proposition in the section 4.5

**Proposition 4.7.** Suppose we have the notation as in the proposition 4.6, then we have

$$\tilde{R}_{ijkl} = A_{ijkl} + B_{ijkl}$$

where

$$A_{ijkl} = 2\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{kj} - 4 \sum_s F_{iks}\overline{F}_{jls} + 2 \sum_{mnpq} F_{qkm}\overline{F}_{plm}\overline{F}_{inp}\overline{F}_{jnq}$$

$$B_{ijkl} = \sum_{rs}(F_{irs,k} - \sum_{mn} A_{ikm} F_{nrs} h^{nm})(F_{jrs,l} - \sum_{mn} A_{jlm} F_{nrs} h^{nm}) + 2 \sum_{mno1} A_{ikm} h^{nm1} A_{jlm} h^{m1o}$$
here we define

\[ A_{ikm} = \sum_{rs} F_{irs,k} \overline{F_{mrs}} \]

Proof: A straightforward computation.

Lemma 4.4. Suppose that \( \xi, \eta \in T_p^{(1,0)} M \) and define \( \|\xi\|^2 = h_{ij} \xi^i \overline{\xi^j} \), then

\[ |\tilde{R}_{ijkl} \xi^i \xi^k \eta^l \eta^j| \leq (4 + C_p) \|\xi\|^2 \|\eta\|^2 \]

Proof: Note that the holomorphic bisectional curvature of \( M \) is nonpositive. We know that the holomorphic sectional curvature is bounded by \( C_p \), i.e.

\[ |\tilde{R}_{ijkl} \xi^i \xi^k \eta^l \eta^j| \leq C_p \|\xi\|^4 \]

We have

\[ \sum_{pq} \left| \sum_{ni} \sum_{ij} (\sum_{n} F_{inp} \xi^i) (\sum_{j} F_{jnq} \eta^j) \right|^2 \]

\[ = \sum_{pq} \left| \sum_{n} (\sum_{i} F_{inp} \xi^i) (\sum_{j} F_{jnq} \eta^j) \right|^2 \]

\[ \leq \sum_{pq} \left| \sum_{n} (\sum_{i} F_{inp} \xi^i)^2 \sum_{n} (\sum_{j} F_{jnq} \eta^j)^2 \right| \]

\[ = \sum_{pn} \left| \sum_{i} F_{inp} \xi^i \sum_{qn} \sum_{j} F_{jnq} \eta^j \right|^2 \]

\[ \leq \|\xi\|^2 \|\eta\|^2 \]

here we use the fact that

\[ h_{ij} = 2 \delta_{ij} + \sum_{rs} F_{irs,k} \overline{F_{mrs}} \]

and

\[ \sum_{pn} \left| \sum_{i} F_{inp} \xi^i \right|^2 = \sum_{i,j} \sum_{pn} F_{inp} \xi^i \overline{F_{jnq} \xi^j} \leq \|\xi\|^2 \]
Thus

\[ \sum_{ijkl} \sum_{mnpq} F_{qkm} F_{plm} F_{ijnq} \xi^i \xi^k \eta^j \eta^l \]
\[ = \sum_{pq} \left( \sum_{nij} F_{ijnq} \xi^j \eta^l \right) \left( \sum_{mkl} F_{qkm} \xi^k \eta^l \right) \]
\[ \leq \sqrt{\sum_{pq} \left| \sum_{nij} F_{ijnq} \xi^j \eta^l \right|^2 \left| \sum_{mkl} F_{qkm} \xi^k \eta^l \right|^2} \]
\[ \leq ||\xi||^2 ||\eta||^2 \]

We also have

\[ \sum_s \left| \sum_{ik} F_{iks} \xi^i \xi^k \right|^2 \leq \sum_s \left( \sum_k \left| \sum_i F_{iks} \xi^i \xi^k \right|^2 \sum_k \left| \xi^k \right|^2 \right) \leq ||\xi||^4 \]

Thus by proposition 4.7, we have

\[ \left| \sum_{ijkl} A_{ijkl} \xi^i \xi^k \eta^j \eta^l \right| \leq 4 ||\xi||^2 ||\eta||^2 \]

We also have

\[ \sum_{ijkl} B_{ijkl} \xi^i \xi^k \eta^j \eta^l \leq \sqrt{\sum_{ijkl} B_{ijkl} \xi^i \xi^k \xi^j \xi^l \sum_{ijkl} B_{ijkl} \eta^i \eta^k \eta^j \eta^l} \]

Combining the above two inequalities we proved the lemma. \( \square \)

**Proof of theorem 4.9:**

Let

\[ \xi = X - \sqrt{-1}JX \]
\[ \eta = Y - \sqrt{-1}JY \]

Then

\[ R(X, Y, X, Y) = \frac{1}{8} \left( Re(R(\xi, \eta, \xi, \eta)) - R(\xi, \xi, \eta, \eta) \right) \]
The bisectional curvature is bounded by $C_p$

$$|R(\xi, \bar{\xi}, \eta, \bar{\eta})| \leq C_p||\xi||^2||\eta||^2$$

Thus

$$|\bar{R}(X, Y, X, Y)| \leq \frac{1}{4}(2 + C_p)||\xi||^2||\eta||^2 = (2 + C_p)||X||^2||Y||^2$$

### 4.7 An Asymptotic Estimate

In this section, we make use of the estimates in the previous sections to prove an asymptotic estimate of the Weil-Petersson metric of the degeneration of a Calabi-Yau threefold. The motivation for the estimate is from a result of G. Tian [25]. Although the argument can be generalized to study the Weil-Petersson metric of a horizontal slice near infinity, we restrict ourselves to the degeneration of Calabi-Yau threefold.

We say $\pi : X \to \Delta$ is a degeneration of Calabi-Yau threefold if $X$, $\Delta$ are complex manifolds and $\pi$ is holomorphic, and $\Delta$ is the unit disk in $\mathbb{C}$. $\forall t \in \Delta$, $t \neq 0$, $\pi^{-1}(t)$ is a smooth Calabi-Yau threefold while $\pi^{-1}(0)$ is a divisor of normal crossing. We also denote $\Delta^*$ to be the punctured unit disk.

