THE LOG TERM OF THE SZEGÖ KERNEL

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Abstract
In this paper, we study the relations between the log term of the Szegö kernel of the unit circle bundle of the dual line bundle of an ample line bundle over a compact Kähler manifold. We prove a local rigidity theorem. The result is related to the classical Ramadanov conjecture.

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1. Introductions
Prescribing geometric structures of a complex manifold often introduces interesting and important partial differential equations. A typical example of this kind is the problem of finding Kähler metrics with constant scalar curvature on a Kähler manifold. Such a problem defines a fourth-order elliptic partial differential equation. The study of these partial differential equations, including the Kähler-Einstein equations, forms one of the richest topics in complex geometry.

In this paper, we introduce a new set of equations coming from the Szegö kernel (resp., Bergman kernel) of a unit circle (resp., unit disk) bundle. We prove that these equations, which generalize the equation for finding Kähler metrics with constant scalar curvature, are all elliptic. As an application of the result, we relate the
Ramadanov conjecture to these equations and prove a local rigidity theorem concerning the log term of the Szegő kernel.

Our basic setting is as follows. Let \((L, h) \to M\) be a positive Hermitian line bundle over the compact complex manifold \(M\) of dimension \(n\). The pair \((M, L)\) is called a polarized manifold. The Kähler metric \(\omega\) of \(M\) is defined to be the curvature of the Hermitian metric \(h\). Let \(L^*\) be the dual bundle of \(L\). The unit circle bundle \(X\) of \(L^*\) is a strictly pseudoconvex manifold with the natural measure defined by the \(S^1\) action and the polarization of \(M\). That is, the measure is \(dV = \frac{1}{n!} \pi^* (\omega^n) \wedge d\theta\), where \(\frac{\partial}{\partial \theta}\) is the infinitesimal \(S^1\)-action on the unit circle bundle. The Szegő projection \(\Pi\) is a linear map from \(L^2(X)\) to the Hardy space \(H^2(X)\), which is the space of \(L^2\) boundary functions of holomorphic functions of the unit disk bundle \(D\). Let \(\Pi(x, y)\) be the Szegő kernel of \(X\); that is, let \(\Pi(x, y)\) be the function on \(X \times X\) such that for any \(f \in L^2(X)\), \(\int_X \Pi(x, y) f(y) dy \in H^2(X)\), where \(dy = dV\) is the measure defined above. Then, by [5], there is a paramatrix

\[
s(x, y, t) \sim \sum_{k=0}^{\infty} t^{n-k} s_k(x, y),
\]

where \(s_k(x, y)\) \((k \in \mathbb{Z}_+)\) are smooth functions on \(X \times X\) and \(t \in \mathbb{R}\), such that

\[
\Pi(x, y) = \int_0^\infty e^{it\psi(x, y)} s(x, y, t) dt
\]

(1.1)

for some suitable complex phase function \(\psi(x, y)\) of \(X \times X\).

In general, the paramatrix of the Szegő kernel of a pseudoconvex manifold is quite difficult to compute. However, since the bundle \(X\) is \(S^1\)-invariant, we may split the Szegő kernel into several pieces. More precisely, let \(\frac{\partial}{\partial \theta}\) be the infinitesimal \(S^1\)-action of \(X\). Define

\[
H^2_m(X) = \left\{ f \in H^2(X) \mid \frac{\partial}{\partial \theta} f = \sqrt{-1} m f \right\}.
\]

Let \(\Pi_m\) be the projection of \(H^2(X)\) to \(H^2_m(X)\). Then the kernel \(\Pi_m(x, y)\) of \(L^2(X) \to H^2_m(X)\) is the Fourier coefficient of \(\Pi(x, y)\):

\[
\Pi_m(x, y) = \frac{1}{2\pi} \int_{S^1} \Pi(x, e^{i\theta} y) e^{m\sqrt{-1} \theta} d\theta.
\]

Using the paramatrix of the Szegő kernel, Zelditch [26] (and Catlin [6] independently for the Bergman kernel) was able to prove that there is an asymptotic expansion of \(\Pi_m(x, x)\) (cf. Theorem 2.1),

\[
\Pi_m(x, x) \sim m^n \left( a_0 + \frac{a_1}{m} + \cdots \right),
\]

(1.2)

where the \(a_k\)’s are all smooth functions of \(M\). The expansion is called the Tian-Yau-Zelditch expansion. In [19], Lu was able to prove that all \(a_k\)’s are polynomials of the
curvature and its derivatives. In particular, $a_0 = 1$ and $a_1 = \rho$, the scalar curvature of the Kähler manifold. Thus the equation for finding the metrics such that $a_1 = \text{const}$ is the equation for finding the Kähler metrics with constant scalar curvature.

Because of the work of Donaldson [10], it is natural to study metrics with $a_k$ prescribed for $k \geq 2$. Donaldson was interested in modifying $h^m$ to a Hermitian metric $h'$ for some large $m$ such that the metric $h'$ is balanced. As a corollary of his result, Donaldson was able to give a proof of the uniqueness of the Kähler metrics of constant scalar curvature. Since $a_1 = (1/2)\rho$, where $\rho$ is the scalar curvature, $\int_M (a_1 - \bar{a}_1) \theta$ defines the Futaki invariants, where $\bar{a}_1$ is the average of $a_1$ and $\theta$ is the Hamiltonian function of a holomorphic vector field. Nonlinearizing the Futaki invariants, we get the Mabuchi $K$ energy, whose convexity plays the key role in proving the uniqueness of the metrics of constant scalar curvature (cf. [7]).

We wish to study the analogous problems for $a_k$ when $k > 1$. In this paper, among the other results, we prove that for any given $k$ and function $f$, for a fixed metric $\omega$, the equation for finding the function $\phi$ such that $a_k (\omega - (\sqrt{-1}/(2\pi)) \partial \bar{\partial} \phi) = f$ is an elliptic equation of order $2k + 2$. Thus prescribing $a_k$ gives an interesting set of new elliptic equations.

Since the fact that the Bergman potential $\Pi_m(x, x)$ is a constant implies stability (cf. [20], [27]), we are particularly interested in the question of finding metrics such that $a_k = 0$ for $k > n$. Such a question is related to the Ramadanov conjecture (see [22]). The conjecture, in terms of the Bergman kernel, can be stated as follows.

**Conjecture (Ramadanov [22])**

*Let $\Omega$ be a bounded strongly pseudoconvex domain of $\mathbb{C}^n$. Assume that the log term of the Bergman kernel is zero; then $\Omega$ is biholomorphic equivalent to the unit ball of $\mathbb{C}^n$.*

Not much is known about the conjecture for $n > 1$. If $\Omega$ is a complete Rienhardt domain of $\mathbb{C}^2$, the conjecture was proved by Nakazawa [21]. The conjecture was proved to be true for any strongly pseudoconvex domain in $\mathbb{C}^2$ by Graham [12] using an unpublished note of Burns. In [16], the computation needed in Graham’s proof was given. Boutet de Monvel [4] gave an independent proof of Graham’s result around the same time.

There are only partial results in higher dimensions. K. Hirachi proved (see [15]) that the Radamanov conjecture is true for real ellipsoids that are sufficiently close to the unit ball. In the Szegö kernel case, he proved (see [14]) that if $n = 2$ with tranversal symmetry and if the log term of the Szegö kernel vanishes to the third order, then the boundary is spherical. Hanges [13] proved a similar result with an additional assumption on the choice of volume element on the boundary (see [15] for
further references for the conjecture).

One can form a similar conjecture for the Szegö kernel as well. The same conjecture makes sense if we replace the bounded strongly pseudoconvex domains by strongly pseudoconvex manifolds.

Let $H^\ast$ be the universal line bundle of the complex projective space $\mathbb{C}P^n$. The unit circle bundle $X$ of $H^\ast$ is the unit sphere in $\mathbb{C}^n$. Using this observation, we form the following Ramadanov conjecture for the unit circle bundle $X$.

**CONJECTURE**

Let $\omega \in [\omega_{FS}]$ be a Kähler metric on $\mathbb{C}P^n$ which is in the same cohomology class as the Fubini-Study metric $\omega_{FS}$. Let $(H, h)$ be the hyperplane bundle whose curvature is $\omega$. Let $X$ be the unit circle bundle of the universal line bundle $H^\ast$. If the log term of the Szegö kernel of $X$ vanishes, then there is an automorphism $\varphi : \mathbb{C}P^n \to \mathbb{C}P^n$ such that $\varphi^* \omega = \omega_{FS}$.

We confirm the above conjecture for $n = 1$ in this paper. In the case $n = 1$, the unit circle bundle $X$ is of dimension 3 and the result is parallel to the case of strongly pseudoconvex domains in $\mathbb{C}^2$ in the Ramadanov conjecture.

**THEOREM 1.1** (The case $n = 1$)

Let $\omega \in [\omega_{FS}]$ be a Kähler form on $\mathbb{C}P^1$ which is in the same cohomology class as the Fubini-Study metric $\omega_{FS}$. Let $(H, h)$ be the hyperplane bundle whose curvature is $\omega$. Let $X$ be the unit circle bundle of the universal line bundle $H^\ast$. If the log term of the Szegö kernel of $X$ vanishes, then there is an automorphism $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $\varphi^* \omega = \omega_{FS}$.

The conjecture is still open in high dimensions. The main result of this paper is the following local rigidity theorem.

**THEOREM 1.2**

Let $h$ be the standard metric on $(\mathbb{C}P^n, H)$. Let $h'$ be another metric on the hyperplane bundle $H$ over $\mathbb{C}P^n$. Then there is an $\varepsilon > 0$, depending only on $n$, such that for any $h'$ with

$$\left\| \frac{h'}{h} - 1 \right\|_{C^{2n+4}} < \varepsilon,$$

where the log term of the Szegö kernel of the unit circle bundle of $h'$ is zero, there is an automorphism $f$ of $\mathbb{C}P^n$ such that $f^* (\omega_{h'}) = \omega_{FS}$, where $\omega_{h'}$ is the curvature of $h'$.

The organization of this paper is as follows. In Section 2, we prove that if the log
term is zero, then \( a_k = 0 \) for \( k > n \). This result lets us use the methods of partial differential equations to study the Ramadanov conjecture. In Section 3, by tracing the terms in \( a_k \) of the highest Weyl weight, we prove that the equations \( a_k = f \) are all elliptic equations. As a corollary, we get the Schauder estimate (see Corollary 3.1).

In Theorems 4.1 and 4.2, we study the uniformity of the Tian-Yau-Zelditch expansion. Using the elliptic estimate, we have the following general result.

**THEOREM 1.3**

If \( h \) is the metric such that the log term of the Szegő kernel (of the unit circle bundle \( X \) of \( L^* \), the dual bundle of the ample line bundle \( L \) over \( M \)) vanishes, then there is a finite-dimensional vector subspace \( V \) of the space of smooth functions on \( M \) such that if \( \varphi / \notin V \), the log term of the Szegő kernel is not zero for the metric \( h + \varepsilon \varphi \) for sufficiently small \( \varepsilon \).

This proves that, generically, the log term is not zero.

The technical heart of this paper is in Sections 6 and 7. In Section 6, we compute concretely the vector space \( V \) in Theorem 1.3. In order to determine the vector space \( V \) in Theorem 1.3, we use the fact that the orthonormal basis of \( H^0(\mathbb{C}P^n, H^m) \) can always be explicitly written. Then Lemma 6.1 and Proposition 6.1 become purely combinatoric. From these results, we prove that \( V \) must be the eigenspace of some eigenvalue of \( \mathbb{C}P^n \). In Theorem 6.2, we further confirm that the eigenvalue must be \((n + 1)\). It is well known that the eigenfunctions with respect to the eigenvalue \((n + 1)\) of \( \mathbb{C}P^n \) are the Hamiltonian functions of holomorphic vector fields on \( \mathbb{C}P^n \). Thus, in order to prove Theorem 1.2, we have to get rid of the actions of the automorphism group of \( \mathbb{C}P^n \) generated by the holomorphic vector fields. (The idea was first used by Bando and Mabuchi [1].) In [9], the authors introduced the concept of a centrally positioned Kähler metric. In our case, we need to know, when a metric is not far away from the Fubini-Study metric, whether we can find a small automorphism under which the metric is centrally positioned. This is done in Lemma 7.2 using the contraction principle.

**2. Pseudoconvex manifolds with zero log term**

In this section, we study the relations between the vanishing of the log term in the Szegő or Bergman kernels and the coefficients in the Tian-Yau-Zelditch expansion (see [19]). Except for Theorem 2.4, or where otherwise stated, most results of this section were known from the work of Zelditch [26] and Catlin [6]. We begin with the following standard settings.

Suppose that \((L, h) \rightarrow M\) is a positive Hermitian line bundle over the compact complex manifold \( M \) as in Section 1. Let \((L^*, h^{-1})\) be the dual bundle. We define a
smooth function $\rho : L^* \to \mathbb{R}$ as follows. Let $U \subset M$ be an open neighborhood of $M$ such that $L^*|_U \overset{\varphi}{=} U \times \mathbb{C}$ is a local trivialization. Let

$$\rho(x) = \frac{1}{h(z)}|v|^2 - 1,$$

where $h(z)$ is the local representation of the Hermitian metric $h$ under the trivialization $\varphi$ and $\varphi(x) = (z, v)$. It is not hard to see that $\rho(x)$ does not depend on the choice of the local trivialization.

**Definition 2.1**

Let $D = \{ x \in L^* \mid \rho(x) \leq 0 \}$. Let $X$ be the boundary $\partial D$ of $D$. We call $D$ and $X$ the unit disk bundle and the unit circle bundle, respectively, of $L^*$.

In what follows, we take the Szegő kernel as an example in our proof. The results of the Bergman kernel are similar, and we state only the theorem, without proof.

By definition, the curvature of $h$ is positive, and thus $X$ is a strongly pseudoconvex manifold.

$X$ is $S^1$-invariant. Let $r_\theta : X \to X$ be defined by $r_\theta(z, v) = (z, ve^{\sqrt{-1}\theta})$. Let $\frac{1}{\partial \theta}$ be the infinitesimal action of $S^1$ on $X$ defined by $r_\theta$. Let $\pi : L^* \to M$ be the projection. Define the measure $d\mu = (1/n!)\pi^*\omega^n \wedge d\theta$ on $X$. The Szegő kernel is defined as the kernel of the projection of the space $L^2(X)$ to the space $H^2(X)$, the Hardy space. By definition, $H^2(X)$ is the space of $L^2$-functions on $X$ which are the boundary values of holomorphic functions on $D$.

Let $\Pi$ be the Szegő projection, and let $\Pi(x, y)$ be its kernel. Then

$$\Pi : L^2(X) \to H^2(X)$$

such that for any $f \in L^2(X)$,

$$\Pi f = \int_X \Pi(x, y) f(y) d\mu(y)$$

is in $L^2(X)$ and is the boundary function of a holomorphic function on $D$.

Suppose that $M$ is covered by finite coordinate charts $\{U_a\}_{a \in I}$. The transition functions of the line bundle $L$ are a set of holomorphic functions $g_{a\beta}$ on $U_a \cap U_\beta \neq \emptyset$. Let $h_a(x)$ be the local representation of the Hermitian metric $h$. Then on $U_a \cap U_\beta \neq \emptyset$ we have

$$h_a|g_{a\beta}|^2 = h_\beta.$$

For each $h_a$, we define a $C^\infty$-function $\tilde{h}(x, y)$ on $U_a \times U_a$ such that $\tilde{h}_a(x, x) = h_a(x)$ and $\tilde{h}_a(x, y)$ is almost analytic in the sense that $\partial_x \tilde{h}_a(x, y)$ and $\partial_y \tilde{h}(x, y)$ vanish at $x = y$ to infinite order. Such a function $\tilde{h}_a(x, y)$ exists by [5].
Let \( \sigma_\alpha \) be the partition of the unity subordinated to the covering \( \bigcup \{ U_\alpha \} \) such that \( \sqrt{\sigma_\alpha} \) are all smooth. Define

\[
h_\alpha(x, y) = \sum_{U_\gamma \cap U_\alpha \neq \emptyset} \sqrt{\sigma_\gamma(x)} \sqrt{\sigma_\gamma(y)} g_{\gamma \alpha}(x) \tilde{g}_{\gamma \alpha}(y) \tilde{h}_{\gamma}(x, y).
\]

It is then easy to check that

\[
h_\alpha(x, y) = h_\beta(x, y) g_{\beta \alpha}(x) g_{\beta \alpha}(y)
\]

on \( U_\alpha \cap U_\beta \neq \emptyset \). Furthermore, \( h_\alpha(x, x) = h_\alpha(x) \), and \( \tilde{\partial}_x h_\alpha(x, y), \tilde{\partial}_y h_\alpha(x, y) \) vanish at \( x = y \) to infinite order.

Let \( x, y \in L^*|U_\alpha \), whose local coordinates are \((z, v)\) and \((w, v')\), respectively. Define a global function

\[
\psi(x, y) = \psi(z, v, w, v') = \frac{1}{i} \left( \frac{1}{h_\alpha(z, w)} v^* v' - 1 \right).
\]

If \( x, y \in X \), write

\[
v = \sqrt{h(z)} e^{i\theta}, \quad v' = \sqrt{h(w)} e^{i\theta'},
\]

where \( \theta, \theta' \) are real numbers. Thus on \( X \) we have

\[
\psi(x, y) = \psi(z, \theta, w, \theta') = \frac{1}{i} \left( \frac{\sqrt{h(z)} \sqrt{h(w)}}{h(z, w)} e^{i(\theta - \theta')} - 1 \right).
\] (2.1)

Let \( \hat{S}_1, \ldots, \hat{S}_d \) be functions on \( H^2(X) \) such that

\[
\frac{\partial}{\partial \theta} \hat{S}_i = \sqrt{-1} m \hat{S}_i
\]

for \( 1 \leq i \leq d \). Using [26], we can identify the functions \( \hat{S}_i \) to the holomorphic sections \( S_i \) of \( L^m \). We assume that \( \hat{S}_1, \ldots, \hat{S}_d \) is an orthonormal set with respect to the measure \( d\mu = (1/n!) \pi^* \omega_R^d \wedge d\theta \). That is,

\[
\int_X \hat{S}_i \hat{S}_j d\mu = \delta_{ij}
\]

for \( 1 \leq i, j \leq d \). Define

\[
\Pi_m(x, y) = 2\pi N \sum_{i=1}^N \hat{S}_i(x) \hat{S}_i(y).
\] (2.2)

The basic identity relating \( \Pi_m \) and \( \Pi \) is the following (cf. [26]):

\[
\Pi_m(x, y) = \int_{S^1} \Pi(x, r\theta y) e^{\sqrt{-1} m \theta} d\theta.
\] (2.3)
It is also well known (cf. [2]∗) that for the pseudoconvex manifold $X$, the Szegő kernel can be written as

$$
\Pi(x, y) = \frac{u(x, y)}{\psi(x, y)^{n+1}} + v(x, y) \log \psi(x, y),
$$

(2.4)

where $u, v$ are $C^\infty$-functions defined on $D \times D$.

Using the above notation, Zelditch [26] (and Catlin [6] for the similar result for the Bergman kernel) proved the following.

**THEOREM 2.1**

With notation as above, we have the asymptotic expansion

$$
\Pi_m(x, x) \sim m^n \left( a_0(x) + \frac{a_1(x)}{m} + \frac{a_2(x)}{m^2} + \cdots \right),
$$

(2.5)

where $a_i$ (i ≥ 1) are smooth functions on $M$ and $a_0(x) = 1$. The expansion is convergent in the sense that

$$
\left\| \Pi_m(x, x) - m^n \left( 1 + \frac{a_1(x)}{m} + \cdots + \frac{a_k(x)}{m^k} \right) \right\|_{C^l} \leq C \frac{1}{m^{k+1}},
$$

(2.6)

where $C$ is a constant depending on $k, l$, and the manifold $M$.

Using Zelditch's result, we prove the following (see [19]).

**THEOREM 2.2**

The coefficients $a_i$ can be written as polynomials of the curvature and their derivatives of $M$. The Weyl weight of $a_k$ is $2k$ for $k = 1, 2, \ldots$. In particular, we have

$$
\begin{align*}
  a_0 &= 1, \\
  a_1 &= \frac{1}{2} \rho, \\
  a_2 &= \frac{1}{3} \Delta \rho + \frac{1}{24} (|R|^2 - 4|\text{Ric}|^2 + 3 \rho^2), \\
  a_3 &= \frac{1}{8} \Delta \Delta \rho + \frac{1}{24} \text{div div}(R, \text{Ric}) - \frac{1}{6} \text{div div}(\rho \text{Ric}) \\
  &\quad + \frac{1}{48} \Delta (|R|^2 - 4 |\text{Ric}|^2 + 8 \rho^2) + \frac{1}{48} \rho (\rho^2 - 4 |\text{Ric}|^2 + |R|^2) \\
  &\quad + \frac{1}{24} (\sigma_3(\text{Ric}) - \text{Ric}(R, R) - R(\text{Ric}, \text{Ric})),
\end{align*}
$$

where $R, \text{Ric}$, and $\rho$ represent the curvature tensor, the Ricci curvature, and the scalar curvature of $g$, respectively, and $\Delta$ represents the Laplacian of $M$. For the precise definition of the terms in the expression of $a_3$, see [19, Section 5].

*In [2], the result was written in terms of strongly pseudoconvex domains of $C^n$, but it is also true for strongly pseudoconvex manifolds.*
For the above settings, the famous Ramadanov conjecture [22] (in terms of the Szegő kernel) states that if the function $v(x, y)$ in (2.4) is identically zero, then the manifold $X$ must be the sphere.

The main result of this section is the following.