**Theorem 4.10.** Suppose $X \to \Delta$ is a degeneration of Calabi-Yau threefolds. Suppose $\omega$ be the Weil-Petersson metric on $\Delta^*$. Then if

$$\lim_{r \to 0} \frac{\log \omega}{\log \frac{1}{r}} = 0$$

Then

$$\lambda(z) \leq C_1(\log \frac{1}{r})^{4c(n)}$$

where $c(n) = ((\sqrt{n} + 1)^2 + 1)$ and $C_1$ is a constant.

**Remark 4.3.** $\omega$ is a Kähler metric on $\Delta^*$. So there is a function $\lambda(z) > 0$ on $\Delta^*$ such that

$$\omega = \sqrt{-1}\lambda(z)dz \wedge d\bar{z}$$

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The assumption is understood as
\[
\lim_{r \to 0} \frac{\log \lambda}{\log \frac{1}{r}} = 0
\]
and the conclusion of the theorem is understood as
\[
\lambda(z) \leq C_1 (\log \frac{1}{r})^{4\varepsilon(n)}
\]

**Proof:** By a theorem of Tian, there are no obstruction towards the deformation of a Calabi-Yau three-fold. Suppose \( M \in \Delta^* \) is a fiber. Let
\[
n = \dim H^1(M, \Theta)
\]
Then there is an \( n \)-dimensional complex manifold \( S \) such that
\[
\Delta^* \subset S
\]
Now consider the cubic form
\[
F = F_{\alpha\beta\gamma}
\]
of \( S \). Suppose \( z_1 = z \), and suppose \( \Delta^* \) is defined by \( z_2 = \cdots = z_n = 0 \) near a point \( p \).
Then by the Strominger’s formula, we have
\[
R_{1\Omega 1\Omega} = 2g_{1\Omega}^2 - \frac{1}{(\Omega, \Omega)^2} F_{1\xi \eta} \overline{F_{1\eta \xi}} g^{\xi \eta}
\]
(4.4.19)
Here \( (\Omega, \overline{\Omega}) = \sqrt{-1} Q(\Omega, \overline{\Omega}) \)

**Lemma 4.5.** Let \( \lambda = g_{1\Omega} \), then
\[
\lambda^{-1} \frac{1}{(\Omega, \overline{\Omega})^2} F_{1\xi \eta} \overline{F_{1\eta \xi}} g^{\xi \eta} \leq \frac{1}{(\Omega, \overline{\Omega})^2} F_{1\alpha \xi} \overline{F_{1\beta \eta}} g^{\alpha \beta} g^{\xi \eta}
\]

**Proof:** Of course we can prove this inequality by using a version of Cauchy inequality. But here we use another proof: If \( g_{1\overline{\eta}} = 0 \) for \( \beta \neq 1 \), then the inequality is trivial. So we would like to choose a coordinate such that \( g_{1\overline{\eta}} = 0, \beta \neq 1 \).
Let $A$ be an $(n-1) \times (n-1)$ matrix. Let

$$w_i = A_{ij} z_j$$

for $j = 2, \cdots, n$. If $A$ is a nonsingular matrix, then $(z_1, w_2, \cdots, w_n)$ will be again local holomorphic coordinate and $\Delta^*$ is again be defined by $w_2 = \cdots = w_n = 0$. Now we choose an $A$ such that

$$\left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial w^k} \right) = 0$$

for $k = 2, \cdots, n$. Suppose $\tilde{g}_{\alpha\beta}$ is the matrix under the coordinate $(z_1, w_2, \cdots, w_n)$, $(\tilde{g}_{1\alpha}) = 0$ for $\alpha \neq 1$.

\[\square\]

Using the lemma, from equation 4.4.19, we have

$$R_{1TT} \geq 2\lambda^2 - \lambda(h_{1T} - 2g_{1T}) \geq -\lambda h_{1T}$$

Suppose $\Delta = \frac{\partial^2}{\partial z_1 \partial z_1}$, then the Gauss curvature of $\Delta^*$ is

$$K = -\frac{4}{\lambda} \Delta \log \lambda$$

On the other hand

$$\Delta \log \log \frac{1}{r} = -\frac{1}{4r^2(\log \frac{1}{r})^2}$$

We also have the Gauss formula

$$4R_{1TT} \leq -K \lambda^2$$

So we have

$$\Delta \log \lambda = -\frac{1}{4} \lambda K \geq \frac{1}{\lambda} R_{1TT} \geq -h_{1T}$$

As we have proved, the holomorphic sectional curvature of $(h_{1T})$ is less than $-\frac{1}{c(n)}$.

Thus by the Schwartz lemma

$$h_{1T} \leq \frac{c(n)}{(r \log \frac{1}{r})^2}$$

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Thus
\[ \Delta \log \frac{\lambda}{(\log \frac{1}{r})^{4c(n)}} \geq 0 \]

The rest of the proof is quite elementary: let
\[ f = \log \frac{\lambda}{(\log \frac{1}{r})^{4c(n)}} \]

Then
\[ \lim_{r \to 0} \frac{f}{\log \frac{1}{r}} = 0 \]

So for any \( \varepsilon \), there is a \( \delta \) such that \( r < \delta \) implies \(-f + \varepsilon \log \frac{1}{r}\) large enough. Now
\[ \Delta (-f + \varepsilon \log \frac{1}{r}) \leq 0 \]

So the minimum point must be obtained at \( |r| = \frac{1}{2} \). So
\[ f - \varepsilon \log \frac{1}{r} \leq C_1 - \varepsilon \log 2 \leq C_1 \]

for any \( \varepsilon \) small. thus letting \( \varepsilon \to 0 \), we have \( f \leq C_1 \), which completes the proof.

Remark 4.4. In Hayakawa [12], the author claimed a relation between the degeneration of the Calabi-Yau manifold and the noncompleteness of the Weil-Petersson metric. But her proof was incomplete. C-L. Wang [27] gave a proof of this and studied the Weil-Petersson metric in great detail. In particular, he proved an asymptotic estimate for the degeneration of Calabi-Yau manifold which is slightly sharper then our estimate independently using a different method.
Chapter 5

Further Results

In this chapter we discuss some global results of the moduli space or more generally, the horizontal slices. My main prototype is the moduli space of Calabi-Yau threefold. But most results in this chapter are on arbitrary dimensional Calabi-Yau manifolds.