**THEOREM 2.3**

*Let $X$ be the unit circle bundle of $L^*$ over $M$. If $v(x, y)$ (i.e., the log term) of the Szegő kernel of $X$ vanishes, then the coefficients $a_k$ in Theorem 2.1 vanish for $k > n$.***

**Proof**

Using (2.1) and (2.3), if $v \equiv 0$ in (2.4), then we have the identity

$$\Pi_m(x, x) = \int_{S^1} \frac{(\sqrt{-1})^{n+1}u(x, r_0 x)}{(e^{-\sqrt{-1} \theta} - 1)^{n+1}} e^{\sqrt{-1} m \theta} d\theta.$$  

We prove that the above expression expands to a polynomial of $m$. Let $b > 1$ be a real number; then the above integration is understood as

$$\Pi_m(x, x) = \lim_{b \to 1} \int_{S^1} \frac{(\sqrt{-1})^{n+1}u(x, r_0 x)}{(e^{-\sqrt{-1} \theta} - b)^{n+1}} e^{\sqrt{-1} m \theta} d\theta.$$  

Using integration by parts $n$ times, we get

$$\Pi_m(x, x) = \lim_{b \to 1} \int_{S^1} \frac{\xi(x, \theta, m)}{e^{-\sqrt{-1} \theta} - b} e^{\sqrt{-1} m \theta} d\theta,$$

where $\xi(x, \theta, m)$ is a polynomial of $m$ and the coefficients are smooth functions of $x$ and $\theta$. By the Riemann-Lebesgue lemma, we know that the above expression has the same asymptotic expansion as

$$\Pi_m(x, x) = \xi(x, 0, m) \cdot \lim_{b \to 1} \int_{S^1} \frac{1}{e^{-\sqrt{-1} \theta} - b} e^{\sqrt{-1} m \theta} d\theta.$$  

Thus there is a polynomial $P(x, m)$ of $m$ of degree less than or equal to $m$ such that

$$\Pi_m(x, x) \sim P(x, m)$$

in the sense that

$$\left| \Pi_m(x, x) - P(x, m) \right| < \frac{C}{m^k}$$

for any $k$. Comparing the above result with the expansion in Theorem 2.1, we get the conclusion.

We can also prove the above result using the stationary phase theorem as follows: the theorem of Boutet de Monvel and Sjöstrand [5, Theorem 1.5, Section 2.c] (see
also [26]) states that there exists a symbol \( s \in S^n(X \times X \times \mathbb{R}^+) \) of the type
\[
s(x, y, t) \sim \sum_{k=0}^{\infty} t^{n-k} s_k(x, y)
\]
such that
\[
\Pi(x, y) = \int_0^\infty e^{it\psi(x, y)} s(x, y, t) \, dt.
\] (2.7)

Using (2.3), we have (see [26])
\[
\Pi_m(x, x) \sim \sum_{k=0}^{\infty} m^{n-k+1} \int_0^\infty \int_{S^1} e^{im((t/i)(e^{i\theta} - 1) - \theta) - 2(t - 1)\theta - i\theta^2} t^{n-k} s_k(r\theta x, x) \, dt \, d\theta.
\] (2.8)

By the stationary phase method (see [18, Theorem 7.7.5]), we have
\[
\int_0^\infty \int_{S^1} e^{im((t/i)(e^{i\theta} - 1) - \theta) - 2(t - 1)\theta - i\theta^2} t^{n-k} s_k(r\theta x, x) \, dt \, d\theta \sim 2\pi \sum_j m^{-j-1} L_j(t^{n-k} s_k(r\theta x, x)),
\]
where
\[
L_j(t^{n-k} s_k(r\theta x, x)) = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu \mu} \left( \frac{\partial^2}{\partial t \partial \theta} - i \frac{\partial^2}{\partial t^2} \right)^\nu \left( g(t) \frac{t^{n-k} s_k(r\theta x, x)}{\mu!} \right)\]
for
\[
g(t, \theta) = \frac{t}{i}(e^{i\theta} - 1) - 2(t - 1)\theta - i\theta^2
\]
(cf. [26]). If \( j > n - k \), then
\[
\left( \frac{\partial}{\partial t} \right)^\nu \left( g(t) \frac{t^{n-k} s_k(r\theta x, x)}{\mu!} \right) \equiv 0.
\]
Thus if \( j > n - k \), \( L_j(t^{n-k} s_k(r\theta x, x)) \equiv 0 \). From (2.8), we see that
\[
\Pi_m(x, x) \sim C \sum_{j+k \leq n} m^{n-k-j} L_j(t^{n-k} s_k(r\theta x, x)).
\]
Comparing this to (2.5), we have \( a_k = 0 \) for \( k > n \). \( \square \)

For the Bergman kernel of \( D \), we have the following parallel result.

**THEOREM 2.4**

If the log term of the Bergman kernel \( B(x, y) \) of \( D \) vanishes, then the coefficients \( a_k \) in Theorem 2.1 vanish for \( k > n \).
3. Order of the coefficients

Let \( d = d_m = \dim \mathbb{C} H^0(M, L^m) \) for a fixed integer \( m \). Let \( \{ S_0, \ldots, S_{d-1} \} \) be a basis of \( H^0(M, L^m) \). The metrics \( (h, \omega_g) \) define the \( L^2 \) inner product \( (\cdot, \cdot) \) on \( H^0(M, L^m) \) as

\[
(S_A, S_B) = \int_M (S_A, S_B) \, dV_g, \quad A, B = 0, \ldots, d - 1,
\]

where \( (S_A, S_B) \) is the pointwise inner product with respect to \( h_m \) and \( dV_g = (1/n!) \omega_g^n \). If \( T_0, \ldots, T_{d-1} \) is an orthonormal basis of \( H^0(M, L^m) \) with respect to the above inner product, then we define the sum of the pointwise norm

\[
\|T_0\|^2(z) + \cdots + \|T_{d-1}\|^2(z) = \langle T_0, T_0 \rangle(z) + \cdots + \langle T_{d-1}, T_{d-1} \rangle(z)
\]

to be the Bergman potential of the metric. The key link between the Bergman potential and the Szegő kernel is (cf. [26])

\[
\Pi_m(x, x) = \|T_0\|^2(z) + \cdots + \|T_{d-1}\|^2(z),
\]

where \( \pi(x) = z \), and \( \Pi_m(x, x) \) is defined in (2.2).

Now we assume that \( S_0, \ldots, S_{d-1} \) is a basis of the space \( H^0(M, L^m) \). We further assume that at a point \( z \in M \),

\[
S_0(z) \neq 0, \quad S_A(z) = 0, \quad A = 1, \ldots, d - 1.
\]

Suppose

\[
F_{AB} = (S_A, S_B), \quad A, B = 0, \ldots, d - 1.
\]

Then \( (F_{AB}) \) is the metric matrix that is positive Hermitian. Let \( (I_{AB}) \) be the inverse matrix of \( (F_{AB}) \). Let \( x \in X \) such that \( \pi(x) = z \). Then by linear algebra, we have (cf. [19])

\[
\Pi_m(x, x) = I_{00}\|S_0(z)\|^2_{h_m}, \quad (3.1)
\]

where \( \|S_0(z)\|^2_{h_m} = \langle S_0(z), S_0(z) \rangle \) is the pointwise norm of the section \( S_0(z) \).

The main result of this section is to prove that all the coefficients \( a_k \) in the theorem of Zelditch (see Theorem 2.1) can be represented by \( C \Delta^{k-1} \rho + \text{lower-order terms} \), where \( C \neq 0 \) is a constant depending only on \( k \) and \( n \), and \( \rho \) is the scalar curvature of \( M \). To make the above statement rigorous, we need the following definition.

**Definition 3.1**

Let \( R \) be a component of the \( i \)th-order covariant derivative of the curvature tensor, or the Ricci tensor, or the scalar curvature at a fixed point where \( i \geq 0 \). Define the weight \( w(R) \) and the order \( \text{ord}(R) \) of \( R \) to be the numbers \( (1 + i/2) \) and \( i/2 \), respectively. For example,

\[
w(R_{ijkl}) = w(R_{ij}) = w(\rho) = 1,
\]

\[
\text{ord}(R_{ijkl}) = \text{ord}(R_{ij}) = \text{ord}(\rho) = 0
\]
and
\[ w(R_{ijkl,m}) = \frac{3}{2}, \]
\[ \text{ord}(R_{ijkl,m}) = \frac{1}{2}. \]

In particular, the weight and the order of a constant are zero. The concepts of weight and order can be extended to monomials of the curvature and its derivatives by assuming that
\[ w(f_1 f_2) = w(f_1) + w(f_2), \]
\[ \text{ord}(f_1 f_2) = \text{ord}(f_1) + \text{ord}(f_2), \]
where \( f_1, f_2 \) are monomials. If \( f = \sum f_i \) with \( f_i \) monomials of the same weight or order, then we define \( w(f) = w(f_1) \) and \( \text{ord}(f) = \text{ord}(f_1) \), respectively.

**Remark 3.1**
The definition of the weight here is half of the Weyl weight in Fefferman’s paper [11].

Let \( A \) be the set of all monomials of the curvature and its derivatives at a fixed point \( z \in M \). Define
\[ A' = \{ f \in A \mid \text{ord}(f) \leq w(f) - 2 \}. \]
Let \( B \) and \( B' \) be the complex vector spaces generated by \( A \) and \( A' \), respectively. We have the following simple relation between weight and order.

**Lemma 3.1**
*For any \( f_1, f_2 \in A \) with \( w(f_1), w(f_2) \neq 0 \), we have \( f_1 f_2 \in A' \). In particular, \( B' \) is an ideal of \( B \).*

**Proof**
If \( w(f_1), w(f_2) \neq 0 \), then we have
\[ \text{ord}(f_i) \leq w(f_i) - 1, \quad i = 1, 2. \]
Thus
\[ \text{ord}(f_1 f_2) = \text{ord}(f_1) + \text{ord}(f_2) \leq w(f_1) + w(f_2) - 2 = w(f_1 f_2) - 2. \]

The main result of this section is the following.
THEOREM 3.1
With notation as above, for any $k \geq 1$, there is a constant $C = C(k, n) \neq 0$ such that

$$a_k \equiv C \Delta^{k-1} \rho \pmod{B'},$$

where $\rho$ is the scalar curvature of $M$ and $\Delta$ is the Laplace operator of $M$.

Proof
In order to prove the theorem, we must estimate the quantities in (3.1). We construct peak sections of $L^m$ for $m$ large. So let us quickly review the concept of peak sections which was introduced in [25].

Choose a local normal coordinate $(z_1, \ldots, z_n)$ centered at $z$ such that the Hermitian matrix $(g_{\alpha\bar{\beta}})$ satisfies

$$g_{\alpha\bar{\beta}}(z) = \delta_{\alpha\beta}, \quad \frac{\partial^{p_1+\cdots+p_n} g_{\alpha\bar{\beta}}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(z) = 0$$

for $\alpha, \beta = 1, \ldots, n$ and any nonnegative integers $p_1, \ldots, p_n$ with $p_1 + \cdots + p_n \neq 0$. Such a local coordinate system, which is known as the $K$-coordinate system (cf. [3] or [23] for details) exists and is unique up to an affine transformation. We choose a local holomorphic frame $e_L$ of $L$ at $z$ such that the local representation function $a$ of the Hermitian metric $h$ has the properties

$$a(z) = 1, \quad \frac{\partial^{p_1+\cdots+p_n} a}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(z) = 0$$

for any nonnegative integers $(p_1, \ldots, p_n)$ with $p_1 + \cdots + p_n \neq 0$.