The moduli space of $K-3$ surface is a locally Hermitian symmetric space with finite volume. By Margulis [18], Mok-Siu-Yeung [19], Jost-Yau [14] and Corlette [6], we know that such a moduli space is superrigid (see Corlette [6]). It is then a natural question to ask that if such a super rigidity property or any kind of rigidity property holds true in high dimensions. The difficulty of the problem using differential geometric method is that we don’t have a canonical Kähler metric on the moduli space with good curvature properties. For example, in the case of Calabi-Yau threefold, by the Strominger’s formula [23], the curvature of the Weil-Petersson 4.1 metric only has lower bound, while the curvature of the metric of the Variational of Hodge Structure (VHS metric) (definition 3.5) only has upper bound for the Ricci curvature. The bad properties of curvature prevent us from using analytic tools such as harmonic maps in Mok-Siu-Yeung [19], Jost-Yau [14] and Corlette [6].

In this point of view, moduli space is harder to study then Hermitian symmetric space. On the other hand, however, moduli space can be viewed as complex submanifold
of the Classifying space, and since we have already known the properties of the immersion to the classifying space [cf. Chapter 3], the second variational formula for the arc-length will have already told us some information.

In the first section, we will see an application of the second variational formula gives us the uniqueness of the VHS metric and the Weil-Petersson metric in certain class.

In section two, we study the local rigidity of the monodromy group representation. We use some ideas from Sidney Frankel [8]. This theorem is an intermediate step in our attempts to prove the local rigidity theorem of horizontal slices.

## 5.1 A Metric Rigidity Theorem

In this section, we proved that, for concave horizontal slices (definition 3.2), the VHS metric is intrinsically defined. That is, the VHS metric does not depend on the canonical embedding to the classifying space. The Weil-Petersson metric has the similar property in dimension 3.

To be precise, suppose $\Gamma \backslash \mathcal{M} \rightarrow \Gamma \backslash \mathcal{D}$ is a horizontal slice. Then we can define the VHS metric and the Weil-Petersson metric on $\Gamma \backslash \mathcal{M}$. But as a complex manifold, the horizontal embedding $\Gamma \backslash \mathcal{M} \rightarrow \Gamma \backslash \mathcal{D}$ may not be unique. We are going to prove, the metrics defined on $\Gamma \backslash \mathcal{M}$ is independent of the choice of the embedding for certain class of Kähler manifolds. If the metric has this property, we say such a metric is defined intrinsically.

If $\mathcal{M}$ is the moduli space of some Calabi-Yau manifold, then there is nothing mysterious in it because the Weil-Petersson metric can be defined via the universal family of the Calabi-Yau manifolds — the Ricci flat metric on a Calabi-Yau manifold $X$ has already defined a Hermitian metric on the tangent space $H^1(X, \Theta)$, where $\Theta$ is the holomorphic tangent bundle. But for an abstract horizontal slice, it is an interesting property and it is also interesting to ask if there are any restrictions on a Kähler metric to be
Weil-Petersson metric on an abstract complex manifold.

The assumption to the Kähler metric we give here is that the manifold is concave. At this moment we can not verify it for moduli space or for complete horizontal slices with finite VHS volume. But we believe that it is true at least in some kind of version. It is true in the case of K-3 surfaces.

**Definition 5.1.** We say a complex manifold \( M \) is concave, if there is an exhaustion function \( \varphi \) on \( M \) such that the Hessian of \( \varphi \) has at least two negative eigenvalues at each point outside some compact set.

Suppose \( f_i : \mathcal{M} \to D, i = 1, 2 \) are two horizontal slices. Suppose we have \( \Gamma \in \text{Aut}(\mathcal{M}), \Gamma_0 \in \text{Aut} D \) and we have the group homomorphism

\[ \rho : \Gamma \to \Gamma_0, \]

such that

\[ f_i(\gamma x) = \rho(\gamma)f_i(x), \quad i = 1, 2 \]

Here the action \( \rho(\gamma) \) on \( D \) is just the left translation.

**Theorem 5.1.** With the notations as above. Suppose that \( \Gamma \backslash \mathcal{M} \) has no nonconstant plurisubharmonic functions. Then there is a biholomorphic isometry with respect to the VHS metric \( f : \mathcal{M}_1 \to \mathcal{M}_2 \) such that \( f \circ f_1 = f_2 \).

This proves the uniqueness intrinsic-definedness of the VHS metric. Furthermore we have

**Theorem 5.2.** With the notations as above. Assuming \( M \) is a concave manifold, then the VHS metric is intrinsically defined. Furthermore, in the case of dimension 3, the Weil-Petersson metric is also intrinsically defined.
Proof of theorem 5.1: Let $D_1 = G/K$ be the base symmetric space. For the sake of simplifying notations, we also denote $f_1: M \to G/K$ and $f_2: M \to G/K$ to be the two canonical projections. Let

$$g: M \to R, \quad g(x) = d(f_1(x), f_2(x))$$

where $d(\cdot, \cdot)$ is the distance function of $G/K$. Thus since $G/K$ is a Cartan-Hardamad manifold, $g(x)$ is smooth if $g(x) \neq 0$.