Suppose that the local coordinate $(z_1, \ldots, z_n)$ is defined on an open neighborhood $U$ of $x_0$ in $M$. Define the function $|z|$ by $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for $z \in U$.

Let $\mathbb{Z}_n^m$ be the set of $n$-tuples of integers $(p_1, \ldots, p_n)$ such that $p_i \geq 0$ ($i = 1, \ldots, n$). Let $P = (p_1, \ldots, p_n)$. Define

$$z^P = z_1^{p_1} \cdots z_n^{p_n}$$

and

$$p = |P| = p_1 + \cdots + p_n.$$

The following lemma is proved in [24] using the standard $\bar{\partial}$-estimates (see, e.g., [17]).

*Note that there is a little ambiguity about the notation here. We use $z$ to denote both the point and its local coordinates. However, this should be clear from the context.*
LEMMA 3.2
For \( P = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n \) and an integer \( p' > p = p_1 + \cdots + p_n \), there exists an \( m_0 > 0 \) such that for \( m > m_0 \), there is a holomorphic global section \( S_{P,m}^{p'} \) in \( H^0(M, L^m) \), satisfying
\[
\int_M \| S_{P,m}^{p'} \|^2_{h_m} dV_g = 1,
\]
\[
\int_{M \setminus \{ r \leq \log m/\sqrt{m} \}} \| S_{P,m}^{p'} \|^2_{h_m} dV_g = O\left( \frac{1}{m^{2p'}} \right),
\]
and \( S_{P,m}^{p'} \) can be decomposed as
\[
S_{P,m}^{p'} = \tilde{S}_{P,m} + u_{P,m} \quad (\tilde{S}_{P,m} \text{ and } u_{P,m} \text{ not necessarily continuous}),
\]
so that
\[
\tilde{S}_{P,m}(x) = \begin{cases} \lambda p z^P e^{m} (1 + O\left( \frac{1}{m^{2p'}} \right)), & x \in \{ r \leq \frac{\log m}{\sqrt{m}} \}, \\ 0, & x \in M \setminus \{ r \leq \frac{\log m}{\sqrt{m}} \}, \end{cases}
\]
\[
u_{P,m}(x) = O\left( |z|^{2p'} \right), \quad x \in U,
\]
and
\[
\int_M \| u_{P,m} \|^2_{h_m} dV_g = O\left( \frac{1}{m^{2p'}} \right),
\]
where \( O(1/m^{2p'}) \) denotes a quantity dominated by \( C/m^{2p'} \) with the constant \( C \) depending only on \( p' \) and the geometry of \( M \). Moreover,
\[
\lambda_{P}^{-2} = \int_{r \leq \log m/\sqrt{m}} |z|^P a_m^m dV_g.
\]

Define an order \( \geq \) on the multiple indices \( P \) as follows: \( P \geq Q \) if
1. \( |P| > |Q| \) or
2. \( |P| = |Q| \) and \( p_j = q_j \) but \( p_{j+1} > q_{j+1} \) for some \( 0 \leq j \leq n \).

Using this order, there is a one-to-one order-preserving correspondence \( \kappa \) between \( \{ 0, 1, 2, \ldots \} \) and \( \{ P \mid P \in \mathbb{Z}_+^n \} \).

We need the following proposition from [19, Proposition 2.1].

PROPOSITION 3.1
We have the following expansion for any \( p' > t + 2(n + p + q) \):
\[
(S_{P,m}^{p'}, S_{Q,m}^{p'}) = \frac{1}{m^q} \left( a_0 + \frac{a_1}{m} + \cdots + \frac{a_t-1}{m^{t-1}} + O\left( \frac{1}{m^t} \right) \right),
\]
where \( \delta = 1 \) or \( 1/2 \) and where all the \( a_i \)'s are polynomials of the curvature and its derivatives such that
\[
\text{ord}(a_i) = i + \delta.
\]
It has been proved in [19, Theorem 3.1] that for any $t > 0$, there is an $s > 0$ such that up to $O(1/m^t)$, $I_{00}$ depends only on $S_{\kappa(i)} (i = 0, \ldots, s)$. More precisely, let

$$F'_{ab} = (S_{\kappa(a)}, S_{\kappa(b)}), \quad 0 \leq a, b \leq s.$$ 

Let $(F'_{ab})$ be the inverse matrix of $(F_{ab})$; then

$$I_{00} = I'_{00} + O\left(\frac{1}{m^t}\right).$$

Let

$$(F'_{ab}) = \begin{pmatrix} 1 & M_{21} \\ M_{12} & M_{22} \end{pmatrix},$$

where $M_{12} \in \mathbb{C}^s$, $M_{21}^T \in \mathbb{C}^s$, and $M_{22}$ is an $(s \times s)$-matrix. By an elementary computation, we have

$$I'_{00} = 1 + M_{12}^T(M_{22} - M_{21}M_{12})^{-1}M_{21}. \quad (3.5)$$

From [19, Lemma 2.2], we know that $M_{12} = O(1/m)$. In particular, for any monomial in any entry $e$ of $M_{12}$, $\text{ord}(e) \leq w(e) - 1$. Thus by (3.5), Proposition 3.1, and Lemma 3.1, we have

$$I_{00} \equiv 1 \pmod{B'} \quad (3.6)$$

Next, let us consider $\|S_0(z)\|_{h_m}^2$. By Lemma 3.2, we see that

$$\|S_0(z)\|_{h_m}^2 = \lambda_0^2 + O\left(\frac{1}{m^N}\right)$$

for any $N$, where

$$\lambda_0^{-2} = \int_{r \leq \log \sqrt{m}} a^m dV_g.$$ 

Let $\xi = \log \alpha + |z|^2$, $\eta = \log \det g_{\alpha\beta}$. We have

$$\lambda_0^{-2} = \int_{r \leq \log \sqrt{m}} e^{m\xi + \eta} e^{-m|z|^2} dV_0$$

for the Euclidean volume form $dV_0$.

By Lemma 3.1, we see that

$$\lambda_0^{-2} \equiv \int_{r \leq \log \sqrt{m}} (1 + m\xi + \eta)e^{-m|z|^2} dV_0 \pmod{B'}.$$ 

Using the fact that

$$\int_{\mathbb{C}^n} |z^{p_1} \cdots z^{p_n}|^2 e^{-m|z|^2} dV_0 = \frac{p_1! \cdots p_n!}{m^{n+p}},$$
we have
\[ \lambda_0^{-2} \equiv \frac{1}{m^n} \]
\[ + \frac{1}{m^n} \sum_{k=1}^{N} \left( \frac{1}{(k+1)!} \Delta_c^{k+1} \xi + \frac{1}{k!} \Delta_c^{k} \eta \right) \frac{1}{m^k} + O\left( \frac{1}{m^{N+n}} \right) \quad \text{(mod } B') \] (3.7)
for any \( N \), where \( \Delta_c \) is the complex Laplace operator on \( \mathbb{C}^n \), defined by
\[ \Delta_c = \sum_{i=1}^{n} \partial_{\bar{z}_i} \partial_{z_i}. \] (3.8)

As before, \( \Delta \) is the Laplace operator on \( M \). It is not hard to see that
\[ \Delta_c^j \eta \equiv - \Delta^{j-1} \rho \quad \text{(mod } B') \] (3.9)
for \( j \geq 1 \). Using the same method, we have
\[ \Delta_c^{j+1} \xi \equiv \Delta^{j-1} \rho \quad \text{(mod } B'). \] (3.10)

Combining (3.7), (3.9), and (3.10), we have
\[ \lambda_0^{-2} \equiv \frac{1}{m^n} \left( 1 - \sum_{k=1}^{N} \frac{k}{(k+1)!m^k} \Delta_c^{k-1} \rho \right) + O\left( \frac{1}{m^{N+n}} \right) \quad \text{(mod } B'). \]

Thus
\[ \lambda_0^2 \equiv m^n \left( 1 + \sum_{k=1}^{N} \frac{k}{(k+1)!m^k} \Delta_c^{k-1} \rho \right) + \left( \frac{1}{m^{N+n}} \right) \quad \text{(mod } B'). \] (3.11)

Comparing the above equation with (3.6), we have
\[ a_k = \frac{k}{(k+1)!} \Delta_c^{k-1} \rho \quad \text{(mod } B'), \]
and Theorem 3.1 is proved. \( \square \)

For the rest of this paper, we study the coefficients \( a_k \) in the Tian-Yau-Zelditch expansion for different metrics. We thus use the notation \( a_k(x, h) \), where \( x \in M \) and \( h \) is the Hermitian metric on \( L \), to explicitly represent the dependence of the coefficients to the metric.

Let \( \varphi \in C^\infty(M) \). Then \( h e^{\varphi} \) defines a Hermitian metric on \( L \). Let
\[ \omega_\varphi = \left( \frac{\sqrt{-1}}{2\pi} \right) (g_{i\bar{j}} - \partial_i \bar{\partial}_j \varphi) \, dz_i \wedge d\bar{z}_j \]
be the corresponding Kähler form. We have the following.

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COROLLARY 3.1
Using notation as in Theorem 3.1, let
\[ \omega \phi > \frac{1}{2} \omega. \]
Then if
\[ a_k(x, h \phi^+) = 0, \]
there is a constant \( C(l) \), depending on the \( C^{l-4} \)-bound of the curvature of \( \omega \), such that
\[ \| \phi \|_{C^l} \leq C(l) \| \phi \|_{C^{2k+2}} \]
for \( l > 2k+2 \).

Proof
We have
\[ a_k \equiv C \Delta^{k-1} \rho \quad \text{(mod } B'), \]
where \( \rho \) is the scalar curvature of \( \omega \phi \). By the Schauder estimate, we have
\[ \| \rho \|_{C^{2k-2,1/2}} \leq C \| \phi \|_{C^{2k+1,1/2}}. \]
Since
\[ \rho = -\Delta \log \frac{\omega^n}{\omega^+} - \sum j \overline{c}_j \log \omega^n, \]
we have
\[ \| \frac{\omega^n}{\omega^+} \|_{C^{2k,1/2}} \leq C \| \phi \|_{C^{2k+1,1/2}}. \]
By the Schauder estimate again, we have
\[ \| \phi \|_{C^{2k+2,1/2}} \leq C \| \phi \|_{C^{2k+1,1/2}}. \]
The bootstrapping method gives the higher-order estimates.