Let $p \in M$ and $X \in T_pM$. Let $X_1 = (f_1)_pX, X_2 = (f_2)_pX$. Let $\sigma$ be the geodesic ray starting at $p$ with vector $X$. i.e.

$$\left\{ \begin{array}{l}
\sigma''(t) = 0 \\
\sigma(0) = p, \sigma'(0) = X
\end{array} \right.$$ 

Suppose the smooth function $\sigma(s, t)$ is defined as follows: for fixed $s$, $\sigma(s, t)$ is the geodesic in $G/K$ connecting $f_1(\sigma(s))$ and $f_2(\sigma(s))$. Furthermore, we assume that $\sigma(0, t)$ is normal. i.e. $t$ is the arc length. define

$$\tilde{X}(s) = \left. \frac{d}{ds} \right|_{s=0} \sigma(s, t)$$

be the Jacobi field of the variation. In particular

$$\left\{ \begin{array}{l}
\tilde{X}(0) = X_1 \\
\tilde{X}(l) = X_2
\end{array} \right.$$ 

where $l = g(x)$. Suppose $T$ is the tangent vector of $\sigma(0, t)$, we have the second variation formula

$$XX(g)|_p = \left< \nabla_{X_2} X_2, T \right> - \left< \nabla_{X_1} X_1, T \right>$$

$$+ \int_0^l \| \nabla_T \tilde{X} \|^2 - R(T, \tilde{X}, T, \tilde{X}) - (T < \tilde{X}, T>)^2$$

where $\nabla$ is the connection operator on $G/K$ and $R(\cdot, \cdot, \cdot, \cdot)$ is the curvature tensor.
We also have the first variation formula
\[ Xg = < (f_2)_*X, T > - < (f_1)_*X, T > \]

Now by theorem 3.5, we have
\[ \nabla_{(f_i)_*X}(f_i)_*X + \nabla_{(f_i)_*JX}(f_i)_*JX + (f_i)_*[X, JX] = 0 \]
for \( i = 1, 2 \). Here \( \nabla \) is the invariant connection on the symmetric space.

Define
\[ D(X, X) = XXg + (JX)(JX)g + J[X, JX]g \]

Using the fact that \( J \) is \( \nabla \)-parallel, we see
\[
D(X, X)g = \int_0^t |\tilde{X}'|^2 - R(T, \tilde{X}, T) - (T < \tilde{X}, T >)^2 \\
+ \int_0^t |\tilde{JX}'|^2 - R(T, \tilde{JX}, T) - (T < \tilde{JX}, T >)^2
\]

where \( \tilde{JX} \) is the Jacobi connecting \( f_1(J\sigma(t)) \) and \( f_2(J\sigma(t)) \).

Claim: If \( g(x) \neq 0 \), then Hessian of \( g \) at \( x \) is semipositive.

Proof: Let \( (\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}) \) be the holomorphic normal frame at \( p \in M \). In order to prove \( g \) is plurisubharmonic, it suffices to prove that \( \frac{\partial^2 g}{\partial z_i \partial z_i} \geq 0 \). But
\[
4 \cdot \frac{\partial^2 g}{\partial z_i \partial z_i} = \frac{\partial^2 g}{\partial x_i^2} + \frac{\partial^2 g}{\partial y_i^2} = D(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i})g
\]

Let \( X = \frac{\partial}{\partial x_i} \) in the second variation formula. Since the curvature of the symmetric space is nonpositive,
\[
D(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i})g \\
\geq \int_0^t |\tilde{X}'|^2 - (T < \tilde{X}, T >)^2 + |\tilde{JX}'|^2 - (T < \tilde{JX}, T >)^2
\]
because the curvature operator is nonpositive. On the other hand
\[
|X'|^2 - (T < \tilde{X}, T >)^2 = |\tilde{X}' - < \tilde{X}', T >|^2 \geq 0
\]
\[
|JX'|^2 - (T < J\tilde{X}, T >)^2 = |J\tilde{X}' - < J\tilde{X}', T >|^2 \geq 0
\]
Thus \(g\) is plurisubharmonic if \(g(x) \neq 0\).

Now \(g^2(x)\) is a smooth function on \(M\). It is easy to see that \(g^2\) is a plurisubharmonic function. But \(g^2\) is also \(\Gamma\)-invariant so it descends to a function on \(\Gamma \backslash M\). Thus \(g^2\) and \(g\) must be a constant.

Now that \(g\) is a constant, we must have
\[
\begin{cases}
\tilde{X}' - < \tilde{X}', T > = 0 \\
R(T, \tilde{X}, T, \tilde{X}) = 0
\end{cases}
\]
On the other hand, we have
\[
< \tilde{X}, T > (0) = < \tilde{X}, T > (l)
\]
from the first variational formula. And we also know \(\tilde{X}\) is a Jacobi field. Thus \(\tilde{X}'' \equiv 0\).

From these information, we know that \(\tilde{X}' \equiv 0\).

This proves that
\[
f : M_1 \to M_2, \quad f_1(x) \mapsto f_2(x)
\]
is a biholomorphic isometry with respect to the VHS metric (see Chapter 2).

In what follows we give a sufficient condition for a manifold with no nonconstant (sub)solutions of certain elliptic equations.

**Definition 5.2.** Suppose \(\Gamma \backslash M\) is a concave manifold with the exhaustion function \(\varphi\).
A real analytic function \(f\) satisfies the Hopf condition, if it is a constant or if there are
constants $c_1, c_2, \cdots, c_i \to +\infty$ for each $z_i \in \{\varphi \leq c_i\}$ with

$$f(z_i) = \max_{\varphi \leq c_i} f(z)$$

we have $\nabla f(z_i) \neq 0$.

**Proposition 5.1.** Suppose $\mathcal{M}$ is concave with the exhaustion function $\varphi$. Then for any real analytic function $f$ satisfying the Hopf condition, $f$ is a constant.

**Proof:** Assuming $f$ is not a constant. We know that the set

$$A = \{c | \varphi(x) = c, \nabla \varphi(x) = 0\}$$

is of zero measure. So we can choose a $c \notin A$ and sufficient big enough such that

$$U = \{x | \varphi(x) \leq c\}$$

is compact and $\partial U$ is smooth and the Hessian of $\varphi$ on has two negative eigenvalues.

Let $z_0 \in U$ such that $f(z_0) = \max_{\partial U} f(x)$. Then $z_0 \in \partial U$, otherwise $\nabla f(z_0) = 0$, a contradiction.