4. The uniformity of the expansion
As in the previous sections, let \( h \) be a Hermitian metric on the line bundle \( L \) over \( M \). Let \( \phi \) be a smooth function such that \( h_t = h e^{t \phi} \) is a family of Hermitian metrics with \(-\bar{c} \log h_1 > 0\). Assume that \( S_0, \ldots, S_{d-1} \) is a basis for the Hermitian vector space \( H^0(M, L^m) \) such that at a fixed point \( x \), \( S_0(x) \neq 0 \) but \( S_j(x) = 0 \) for \( j \neq 0 \). We use \( \langle S_i, S_j \rangle_t \) and \( (S_i, S_j)_t \) to denote the pointwise and the \( L^2 \) inner product, respectively, with respect to the metric \( h_t \). Let \( F_{t, \alpha \beta} = (S_\alpha, S_\beta)_t \). Let \( I_{t, \alpha \beta} \) be the inverse matrix of \( F_{t, \alpha \beta} \). Then by (3.1), the Bergman potential at \( x \) with respect to the metric \( h_t \) is
\[ \sigma(t) = I_{t,00} \| S_0 \|^2_t(x). \]
(4.1)
The following result was pointed out by Zelditch [26] (cf. [10, Proposition 6]).
PROPOSITION 4.1
In Theorem 2.1, for any $s$ and $k$, there is a number $N = N(s, k)$ such that if the metric $\omega$ is bounded from below and is bounded in $C^N$ by a constant $C_1$ with some reference metric, then the constant $C$ in (2.6) depends only on $s, k$, and the constant $C_1$.

Let $\varphi \in C^\infty$. Then in the expansion of $\sigma(t)$, the constant $C$ is independent of $t$ for $0 \leq t \leq 1/2$ by the above proposition.

It is natural to ask whether one can take the derivative of the expansion of $\sigma(t)$ in order to get the expansion of $\sigma'(0)$. The main result of this section confirms that this is indeed the case.

Let $I_{\alpha\beta} = I_{0,\alpha\beta}$. We have the following.

PROPOSITION 4.2
At the fixed point $x$, using the above notation, we have

$$\sigma'(0) = -\sigma(0)I_{00}^{-1}\int_M (m\varphi - \Delta \varphi)I_{\alpha\alpha}\langle S_\alpha, S_\beta \rangle I_{\beta0} \frac{\alpha_0^n}{n!},$$

(4.2)

where we assume that $\varphi(x) = 0$ without losing generality.

Proof
For fixed $m$, we have

$$F_{t,\alpha\beta} = (S_\alpha, S_\beta)_t = F_{0,\alpha\beta} + t\int_M (m\varphi - \Delta \varphi)I_{\alpha\alpha}\langle S_\alpha, S_\beta \rangle I_{\beta0} \frac{\alpha_0^n}{n!} + O(t^2).$$

A straightforward computation gives

$$I_{t,00} = I_{00} - t\int_M (m\varphi - \Delta \varphi)I_{0\alpha}\langle S_\alpha, S_\beta \rangle I_{\beta0} \frac{\alpha_0^n}{n!} + O(t^2).$$

We also have

$$\|S_0\|^2_t = \|S_0\|^2.$$

The proposition follows from (4.1) and the above two equations. \qed

In order to get the uniform estimate, we have to establish a uniform version of the above proposition. The main difficulty here is that the size of the matrix is very large (of the size $m^n$). The technique we use here is to choose a special kind of basis under which the matrix $F_{\alpha\beta}$ takes the form of (4.3).

We take the following definition from [19].

Definition 4.1
We say $N = \{N(m)\}$ is a sequence of $s \times s$ block matrices with block number $t \in \mathbb{Z}$
if for each $m$,

$$N = N(m) = \begin{pmatrix} N_{11}(m) & \cdots & N_{1t}(m) \\ \vdots & \ddots & \vdots \\ N_{t1}(m) & \cdots & N_{tt}(m) \end{pmatrix}$$

such that for $1 \leq i, j \leq t$, $N_{ij}$ is a $(\sigma(i) \times \sigma(j))$-matrix and

$$\sum_{i=1}^{t} \sigma(i) = s,$$

where $\sigma : \{1, \ldots, t\} \to \mathbb{Z}_+$ assigns to each number in $\{1, \ldots, t\}$ a positive integer. We say that $\{N(m)\}$ is of type $A(p)$ for a positive integer $p$ if for any entry $s$ of the matrix $N$, we have the following.

1. If $s$ is a diagonal entry of $N_{ii}$ $(1 \leq i \leq t)$, then we have the Taylor expansion

$$s = 1 + \frac{s_1}{m} + \cdots + \frac{s_{p-1}}{m^{p-1}} + O\left(\frac{1}{m^p}\right).$$

2. If $s$ is not a diagonal entry of $N_{ii}$ $(1 \leq i \leq t)$, then we have the Taylor expansion

$$s = \frac{1}{m^\delta}\left(s_0 + \frac{s_1}{m} + \cdots + \frac{s_{p-1}}{m^{p-1}} + O\left(\frac{1}{m^p}\right)\right),$$

where $\delta$ is equal to 1 or $3/2$.

3. If $s$ is an entry of the matrix $N_{ij}$ for which $|i - j| = 1$, then $s = (1/m^{3/2})$. In addition, if $i \neq t$ or $j \neq t$, then we have the Taylor expansion

$$s = \frac{1}{m^\delta}\left(s_0 + \frac{s_1}{m} + \cdots + \frac{s_{p-1}}{m^{p-1}} + O\left(\frac{1}{m^p}\right)\right),$$

where $\delta$ is equal to 1 or $3/2$.

4. If $s$ is an entry of $N_{ij}$ for which $|i - j| > 1$, then

$$s = O\left(\frac{1}{m^p}\right).$$

For $s$ running from all the entries of $N_{ij}$, where $|i - j| \leq 1$ and $i \neq t$ or $j \neq t$, the set of all quantities $(s_1/m, \ldots, s_{p-1}/m^{p-1})$ or $(s_0/m^\delta, s_1/m^{1+\delta}, \ldots, s_{p-1}/m^{b+\delta-1})$ is called the Taylor data of order $p$.

**Remark 4.1**

Since $N_{ij} = O(1/m^p)$ for $|i - j| > 1$, it can be treated as zero when we are interested only in the expansion of order up to $p - 1$ and when the rank of the matrix is bounded by a constant depending only on $p$. A matrix whose entries $N_{ij} = 0$ for $|i - j| > 1$ is called a tridiagonal matrix. For such a matrix, we have a simple iteration process for finding its inverse matrix (see [19]).
PROPOSITION 4.3
For any positive integer \( p \), there is a number \( \varpi(p) \) such that there is a \( \varpi(p) \times \varpi(p) \) block matrix \( N \) of block number \( (p + 1) \). \( N \) is of type \( A(p) \). Furthermore, the matrix \( (F_{AB}) \) can be represented as
\[
(F_{AB}) = \begin{pmatrix} N & 0 \\ 0 & E(d - \varpi(p)) \end{pmatrix},
\]
where \( E(d - \varpi(p)) \) is the \( (d - \varpi(p)) \times (d - \varpi(p)) \) identity matrix.

Proof
We construct such a matrix using the peak sections in Lemma 3.2. For a multiple index \( (p_1, \ldots, p_n) \), define \( |P| = p_1 + \cdots + p_n \). Suppose
\[
V_k = \{ S \in H^0(M, L^m) \mid D^0 S(x_0) = 0 \text{ for } |Q| \leq k \}
\]
for \( k = 1, 2, \ldots \), where \( Q \in \mathbb{Z}_+^n \) is a multiple index, and \( D \) is a covariant derivative on the bundle \( L^m \). It is not hard to see that \( V_k = \{0\} \) for \( k \) sufficiently large. For fixed \( p \), let \( p' = n + 8p(p - 1) \). Suppose that \( m \) is large enough so that \( H^0(M, L^m) \) is spanned by the \( S_{P,m}^{p'} \)'s for the multiple indices \( |P| \leq 2p(p - 1) \) and \( V_{2p(p-1)} \). Let \( r = d - \dim V_{2p(p-1)} \). Then \( r \) depends only on \( p \) and \( n \). Let \( T_1, \ldots, T_{d-r} \) be an orthonormal basis of \( V_{2p(p-1)} \) such that
\[
(S_{P,m}^{p'}, T_{\alpha}) = 0
\]
for \( |P| \leq 2p(p - 1) \) and \( \alpha > r \). Let \( s(k) = \dim V_k \) for \( k \in \mathbb{Z} \). For any \( 1 \leq i, j \leq p \), let \( N_{ij} \) be the matrix formed by \( (S_{P,m}^{p'}, S_{Q,m}^{p'}) \), where \( 2p(i - 2) \leq |P| \leq 2p(i - 1) \) and \( 2p(j - 2) \leq |Q| \leq 2p(j - 1) \). Furthermore, define \( N_{i(p+1)} \) to be the matrix whose entries are \( (S_P, T_{\alpha}) \) for \( 2p(i - 2) \leq |P| \leq 2p(i - 1) \) and \( 1 \leq \alpha \leq r \). Define \( N_{i(p+1)} \) to be the complex conjugate of \( N_{i(p+1)} \). Finally, define \( N_{i(p+1)(p+1)} \) to be the \( r \times r \) unit matrix \( E(r) \). Then it is easy to check that \( N = (N_{ij}) \) is a sequence of block matrices of type \( A(p) \) with the block number \( p + 1 \) by using the result of Ruan (cf. [19, Lemma 2.2] and Proposition 3.1). Let \( \varpi(p) = 2r = 2(d - \dim V_{2p(p-1)}) \). Then \( \varpi(p) \), which is the rank of the matrix \( N \), depends only on \( p \) and \( n \).

Define an order \( \geq \) on the multiple indices \( P \) as follows: \( P \geq Q \) if
\[
\begin{align*}
(1) & \quad |P| > |Q| \text{ or } \\
(2) & \quad |P| = |Q| \text{ and } p_j = q_j \text{ but } p_{j+1} > q_{j+1} \text{ for some } 0 \leq j \leq n.
\end{align*}
\]
Using this order, there is a one-to-one order-preserving correspondence \( \kappa \) between \( \{0, \ldots, r - 1\} \) and \( \{P \mid |P| \leq 2p(p - 1)\} \).

Define
\[
S_A = \begin{cases} 
S_{\kappa(A),m}^{p'}, & A \leq r - 1, \\
T_{A-r+1}, & A \geq r.
\end{cases}
\]
Comparing the matrix $N$ to the metric matrix $F_{AB} = ((S_A, S_B))$ ($A, B = 0, \ldots, d - 1$), by the choice of the basis we see that

$$(F_{AB}) = \begin{pmatrix} N & 0 \\ 0 & E(d - 2r) \end{pmatrix},$$

(4.3)

where $E(d - 2r)$ is the $(d - 2r) \times (d - 2r)$ identity matrix.

Using the above result, we have the following.

**Theorem 4.1**

There is an expansion of $\sigma'(0)$,

$$\sigma'(0) \sim m^n \left( b_0 + \frac{b_1}{m} + \frac{b_2}{m^2} + \cdots \right),$$

in the sense that for any $k$,

$$\left\| \sigma'(0) - m^n \left( b_0 + \cdots + \frac{b_k}{m^k} \right) \right\|_{C^0} \leq \frac{C}{m^{k+1}},$$

where the constant $C$ depends on $k$ and the manifold $M$ but is independent of $m$. $^*$

**Proof**

By [26], we know that there is an asymptotic expansion of $\sigma(0)$. By [19, Theorem 3.1], we have the asymptotic expansion of $I_{00}$. Thus in order to give the expansion of $\sigma'(0)$, in terms of (4.2), we just need to prove that for any smooth function $\psi$, there is an asymptotic expansion of the expression

$$\sum_{\alpha, \beta = 0}^{d-1} \int_M \psi I_{0\alpha}(S_\alpha, S_\beta) I_{\beta0} d\omega_0^n.$$  (4.4)

We choose the basis $S_0, \ldots, S_{d-1}$ as in Proposition 4.3. By the proposition, we have $I_{0\alpha} = 0$ for $\alpha > 2r$, where $r$ is the size of the matrix $N$ in (4.3). For each fixed $\alpha, \beta$, it is easy to see that there is an asymptotic expansion for the term $\int_M \psi I_{0\alpha}(S_\alpha, S_\beta) I_{\beta0} d\omega_0^n$. The theorem follows from the fact that $r$ is independent of $m$.

We now prove the main result of this section.

$^*$The expansion is convergent even in the $C^\infty$-norm, though we do not need the fact. One may also prove the theorem using the paramatrix of the Szegö kernel, similarly to what Zelditch did in [26]. We may have to cope with the quantity on different circle bundles if we use the Szegö kernel method.
THEOREM 4.2
Suppose that we have the following expansion of $\sigma(0)$ for $t$ small:*

$$\sigma(t) \sim m^n \left( a_0(x, t) + \frac{a_1(x, t)}{m} + \cdots \right)$$

in the sense that

$$\left\| \sigma(t) - m^n \left( a_0(x, t) + \frac{a_1(x, t)}{m} + \cdots + \frac{a_k(x, t)}{m^k} \right) \right\|_{C^0} \leq \frac{C}{m^{k+1}} \quad (4.5)$$

where $k \geq 1$ is an integer and $C$ is independent of $m$ and $t$. Then the expansion of $\sigma'(0)$, if it exists, must be of the form

$$\sigma'(0) \sim m^n \left( \frac{d}{dt} \bigg|_{t=0} a_0(x, t) + \frac{d}{dt} \bigg|_{t=0} \frac{a_1(x, t)}{m} + \cdots \right).$$

Proof
In what follows, we denote by $C$ a general constant that is independent of $m$ and $t$. Suppose that the expansion of $\sigma'(0)$ is

$$\sigma'(0) \sim m^n \left( b_0(x) + \frac{b_1(x)}{m} + \frac{b_2(x)}{m^2} + \cdots \right)$$

with

$$\left\| \sigma'(0) - m^n \left( b_0(x) + \frac{b_1(x)}{m} + \cdots + \frac{b_k(x)}{m^k} \right) \right\|_{C^0} \leq \frac{C}{m^{k+1}}.$$

Using (4.5) and the above inequality, we have

$$m^n \left\| \sum_{i=1}^{k} \frac{1}{m^i} \left( \frac{a_i(x, t) - a_i(x, 0)}{t} - b_i(x) \right) \right\|_{C^0} \leq \frac{3C}{|t|m^{k+1}} + \frac{|\sigma(t) - \sigma(0)|}{t} - \sigma'(0). \quad (4.6)$$

If we choose the basis of $H^0(M, L^m)$ as in Proposition 4.3, then we have

$$\left| \frac{\sigma(t) - \sigma(0)}{t} - \sigma'(0) \right| \leq Cm^2|t|$$

when $mt$ is small.

Thus (4.6) becomes

$$m^n \left\| \sum_{i=1}^{k} \frac{1}{m^i} \left( \frac{a_i(x, t) - a_i(x, 0)}{t} - b_i(x) \right) \right\|_{C^0} \leq \frac{3C}{|t|m^{k+1}} + Cm^2|t|. \quad (4.7)$$

The above inequality holds true for any $k, m$, and $t$. (The constant $C$ depends on $k$.) If we choose $k = 2, |t| = 1/m^{(5/2)},$ then letting $m \to \infty$, we have

$$\left. \frac{d}{dt} \right|_{t=0} a_1(x, t) = b_1(x).$$

*Here $a_i(x, t) = a_i(x, he^{\psi})$ for short.
Now we assume that for any $1 \leq i < j$, we have
\[
\frac{d}{dt} \bigg|_{t=0} a_i(x, t) = b_i(x).
\]
Since all $a_i(x, t)$ are differentiable, for $t$ small there is a constant $C$ such that
\[
\left| \frac{a_i(x, t) - a_i(x, 0)}{t} - b_i(x) \right| \leq C |t|
\]
for $1 \leq i < j$ and
\[
\left| \frac{a_i(x, t) - a_i(x, 0)}{t} - b_i(x) \right| \leq C
\]
for $i \geq j$. Assuming that $k > j$, from (4.7) we have
\[
m^{n-j} \left| \frac{a_j(x, t) - a_j(x, 0)}{t} - b_j(x) \right| \leq \frac{3C}{m^{k+1}|t|} + Cm^2|t| + Cm^{n-1}j|t| + Ckm^{n-j-1}.
\]

We assume $|t| = 1/m^j$ and let $k > 2j$; then the above inequality implies the conclusion of the theorem. $\square$

5. The general case
In this section, we prove Theorem 1.3. First, we establish some general estimates that are used for the rest of the paper.

We use $a_l(x, h)$ to denote the $l$th coefficient in the Tian-Yau-Zelditch expansion (see Theorem 2.2), where $h$ is the Hermitian metric on the bundle $L$ and $x \in M$.

**Lemma 5.1**
Let $l$ be a nonnegative integer. Let $\omega = -(\sqrt{-1}/(2\pi))\partial \bar{\partial} \log h$. Let $\varphi \in C^{2l+2}$ satisfy
\[
\begin{align*}
\|\varphi\|_{C^{2l+2}} &\leq 1, \\
\frac{1}{2} \omega + \sqrt{-1} \partial \bar{\partial} \varphi &> 0.
\end{align*}
\]
Then there is a constant $C$, depending on $l$ and the $C^{2(l-1)}$ curvature bound of the metric $\omega$, such that
\[
\left| a_l(x, h e^{t\varphi}) - a_l(x, h) - \frac{d}{ds} \bigg|_{s=0} a_l(x, h e^{s\varphi})t \right| \leq Ct^2
\]
for $0 \leq t \leq 1$. Furthermore, for a metric $h'$ which is $C^{2(l+1)}$ close to $h$, we have the inequality
\[
\left| a_l(x, h' e^{t\varphi}) - a_l(x, h') \right| \leq C_1 t
\]
for $0 \leq t \leq 1$ and for the constant $C_1$ depending only on $l$, the $C^{2(l-1)}$-bound of the curvature of $h$, and the $C^{2(l+1)}$-norm of $\varphi$. 
Proof
By Theorem 2.2, we know that \( a_l(x, \omega^l) \) is a polynomial of Weyl weight \( 2l \). That means that \( a_l(x, \omega^l) \) is a smooth function of the curvature, of its derivative of \( \omega \) of degree up to \( 2(l - 1) \), and of \( \varphi \), its derivative of degree up to \( 2(l + 1) \). Using the assumption that \( (\sqrt{-1}/(2\pi))(-\text{Tr} \omega + \partial \bar{\partial} \varphi) > 0 \), we can expand \( a_l(x, \omega^l) \) as the Taylor series of \( t \) with the coefficients depending on \( l \), the \( C^2 \)-bound of the curvature of \( \omega \), and the \( C^{2(l+1)} \)-norm of the function \( \varphi \). Thus (5.1) follows from the Taylor expansion.

We note that the constant \( C \) depends only on the bounds of the curvature and the function \( \varphi \). Thus (5.2) follows from this observation. \( \square \)

Proof of Theorem 1.3.
If the theorem is not true, then we have an infinite-dimensional vector space \( V \) such that for any \( \varphi \in V \), the log term for the metric \( \omega^l \) is zero for \( t \) small enough. By Theorem 2.3, we have \( a_{n+1}(x, \omega^l) \equiv 0 \). By Theorem 3.1, we have
\[
a_{n+1}(x, \omega^l) = C \Delta^n \rho + \text{lower-order terms},
\]
where \( \Delta \) and \( \rho \) are, respectively, the Laplacian and the scalar curvature of the metric \( \omega - t(\sqrt{-1}/(2\pi))\partial \bar{\partial} \varphi \). A straightforward computation gives
\[
\frac{d}{dt} \bigg|_{t=0} \rho = \Delta^2 \varphi + R_{ji} \varphi_i \varphi_j,
\]
where \( \Delta \) is the Laplacian of \( \omega \). Thus we have
\[
0 = \frac{d}{dt} \bigg|_{t=0} a_{n+1}(x, \omega^l) = C \Delta^{n+2} \varphi + \text{lower-order terms. (5.3)}
\]
Since the above identity is a linear elliptic equation of \( \varphi \), the solution space is a finite-dimensional space by the Schauder estimates. \( \square \)

6. The cases of complex projective spaces
In this section, we study the unit circle bundle of the universal line bundle of the complex projective space \( \mathbb{C}P^n \). First, we prove Theorem 1.2, which is parallel to the case of a pseudoconvex domain in \( \mathbb{C}^2 \).

Proof of Theorem 1.1
By Theorem 2.3, we must have \( a_2 = 0 \). By Theorem 2.2, we have
\[
a_2 = \frac{1}{3} \Delta \rho + \frac{1}{24} (|R|^2 - 4|\text{Ric}|^2 + 3\rho^2).
\]
Since \( n = 1 \), the above equation is reduced to
\[
\Delta \rho = 0.
\]
Thus the scalar curvature must be constant. Since \( M = \mathbb{C}P^1 = S^2 \), the constant \( \rho \) must be positive, and thus the metric must be the standard one.

We now assume that \((M, L) = (\mathbb{C}P^n, H)\), where \( H \) is the hyperplane bundle of \( \mathbb{C}P^n \). An orthonormal basis of the space \( H^0(M, L^m) \) can be represented by

\[
\sqrt{\frac{(m+n)!}{P!}} z^P
\]

for a multiple index \( P \in \mathbb{Z}^n_+ \) with \(|P| = m\), where \( P! = p_1! \cdots p_n! \) for \( P = (p_1, \ldots, p_n) \).

We first compute concretely the finite-dimensional vector space \( V \) in Theorem 1.3.

Consider the open set \( U_0 \) of \( \mathbb{C}P^n \), where the local coordinate is \((z_1, \ldots, z_n)\) and the homogeneous coordinate is represented by [1, \( z_1, \ldots, z_n \)]. Since \( \mathbb{C}P^n \) is a symmetric space, we only need to consider the expansion at the point \( x_0 = [1, 0, \ldots, 0] \).

The local coordinate of \( x_0 \) is \((0, \ldots, 0)\). Thus in the following we sometimes use zero to represent the point \( x_0 \).