Now suppose $z_0 \in \partial U$. Then for any $z \in \partial U$, $f(z) < f(z_0)$. Let

$$\psi(z) = \frac{\varphi(z_0) - \varphi(z)}{f(z_0) - f(z)}$$

Then $\psi(z) \geq 0$ on $U$ and

**Claim:** Let $P$ be a neighborhood of $z_0$ in $\mathcal{M}$. Then $\psi$ is upper bounded on $P \cap U$.

**Proof:** By the Hopf assumption $\nabla f(z_0) \neq 0$. Then in a neighborhood of $z_0$, $\nabla f \neq 0$. Let $y$ be the point on $\partial U$ such that $\text{dist}(z, \partial U) = \text{dist}(z, y)$. Then we have

$$f(z_0) - f(z) \geq f(y) - f(z) > \varepsilon \text{dist}(y, z)$$

Thus

$$\psi(z) \leq \frac{1}{\varepsilon} (\varphi(z_0) - \varphi(z))/\text{dist}(y, z) = \frac{1}{\varepsilon} (\varphi(y) - \varphi(z))/\text{dist}(y, z) \leq C$$
Now go back to the proof of our proposition, let
\[ f(z) = f(z_0) + a_i(z^i - z_0^i) + b_i(z^i - z_0^i) \]
\[ + c_{ij}(z^i - z_0^i)(z^j - z_0^j) + d_{ij}(z^i - z_0^i)(z^j - z_0^j) \]
\[ + e_{ij}(z^i - z_0^i)(z^j - z_0^j) + O(|z - z_0|^3) \]

Since \( f(z) \) is a real function, we have
\[
\begin{align*}
  a_i &= \overline{b_i} \\
  c_{ij} &= \overline{c_{ij}} \\
  d_{ij} &= \overline{d_{ji}}
\end{align*}
\]

Thus we have
\[
f(z) = f(z_0) + 2\text{Re}(a_i(z^i - z_0^i) + c_{ij}(z^i - z_0^i)(z^j - z_0^j)) \]
\[ + d_{ij}(z^i - z_0^i)(z^j - z_0^j) + O(|z - z_0|^3) \]  
(5.5.1)

Now \( \nabla f(z_0) \neq 0 \). So
\[
S : \quad a_i(z^i - z_0^i) + c_{ij}(z^i - z_0^i)(z^j - z_0^j) = 0
\]
is a complex submanifold of codimension 1. By equation 5.5.1, there is a constant \( C_1 \) such that
\[
f(z) \geq f(z_0) - C_1|z - z_0|^3
\]
for all \( z \in S \).

By the above claim, we know
\[
C_1|z - z_0|^3 \geq f(z_0) - f(z) \geq \frac{1}{C} (\varphi(z_0) - \varphi(z)) \quad z \in S \cap S
\]
So
\[
\varphi(z) \geq \varphi(z_0) - CC_1|z - z_0|^3 \quad z \in S \cap P
\]

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Let $\tilde{\varphi}(z) = \varphi(z) + \varepsilon |z - z_0|^2$. Then $\tilde{\varphi}(z) \geq \tilde{\varphi}(z_0)$, for $z \in S \cap P$ for sufficient small $|z - z_0|$. Thus $\partial \bar{\partial} \tilde{\varphi}(z_0)|_{T_{z_0}S} \geq 0$. So $\partial \bar{\partial} \varphi(z_0)|_{T_{z_0}S} \geq -\varepsilon \to 0$.

On the other hand, $\partial \bar{\partial} \varphi(z_0)$ has two negative eigenvalues. So on $S$, $\partial \bar{\partial} \varphi(z_0)$ has at least 1 negative eigenvalue. This is a contradiction.

Suppose $\Gamma \setminus \mathcal{M}$ is a concave manifold. Then a plurisubharmonic function $f$, $\partial \bar{\partial} f \geq 0$ satisfies the Hopf condition. Thus $f$ is a constant on a concave manifold.

In order to prove that the Weil-Petersson metric is also intrinsically defined in the dimension 3, we need the following Hopf lemma:

**Lemma 5.1.** Let $\mathcal{M}$ be a Kähler manifold. Suppose on an open set $U \subset M$ we have

$$\det(g_{i\bar{j}} + \partial_i \partial_j f) = e^f \det(g_{i\bar{j}})$$

with $(g_{i\bar{j}}) > 0$, $(g_{i\bar{j}} + \partial_i \partial_j f) > 0$. Suppose further that $\partial U$ is smooth. Suppose $z_0 \in \overline{U}$ such that

$$f(z_0) = \max_U f$$

Then if $f$ is not a constant, $\nabla f(z_0) \neq 0$.

**Proof:** Without lossing generality, we assume that $f(z_0) > 0$. Otherwise we can define $f_1 = -f$ and then $f_1$ satisfies

$$\det((g_{i\bar{j}} + f_1\partial_i \partial_j) + (f_1)_{i\bar{j}}) = e^{f_1} \det(g_{i\bar{j}} + f_{i\bar{j}})$$

and $f_1(z_0) > 0$.

Now we see that $z_0 \in \partial U$, otherwise by the maximum principle $f(z_0) \leq 0$.

Since $\partial U$ is smooth, $\partial U$ satisfies the local inner ball condition. Suppose $B_R(x)$ satisfies $B_R(x) \subset U$, $z_0 \in \partial B_R(x)$ and $\partial B_R(x) \cap \partial U = \{z_0\}$. Furthermore, we assume that $R$ is so small that $z \in B_R(x)$ implies $f(z) > 0$. Define

$$h_\beta(z) = f(z_0) - \beta(e^{-d(z,x)^2} - e^{-R^2})$$
Then on $\partial B_R(x)$, we have $h_\beta(z) \geq f(z)$.

**Claim:** $h_\beta(z) \geq f(z)$ on $B_R(x) \setminus B_{\frac{R}{2}}(x)$ for sufficiently small $\beta > 0$.

**Proof:** We know, if $\beta$ is small, then $h_\beta(z) \geq f(z)$ on $\partial B_R(x) \cup \partial B_{\frac{R}{2}}(x)$. If $h_\beta - f$ is not non-negative on $B_R(x) - B_{\frac{R}{2}}(x)$, then there is a minimum point $z'$ inside the ball $B_R(x)$. At $z'$, we have $\partial \partial h(z') \geq \partial \partial f(z')$, Thus

$$e^{f(z')} \det(g_{\overline{j}j}) = \det(g_{\overline{j}j} + f_{\overline{j}j}) \leq \det(g_{\overline{j}j} + h_{\overline{j}j})$$

Letting $\beta \to 0$, we have a sequence $z'_\beta$ such that $\lim_{\beta \to 0} f(z'_\beta) \leq 1$. A Contradiction.