Let \( S_0 = \sqrt{(m+n)!/m!} \) be the section under the standard local trivialization of \( H \) on \( U_0 \). The Hermitian metric on \( L \) is defined by \( h = 1/(1 + |z|^2) \), and the Kähler metric is defined by \( \omega = (\sqrt{-1}/(2\pi)) \partial \bar{\partial} \log(1 + |z|^2) \). The pointwise norm of the section \( S_0 \) at \( x_0 = (0, \ldots, 0) \) is

\[
\|S_0\|^2 = \frac{(m+n)!}{m!} \cdot \frac{1}{(1 + |z|^2)^m}.
\]

Under the basis (6.1), \( I_{0\alpha} = 0, \sigma (0) = (m+n)!/m! \), and \( I^{-1}_{00} = 1 \). Using (4.2), we have

\[
\frac{d}{dt} \bigg|_{t=0} \sigma (t) = -\frac{(m+n)!}{m!} \int_M (m\phi - \Delta \phi)\|S_0\|^2 \omega^n.
\]

Substituting (6.2) into the above equation, we have

\[
\frac{d}{dt} \bigg|_{t=0} \sigma (t) = -\frac{1}{\pi^n n!} \left( \frac{(m+n)!}{m!} \right)^2 \int_{\mathbb{C}^n} (m\phi - \Delta \phi) \frac{1}{(1 + |z|^2)^{m+n+1}} dV_0,
\]

where \( dV_0 \) is the Euclidean volume form of \( \mathbb{C}^n \).

The following identity is elementary and is used repeatedly:

\[
\int_{\mathbb{C}^n} \frac{|z|^P^2}{(1 + |z|^2)^{m+n+1}} dV_0 = \pi^n \frac{P!(m-|P|)!}{(m+n)!},
\]

where \(|P| \leq m\).
LEMMA 6.1
There is an asymptotic expansion
\[
\frac{d}{dt}\bigg|_{t=0} \sigma(t) \sim m^n \left( \tilde{\xi}_0 + \frac{\tilde{\xi}_1}{m} + \cdots \right)
\]
of the right-hand side of (6.3) at the point \( x_0 \) where \( \tilde{\xi}_i \) can be represented as \( \tilde{\xi}_i = f_i(\Delta_c)\varphi(0) \), \( i \geq 1 \) for polynomials \( f_i \). The operator \( \Delta_c \) is defined in (3.8).

Proof
The existence of the expansion is from Theorem 4.1. Assuming \( \varphi(0) = 0 \), the Taylor expansion of \( \varphi \) at \( x_0 \) is
\[
\varphi \sim \sum_{|P|+|Q|>0} \frac{1}{P!Q!} a(P, Q) z^P \bar{z}^Q.
\]
Using (6.4), we have
\[
\int_{\mathbb{C}^n} \frac{1}{(1 + |z|^2)^{m+n+1}} \varphi \, dV_0 \sim \pi^n \sum_P \frac{(m - |P|)!}{P!(m + n)!} a(P, P). \tag{6.5}
\]
We also have
\[
\Delta_c^k \varphi(0) = \sum_{|P|=k} \frac{k!}{P!} a(P, P).
\]
From the above equation, (6.5) becomes
\[
\int_{\mathbb{C}^n} \frac{1}{(1 + |z|^2)^{m+n+1}} \varphi \, dV_0 \sim \pi^n \sum_{k=1}^{\infty} \frac{(m - k)!}{k!(m + n)!} \Delta_c^k \varphi(0).
\]
We thus have the expansion
\[
\int_{\mathbb{C}^n} \frac{1}{(1 + |z|^2)^{m+n+1}} \varphi \, dV_0 \sim \frac{1}{m^{n+1}} \left( \eta_0 + \frac{\eta_1}{m} + \cdots \right), \tag{6.6}
\]
where the coefficients are all polynomials of \( \Delta_c \) acting on \( \varphi \) at zero. The lemma follows from (6.6).

The following proposition is purely combinatoric.

PROPOSITION 6.1
There are polynomials
\[
f_k(t) = \sum_{l=0}^{k} a_{k,l} t^l
\]
of degree \( k \) such that

\[
\begin{align*}
  a_{k,0} &= 0, \\
  a_{k,k} &= 1, \\
  a_{k,k+1} &= 0, \\
  \Delta^k \varphi(0) &= f_k(\Delta_c) \varphi(0),
\end{align*}
\]

(6.7)

where \( \varphi \) is a smooth function, \( \Delta \) is the Laplacian of \( \mathbb{C}P^n \), and \( k \in \mathbb{N} \).

**Proof**

If \( k = 1 \), then (6.7) is valid by choosing \( f_k(t) = t \). Using mathematical induction, we assume that for \( k \geq 1 \),

\[
\Delta^k \varphi(0) = \sum_{l=0}^{k} a_{k,l} \Delta^l_c \varphi(0)
\]

for constants \( a_{k,l} \) (\( 0 \leq l \leq k \)). We wish to construct constants \( a_{k+1,l} \) with \( 0 \leq l \leq k+1 \) such that (6.7) is true for \( k+1 \).

We need the following lemma.

**Lemma 6.2**

Define

\[
\begin{align*}
  a_{k+1,0} &= 0, \\
  a_{k+1,k+1} &= 1, \\
  a_{k+1,k+2} &= 0, \\
  a_{k+1,l} &= a_{k,l-1} + l(2l + n - 1) a_{k,l} + l^2(l + 1)(l + n) a_{k,l+1}, \quad 0 < l < k + 1.
\end{align*}
\]

(6.8)

Then we have

\[
\Delta^{k+1} |z^P|^2(0) = \sum_{l=0}^{k+1} a_{k+1,l} \Delta^l_c |z^P|^2(0)
\]

(6.9)

for \( |P| \leq k + 1 \).

**Proof**

First, if \( |P| = k + 1 \), then

\[
\Delta^{k+1} |z^P|^2(0) = \Delta^{k+1}_c |z^P|^2(0)
\]

and

\[
\Delta^l_c |z^P|^2(0) = 0
\]
for \( l < k + 1 \). Thus, in this case, (6.9) holds true. Note that
\[
\Delta = (1 + |z|^2)(\delta_{ij} + z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.
\]

Now let \(|P| = l < k + 1\). Then we have
\[
\Delta |z^P|^2 = \sum_i p_i^2 |z^P|^2 + \sum_{i,j} p_{ij}^2 |z^{Q_{ij}}|^2 + \sum_i l^2 |z^R_i|^2,
\] (6.10)

where \( P_i, Q_{ij}, \) and \( R_i \) are defined as follows. Let \( e_j = (0, \ldots, 1_j, \ldots, 0) \) for \( 1 \leq j \leq n \). Recall that a multiple index \( S \geq 0 \) if and only if all of its components \( s_i \geq 0 \). For \( 1 \leq i, j \leq n \), define
1. \( P_i = P - e_i \) if \( P_i \geq 0 \); otherwise, \( P_i = P \);
2. \( Q_{ij} = P_i + e_j \);
3. \( R_i = P + e_i \).

The lemma follows from (6.10), the math induction, and the identity
\[
\Delta^l |z^P|^2(0) = !P!.
\] (6.11)

\[ \square \]

**LEMMA 6.3**

If \( P \neq Q \), then
\[
\Delta^k z^P \bar{z}^Q(0) = 0.
\] (6.12)

**Proof**

We use mathematical induction again. If \( k = 1 \), the theorem is true. Thus we assume that (6.12) is true for any \( k \leq s \). A straightforward computation gives
\[
\Delta z^P \bar{z}^Q = \sum R, S z^R \bar{z}^S,
\]

where \( R \neq S \). Thus we have
\[
\Delta^{s+1} z^P \bar{z}^Q(0) = \sum R, S \Delta^s z^R \bar{z}^S(0) = 0.
\]

The lemma is proved. \[ \square \]

**Continuation of the proof of Proposition 6.1**

By linearity, we just need to verify
\[
\Delta^k \varphi(0) = f_k(\Delta \varphi)(0)
\] (6.13)
when \( \varphi = z^P \bar{z}^Q \). If \(|P| + |Q| > 2k\), then both sides of the above equation are zero. If \( P \neq Q \), then by Lemma 6.3, both sides of the above equation are zero. If \( P = Q \), then by Lemma 6.2, (6.13) holds true. \[ \square \]
Using Lemma 6.1, Proposition 6.1, and the homogeneity of $\mathbb{C}P^n$, we have the following.

**THEOREM 6.1**
Let $x \in \mathbb{C}P^n$. Then in the expansion
\[
\frac{d}{dt} \sigma(t) \bigg|_{t=0} \sim m^n \left( b_0 + \frac{b_1}{m} + \cdots \right),
\]
the coefficients can be represented by
\[
b_i(x) = f_i(\Delta)\varphi(x), \quad i \geq 0,
\]
where $\Delta$ is the Laplacian of $\mathbb{C}P^n$.

**Proof**
Let $x = x_0 = [1, 0, \ldots, 0]$. Then by Lemma 6.1, $b_i, i \geq 1$, can be represented as $g_i(\Delta)$ for polynomials $g_i$. Using Proposition 6.1, we can write $b_i = f_i(\Delta)\varphi(x_0)$. The general case follows from the homogeneity of $\mathbb{C}P^n$.

If the log terms of the Szegö kernel with respect to the metrics $h_t = h e^{t\varphi}$ are zero for small $t$, then $b_i(x) \equiv 0$ for $i > n$ by Theorems 2.3 and 4.2. In particular, $b_{n+1} = f_{n+1}(\Delta)\varphi = 0$.

**LEMMA 6.4**
If $f_{n+1}(\Delta)\varphi = 0$, then $\varphi$ is zero or is an eigenfunction of $\Delta$.

**Proof**
We assume that
\[
f_{n+1}(t) = \prod_{i=0}^{s} (t - \mu_i),
\]
where $\mu_i$ are complex numbers. Then we have
\[
\prod_{i=0}^{s} (\Delta - \mu_i I)\varphi = 0.
\]

Define
\[
\psi_i = \prod_{k=i}^{s} (\Delta - \mu_k I)\varphi
\]
for $0 \leq i \leq s$. Then we have $\psi_0 = 0$, and
\[
(\Delta - \mu_{i-1} I)\psi_i = \psi_{i-1}, \quad i \geq 1.
\]
Assuming that $\Delta \psi_{i-1} = -\lambda \psi_{i-1}$ for some $i$, then if $\lambda + \mu_i \neq 0$, we have

$$\psi_i = -\frac{1}{\lambda + \mu_i - 1} \psi_{i-1}.$$ 

In particular, $\Delta \psi_i = -\lambda \psi_i$. If $\lambda + \mu_i = 0$, then we still have $\Delta \psi_i = -\lambda \psi_i$. At this time, $\psi_{i-1} = (\Delta - \mu_i) \psi_i = 0$.

In this case, $\psi_{i-1} = 0$ and $\Delta \psi_i = -\lambda \psi_i$. Using mathematical induction, we have $\Delta \psi_{s+1} = -\lambda \psi_{s+1}$ for some nonnegative real number $\lambda$, where $\psi_{s+1} = \varphi$.