Now return to the proof of the theorem: suppose $\nu$ is the innerward vector at $z_0$, then since $f(z) \leq h_\beta(z)$, we have

$$\nabla f(z_0) \cdot \gamma < \nabla h(z_0) \cdot \gamma < 0$$

**Proof of theorem 5.2:** First, a plurisubharmonic function on a concave manifold is a constant, because it satisfies the Hopf condition. This proves the first part of theorem 5.2.

Now suppose on $\Gamma \setminus \mathcal{M}$. We defined two Weil-Petersson metrics $\omega_1$ and $\omega_2$ by different embedding $f_1$ and $f_2$, we know the corresponding VHS metric is unique by theorem 5.1. So by theorem 4.6 we have

$$(n + 3)\omega_1 + \text{Ric}(\omega_1) = (n + 3)\omega_2 + \text{Ric}(\omega_2)$$

let $f = \log \frac{\omega_1}{\omega_2}$. Then $f$ is a smooth function on $\Gamma \setminus \mathcal{M}$ and

$$(n + 3)\omega_1 = (n + 3)\omega_2 + \frac{\sqrt{-1}}{2} \partial \partial f$$

Let $\omega_2 = \frac{\sqrt{-1}}{2} g_{\overline{j}j} d_i \wedge d\overline{z}_j$. Then in the local coordinate

$$\det(g_{\overline{j}j} + \frac{f_{\overline{j}j}}{n + 3}) = e^f \det(g_{\overline{j}j})$$

By Proposition 5.1 and Lemma 5.1 we see that $\omega_1 \equiv \omega_2$. 

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5.2 Local Rigidity of the Group Representation

In this section we study the monodromy group representation on a horizontal slice. We use the ideas that come from S. Frankel [8] in his study of compact Kähler manifolds of negative Ricci curvature.

We assume that $\mathcal{M}$ is a horizontal slice. Let $\Gamma \subset Aut(\mathcal{M})$ be a discrete group. Suppose $\Gamma \backslash \mathcal{M}$ is of finite volume with respect to the VHS metric.

For the sake of simplicity, we assume that $\Gamma$ is also the subgroup of the left translation of $D = G/V$, the classifying space. There is a natural map $\Gamma \backslash \mathcal{M} \to \Gamma \backslash G/K$ where $G/K$ is the base symmetric space of $D = G/V$ as before. Let

$$\mathcal{G} = \{a \in G|a \in Aut(\mathcal{M})\}$$

Let $\mathcal{G}_0$ be the identity component of $\mathcal{G}$.

The main theorem of this section is

**Theorem 5.3.** Let $\Gamma \backslash \mathcal{M}$ be of finite VHS volume. Suppose further that $\mathcal{G}_0$ is semisimple and $\mathcal{G}_0/K_0$ is a Hermitian symmetric space but is not a complex ball. Here $K_0$ is the maximum compact subgroup of $\mathcal{G}_0$. Then the representation $\Gamma \to \mathcal{G}$ is locally rigid.

By local rigidity we mean that if $\rho_t : \Gamma \to Aut \mathcal{M}$ is a continuous set of representations for $t \in (-\varepsilon, \varepsilon)$. Then if $\varepsilon$ is small enough there is an $a_t$ for any $|t| < \varepsilon$ such that $\rho_t = Ad(a_t)\rho_0$.

Before proving the rigidity theorem, we make an assumption. We postpone the proof of the assumption to the end of this section.

**Assumption 5.1.** Let $K_0$ be a maximal compact subgroup of $\mathcal{G}_0$. Suppose $\Gamma_1 = \Gamma \cap \mathcal{G}_0$.

We assume that $\Gamma_1 \backslash \mathcal{G}_0/K_0$ has finite volume with respect to the standard Hermitian metric on $\mathcal{G}_0/K_0$. In this case, we will call $\Gamma_1$ has finite covolume.

Now we prove a series of lemmas.
Let
\[ G_1 = \Gamma + G_0 \]
be the group generated by \( \Gamma \) and \( G_0 \) in \( G \).

Let
\[ \Gamma_1 = \Gamma \cap G_0 \]

**Lemma 5.2.** Let \( \pi : M \to \Gamma \setminus M \) be the projection. Then for any \( x \in M \), the projection of the \( G_0 \) orbit \( \pi(G_0x) \) is a closed, locally connected, properly embedded smooth submanifold of \( \Gamma \setminus M \).

**Proof (cf, [8]):** \( G_0x \) is a closed properly embedded, locally connected smooth submanifold of \( M \), we claim:

**Claim:** \( \pi^{-1}(\pi(G_0x)) = G_1x \).

**Proof:** We know that \( G \subset N(G_0) \), the normalizer of \( G_0 \) in \( G \). So \( \forall \xi \in G_0, b \in \Gamma \), there is a \( \eta \in G_0 \) such that \( b\xi = \eta b \). Thus \( \forall g \in G_1, g = g_1g_2 \) where \( g_1 \in \Gamma \) and \( g_2 \in G_0 \). So
\[ \pi(gx) = \pi(g_1g_2x) = \pi(g_2x) \in \pi(G_0x) \]

Thus \( gx \in \pi^{-1}\pi(G_0x) \).

On the other hand, if \( y \in \pi^{-1}\pi(G_0x) \), then \( \pi(y) \in \pi(G_0x) \), thus by definition, \( y \in G_1x \).

Now \( G_1x \) is a properly embedded, locally connected smooth submanifold of \( M \) and \( G_1x \) is \( \Gamma \) invariant. So the lemma is proved.

Recall the Margulis theorem:

**Theorem 5.4 (Margulis).** Suppose that \( G_0 \) is defined as above. If \( \Gamma_1 \) is of finite co-volume, then for any homomorphism
\[ \varphi : \Gamma_1 \to \Gamma_1 \]
we have a unique extension

\[ \tilde{\varphi} : G_0 \to G_0 \]

of group homomorphism.

**Lemma 5.3.** If \( x \in G_0 \) such that

\[ xy = yx \]

for all \( y \in \Gamma_1 \). Then \( x = e \)

**Proof:** Let \( \varphi : \Gamma_1 \to \Gamma_1 \) by \( y \to xyx^{-1} \). But \( \varphi \) has an extension \( \tilde{\varphi} : G_0 \to G_0 \). This extension is unique. Thus we get two extensions: first \( \tilde{\varphi}(y) = xyx^{-1} \); second \( \tilde{\varphi}(y) = y \). So they should be equal. \( \square \)

**Lemma 5.4.** Let \( \Gamma_1 = \Gamma \cap G_0 \), then

\[ \frac{Out(\Gamma_1)}{Inn(\Gamma_1)} \]

is a finite group.

Here \( Out(\Gamma_1) \) denotes the group of isomorphisms of \( \Gamma_1 \) and \( Inn(\Gamma_1) \) denotes the group of conjugations of \( \Gamma_1 \).

**Proof:** Let \( \varphi : \Gamma_1 \to \Gamma_1 \) be an element in \( Out(\Gamma_1) \). Since \( \Gamma_1 \) has finite covolume, we know there is a unique extension \( \tilde{\varphi} : G_0 \to G_0 \).

Thus \( \tilde{\varphi} \in Out(G_0) \). Since \( Out(G_0) = Inn(G_0) \) by the Lie group theorem, there is a \( b \in G_0 \) such that \( \tilde{\varphi}(x) = bxb^{-1} \). Define

\[ \tilde{\varphi} : G_0/K_0 \to G_0/K_0 \]

It is a \( \Gamma \)-equivariant holomorphic map. So the lemma follows from the following proposition.
Proposition 5.2. Suppose $\Gamma \backslash G/K$ is of finite volume, then $\text{Aut}(\Gamma \backslash G/K)$ is a finite group.

Proof: Since $\Gamma \backslash G/K$ is a Hermitian symmetric space, we know $\text{Aut}(\Gamma \backslash G/K)$ is the same as $\text{Iso}(\Gamma \backslash G/K)$.

Suppose $\text{Iso}(\Gamma \backslash G/K)$ is not finite. Then we have a sequence of isometries $f_1, f_2, \ldots$. Let $p \in \Gamma \backslash G/K$ be a fixed point and let $U$ be a normal coordinate of $p$. The we know that $f_i(p)$ must be bounded, otherwise there is a subsequence of $f_i$ such that $f_i(U)$ will be mutually disjoint. This will contradict to the fact that $\Gamma \backslash G/K$ has finite volume, because

$$\text{vol}(\Gamma \backslash G/K) \geq \sum \text{vol}(f_i(U)) = \sum \text{vol}(U)$$

So $f_i(p)$ is bounded and there is a point $q \in \Gamma \backslash G/K$ such that $q = \lim f_i(p)$. For any $x \in \Gamma \backslash G/K$, if $i$ is large enough such that $d(f_i(p), q) < 1$, then

$$d(f_i(x), q) \leq d(f_i(x), f_i(p)) + 1 = d(x, p) + 1$$

By Ascoli theorem, there is a subsequence of $f_i$ such that $f_i$ converges to an $f \in \text{Iso}(\Gamma \backslash G/K)$. Thus $\text{Iso}(\Gamma \backslash G/K)$ is not discrete. So there is a holomorphic vector field $X$ on $\Gamma \backslash G/K$.

Suppose $X = X^i \frac{\partial}{\partial z^i}$ in local coordinate, and $||X||^2 = G_{ij}X^i\overline{X^j}$. Suppose the local coordinate is normal, then

$$\partial_k\overline{\partial_l}||X||^2 = R_{ijkl}X^i\overline{X^j} + \partial_k X^l \overline{\partial_l X^j} \quad (5.5.2)$$

where $R_{ijkl}$ is the curvature tensor of the symmetric space. Thus in particular $\partial\overline{\partial}||X||^2 \geq 0$.

By a theorem of [2], $\Gamma \backslash G/K$ is a concave manifold. Thus $||X||^2$ is a constant. On the other hand, from equation (5.5.2), we have

$$\Delta||X||^2 = \text{Ric}(X) + |\nabla X|^2$$

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So $\text{Ric}(X) = 0$ and thus $X \equiv 0$. This is a contradiction. 

Thus there is an integer $n$ such that $a^n = e$ for all

$$a \in \text{Out}(\Gamma_1)/\text{Inn}(\Gamma_1)$$

Let $\tilde{\Gamma}$ be the subgroup of $\mathcal{G}_1$ generated by $\Gamma_1$ and $a^n$ where $a \in \Gamma$. Then we have an exact sequence

$$1 \to \Gamma_1 \to \tilde{\Gamma} \to \tilde{B} \to 1 \quad (5.5.3)$$

where $\tilde{B}$ is the quotient $\tilde{\Gamma}/\Gamma_1$. For any $\tilde{b} \in \tilde{\Gamma}/\Gamma_1$ with $b \in \tilde{\Gamma}$, we have $b\Gamma_1b^{-1} \subset \Gamma_1$. So $b \in \text{Out}(\Gamma_1)$. But by the definition of $\tilde{\Gamma}$, $b$ is a trivial element in $\text{Out}(\Gamma_1)/\text{Inn}(\Gamma_1)$. So there is a $c \in \Gamma_1$ such that $bc$ is commutative to $\Gamma_1$. So we have

$$\eta: B \to \tilde{\Gamma}, \quad \tilde{b} \mapsto bc$$

We can define a homomorphism

$$\xi: \Gamma_1 \times \tilde{\Gamma}/\Gamma_1 \to \tilde{\Gamma}$$

such that

$$\xi(a, b) = a\eta(b)$$

which is an isomorphism. In other words, the exact sequence (5.5.3) splits.

**Lemma 5.5.** Let $\tilde{\mathcal{G}}_1 = \tilde{\Gamma} + \mathcal{G}_0$. Then

$$\tilde{\mathcal{G}}_1 = \mathcal{G}_0 \times \tilde{B}$$

**Proof:** We have

$$\tilde{\Gamma} = \Gamma_1 \times \tilde{B}$$
Define
\[
\varphi : \mathcal{G}_0 \times \tilde{B} \to \tilde{\mathcal{G}}_1, \quad \varphi(a,b) = ab
\]
Then \(\varphi\) is an isomorphism.

Thus we know a family of representation of \(\tilde{\Gamma}\) splits to the representation to the
discrete group \(\tilde{B}\) and Lie group \(\mathcal{G}_0\) respectively.

**Lemma 5.6.** If the representation \(\tilde{\Gamma} \to \tilde{\mathcal{G}}_1\) is locally rigid, then the representation \(\Gamma \to \tilde{\mathcal{G}}\)
is also locally rigid.

**Proof:** Let \(\varphi_t : \Gamma \to \tilde{\mathcal{G}}\) be a local family of representations, \(t \in (-\varepsilon, \varepsilon)\). Then we see
that \(\varphi_t\) restricts to a trivial family of representations on \(\tilde{\Gamma}\). That is, there are \(a_t \in \tilde{\mathcal{G}}_1 \subset \tilde{\mathcal{G}}\)
with \(a_0 = e\) such that \(\varphi_t(x) = a_t \varphi_0(x) a_t^{-1}\) for \(x \in \tilde{\Gamma}\). Let \(\xi_t = \text{Ad}(a_t^{-1})\varphi_t\). Then we
know \(\xi_t(x) = \varphi_0(x)\) for all \(x \in \tilde{\Gamma}\). Now if \(x \in \Gamma\), then \(x^n \in \tilde{\Gamma}\). So \((\xi_t(x))^n = (\varphi_0(x))^n\)
and \(\xi_0(x) - \varphi_0(x)\). Thus \(\xi_t(x) = \varphi_0(x)\).

Now we prove the assumption 5.1.

**Lemma 5.7.** \(\mathcal{G}_1\) is a closed subgroup of \(\mathcal{G}\).

**Proof:** We know that \(\mathcal{G}_1 \subset \mathcal{G}\). Let \(x_m \in \mathcal{G}_1\) such that \(x_m \to x\) for \(x \in \mathcal{G}\). Then
\(x \in \text{Aut}(\mathcal{M})\) so \(x \in \mathcal{G}\). Thus for sufficient large \(m\), \(x_m\) and \(x\) are in the same component.
In particular, we have \(x \in \mathcal{G}_1\)

**Lemma 5.8.** Let \(p \in \mathcal{M}\), we have
\[
\inf_{q \in \mathcal{G}_1 \setminus \mathcal{G}_0} d(qp, \mathcal{G}_0 p) > 0
\]

**Proof:** Suppose the assertion is not true, then we have \(\{q_m\} \in \mathcal{G}_1\) and \(g_m \in \mathcal{G}_0\) such
that
\[
d(q_mp, g_mp) \to 0, \quad m \to +\infty
\]
or

\[ d(g_m^{-1}q_mp, p) \to 0, \quad m \to +\infty \]

It is easy to check that \( G_0p \) is a homogeneous manifold, with compact stable group. Thus, there are \( k_m \in \mathcal{K}_0 \), a compact subgroup of \( G_0 \) such that

\[ g_m^{-1}q_mk_m \to e \]

So by passing a subsequence if necessary, we know

\[ g_m^{-1}q_m \to g \in \mathcal{K}_0 \subset G_0 \]

This contradicts the fact that \( G_0 \) is open.

Let \( x, y \in \mathcal{M} \). Let

\[ L_1 = G_0x, \quad L_2 = G_0y \]

be the two \( G_0 \) orbits. We can define

\[ f(p) = d(p, L_2) \]

be the distance of a point \( p \in L_1 \) to \( L_2 \).

**Lemma 5.9.** \( f(p) \) is a constant.

**Proof:** Let \( p, q \in L_1 \). then there is a \( g \in \mathcal{G}_0 \) such that \( q = gp \). We have

\[ d(q, L_2) \leq d(q, \xi) = d(gp, \xi) = d(p, g^{-1}\xi) \]

So we have

\[ d(q, L_2) \leq d(p, L_2) \]

On the other hand, we also have

\[ d(q, L_2) \leq d(p, L_2) \]
Thus $f(p)$ is a constant.

Thus we can define the distance between two orbits $L_1, L_2$ by $d(L_1, L_2) = f(p)$. We know if $L_1 \neq L_2$, $d(L_1, L_2) > 0$.

Let $\pi \in G_1/G_0$. Then $\pi L$ defines another orbit. So we have a map

$$G_1/G_0 \to \mathbb{R}, \quad \pi \mapsto d(\pi L, L)$$

We know that $d(\pi L, L) > 0$ for $\pi \neq 0$. Furthermore we have

**Lemma 5.10.**

$$\epsilon = \inf_{\pi \neq 0} d(\pi L, L) > 0$$

**Proof:** This is a consequence of the previous two lemmas.

Now we turn to $\Gamma \setminus \mathcal{M}$. In the previous section, we have proved that for any orbit $G_0 p$, $\Gamma \setminus G_0 p$ are closed, properly embedded submanifolds. We fix one of them, say $L$.

Let

$$U = \{ x \in \mathcal{M} | d(x, L) < \frac{\epsilon}{100} \}$$

where $\epsilon$ is defined in the previous lemma. Then for any $a \in G_1 \setminus G_0$, $a U \cap U = \emptyset$. In particular

$$\Gamma \setminus \mathcal{M} = \Gamma_1 \setminus \mathcal{M}$$

in $\Gamma \setminus \mathcal{M}$.

Now

$$\text{vol}(\Gamma \setminus \mathcal{M}) \geq \text{vol}(\Gamma \setminus U) = \text{vol}(\Gamma_1 \setminus U)$$

For any $p \in \Gamma_1 \setminus U$, there is a unique $q \in L$ such that

$$d(p, q) = d(p, L)$$

Now we can prove
Theorem 5.5. If $vol(\Gamma \setminus \mathcal{M}) < +\infty$, then

$vol(\Gamma \setminus L) < +\infty$

Proof: Let $f(p) = d(p, L)$. Then by the coarea formula

$$vol(\Gamma_1 \setminus U) = \int_0^\epsilon \left( \int_{f=c} \frac{1}{|\nabla f|} \right) dc$$

But $|\nabla f| \leq 1$. So

$$vol(\Gamma_1 \setminus U) \geq \int_0^\epsilon vol(f = c) dc$$

so at least there is a $c$ s.t.

$$vol(f = c) < \infty$$

The theorem follows by induction.
Bibliography


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