We wish to prove that $\lambda = -(n + 1)$. For this purpose, we define

$$\varphi_k = \frac{1}{(1 + |z|^2)^k}$$

for $k = 0, 1, 2, \ldots$. A straightforward computation gives

$$\Delta \varphi_k = -k(k + n) \varphi_k + k^2 \varphi_{k-1}$$

for $k \geq 1$. Using integration by parts, we have

$$\left(k(k + n) - \lambda\right) \int_{\mathbb{C}^n} \varphi \varphi_k = k^2 \int_{\mathbb{C}^n} \varphi \varphi_{k-1}.$$ 

(6.16)

If $\lambda \neq k(k + n)$ for any integer $k$, then since

$$\int_{\mathbb{C}^n} \varphi = 0,$$

using (6.16), we see that

$$\int_{\mathbb{C}^n} \varphi \varphi_k = 0$$

for any $k$. By (6.3), this implies that

$$\frac{d}{dt} \sigma(t) \bigg|_{t=0} = 0.$$ 

However, this is not possible because we have

$$\frac{d}{dt} \bigg|_{t=0} a_1 = \Delta (\Delta + (n + 1)I) \varphi = 0,$$

which implies that $\lambda = n + 1$. Now assume that $\lambda = k_0(k_0 + n)$. Then by (6.16), we have

$$\int_{\mathbb{C}^n} \varphi \varphi_m = \frac{(m)!(k_0)!}{(m-k_0)!(m+k_0+n)!(2k_0+n)!} \int_{\mathbb{C}^n} \varphi \varphi_{k_0}.$$ 

*The proof also implies that the eigenvalues of $\mathbb{C}^n$ must be of the form $k(k + n)$ for $k \in \mathbb{Z}$. 
Thus (6.3) becomes
\[ \frac{d}{dt} \bigg|_{t=0} \sigma(t) = \text{const} \cdot \frac{(m+n) \cdots (m-k_0+1)(m+k_0(k_0+n))}{(m+k_0+n) \cdots (m+n+1)}. \]

The above expression is a polynomial of \( m \) if and only if \( k_0 = 1 \). Thus we have proved the following.

THEOREM 6.2
For the complex projective space \( \mathbb{CP}^n \), the vector space \( V \) in Theorem 1.3 is contained in the eigenspace of the first eigenvalue of \( \mathbb{CP}^n \).

7. The proof of the main theorem
In Section 6, we proved that the vector space \( V \) in Theorem 1.3 is contained in the eigenspace of the eigenvalue \((n+1)\). Among all the eigenspaces, the eigenspace of the eigenvalue \((n+1)\) is special. We have the following well-known result (see [8]).

LEMMA 7.1
We use the same notation as in Section 6. Then the first eigenvalue of \( \mathbb{CP}^n \) with the Fubini-Study metric is \((n+1)\). Let \( \varphi \) be an eigenfunction of the eigenvalue \((n+1)\). Let the \((1,0)\) vector field \( X \) on \( \mathbb{CP}^n \) be defined by
\[ X = g^{ij} \varphi_j \frac{\partial}{\partial \bar{z}^i}. \]

Then \( X \) is holomorphic.

The automorphism of \( \mathbb{CP}^n \) can be represented by a nonsingular \(((n+1) \times (n+1))\)-matrix \( a_{ij} \). That is, for any such matrix, the linear map
\[ f : \mathbb{CP}^n \to \mathbb{CP}^n, \quad Z_i \mapsto \sum_j a_{ij} Z_j \quad (7.1) \]
defines an automorphism. The Bergman potential is invariant under the automorphism, and so are the coefficients in the Tian-Yau-Zelditch expansion. Thus, in order to prove the theorem, we must get rid of these automorphisms. The method below is similar to that in Bando and Mabuchi [1] in the study of the uniqueness of Kähler-Einstein metrics on Fano manifolds. However, we use the notation and the results in [9].

Let \( \omega_0 \) be the Kähler form of the standard Fubini-Study metric. Let \( \omega_\rho \) be defined as
\[ \omega_\rho = \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \rho = f^*_\rho \omega_0, \]
where $\rho$ is a real-valued function defined by

$$\rho = \log \frac{\sum_i |\sum_j a_{ij} Z_j|^2}{\sum_i |Z_i|^2}$$

for a nonsingular $((n + 1) \times (n + 1))$-matrix $(a_{ij})$; $f_\rho$ is the automorphism defined by $\rho$ as in (7.1).

**Definition 7.1**

Any Kähler form $\omega_\phi = \omega_0 + (\sqrt{-1}/(2\pi))\partial\bar{\partial}\phi$ is called centrally positioned with respect to the metric $\omega_\rho = \omega_0 + (\sqrt{-1}/(2\pi))\partial\bar{\partial}\rho$ (which is the Fubini-Study metric) if

$$\int_{\mathbb{C}P^n} \left( \rho + f_\rho^*(\phi) \right) f_\rho^*(\theta) \omega_\rho^n = 0$$

for any $\theta \in \Lambda_{n+1}(\omega_0)$, where $\Lambda_{n+1}(\omega_0)$ is the space of eigenfunctions with respect to the first eigenvalue $(n + 1)$ of $\mathbb{C}P^n$.

For any $\rho$, the Kähler metric defined by the function $\rho + f_\rho^*(\phi)$ and $\phi$ differ only by the automorphism $f_\rho$. So they are essentially equivalent. In particular, the vanishing of the log term of the Szegő kernel for one metric implies the vanishing of the log term of the Szegő kernel for the other. We thus need to choose the best representative among all these equivalent metrics.

The following proposition shows that the best representative always exists.

**Proposition 7.1** (see [9])

Using notation as above, for any $\phi$ there is a function $\rho$ such that $\omega_\phi$ is centrally positioned.

For our purpose, we need only the existence of $\rho$ when $\phi$ is small. However, we need to estimate the solution $\rho$. We use the following method of the contraction principle to construct the solution.

**Lemma 7.2**

Using notation as above, for any $\varepsilon > 0$ there is an $\eta$ such that if $\|\phi\|_{C^0} < \eta$, then $\|\rho\| \leq \varepsilon$.

**Proof**

By Lemma 7.1, we know that the dimension of the eigenspace of the first eigenvalue is equal to the dimension of the space of holomorphic vector fields, which is equal to $(n + 1)^2 - 1$, the dimension of the automorphism group of $\mathbb{C}P^n$. 
Let \( p \) be the space of all \((n + 1) \times (n + 1)\) traceless Hermitian matrices. The real dimension of \( p \) is \((n + 1)^2 - 1\). Let \( \theta_1, \ldots, \theta_s \) \((s = (n + 1)^2 - 1)\) be a real orthonormal basis of \( \Lambda_{n+1}(\omega_0) \), the eigenspace of the eigenvalue \((n + 1)\) of the Fubini-Study metric. By direct calculation, we see that

\[
\begin{cases}
Z_i \bar{Z}_j - \frac{1}{2} \sum_k |Z_k|^2, & 0 \leq i, j \leq n, \\
\frac{|Z_i|^2 - |Z_0|^2}{\sum_k |Z_k|^2}, & 1 \leq i \leq n
\end{cases}
\]

are a (complex) basis of eigenfunctions. For any \((n + 1) \times (n + 1)\)-matrix \( A \in p \), consider the real eigenfunction

\[
\sum_{i,j} a_{ij} Z_i \bar{Z}_j \frac{1}{\sum_k |Z_k|^2}.
\]

If we represent the function as

\[
\sum_{i,j} a_{ij} Z_i \bar{Z}_j \frac{1}{\sum_k |Z_k|^2} = \sum_{i=1}^s b_i \theta_i,
\]

then we can define

\[
L : p \to \mathbb{R}^s, \quad L(A) = (b_1, \ldots, b_s).
\]

Since a real basis of \( \Lambda_{n+1}(\omega_0) \) can be represented by \((n + 1)^2 - 1\)-functions

\[
\begin{align*}
\text{Re} \frac{Z_i \bar{Z}_j}{\sum_k |Z_k|^2}, \\
\text{Im} \frac{Z_i \bar{Z}_j}{\sum_k |Z_k|^2}, \\
\frac{|Z_i|^2 - |Z_0|^2}{\sum_k |Z_k|^2}
\end{align*}
\]

\( L \) is an invertible linear map.

Let \( U \) be a small neighborhood of \( p \) at origin. Let \( A \in p \). Let \( e^A = (\tilde{a}_{ij}) \). Define

\[
\rho_A = \log \frac{\sum_i \left| \sum_j \tilde{a}_{ij} Z_j \right|^2}{\sum_k |Z_k|^2}.
\]

Let \( \theta = (\theta_1, \ldots, \theta_s) \). Define the nonlinear operator

\[
T : U \times C^0(M) \to p : (A, \varphi) \mapsto A - \frac{1}{2} L^{-1} \int_{\mathbb{C}^n} (\rho_A + f^*_\rho_A(\varphi)) f^*_\rho_A(\theta) \omega^n_{\rho_A}.
\]

We see that there is a fixed point of the operator if the \( \|\varphi\|_{C^0} \) is fixed and small. Let \( T(A) = T(A, \varphi) \). Then there is a constant \( C \) such that

\[
\|T(0)\| \leq C \|\varphi\|_{C^0}.
\]
We also have
\[
\|T(B) - T(A)\| = \left\| B - A - \frac{L^{-1}}{2} \int_{C P^n} \left( (f_{\rho_A}^{-1})^*(\rho_A) - (f_{\rho_B}^{-1})^*(\rho_B) \right) \theta \omega^n \right\|
\]
If \(A, B\) are small, then there is a constant \(K\) such that
\[
\|T(B) - T(A)\| \leq K \|B - A\|^2.
\]
If \(\|A\|, \|B\| \leq 1/(4K)\), then we have
\[
\|T(B) - T(A)\| \leq \frac{1}{2} \|B - A\|.
\]
We choose \(\|\phi\|_{C^0}\) to be small enough so that
\[
\|T(0)\| \leq \operatorname{Min}\left(\frac{\varepsilon}{2}, \frac{1}{8K}\right).
\]
Then
\[
\|T^k(0) - T^{k-1}(0)\| \leq \frac{1}{2^{k-1}} \operatorname{Min}\left(\frac{\varepsilon}{2}, \frac{1}{8K}\right)
\]
for \(k = 1, 2, \ldots\). Thus we have \(\sum_{k=1}^{\infty} \|T^k(0) - T^{k-1}(0)\| \leq \varepsilon\), from which we have the fact that
\[
\lim_{k \to \infty} T^k(0) = A
\]
exists, \(\|A\| \leq \varepsilon\), and \(A\) is a fixed point of \(T\). The lemma follows from the fact that \(\omega_{\phi}\) is centrally positioned with respect to \(\rho_A\).

Proof of the Theorem 1.2
We assume that there is a sequence \(\varphi_i, i \geq 1\), such that \(\|\varphi_i\|_{C^{2n+4}} \to 0\), and
\[
a_{n+1}(x, he^{\rho_i}) = 0
\]
for \(i \geq 1\). By Lemma 7.2, we can replace \(\varphi_i\) by \(\bar{\varphi}_i = \rho_i - f_{\rho_i}^*(\varphi_i)\), and we still have
\[
a_{n+1}(x, he^{-\rho_i+f_{\rho_i}^*(\varphi_i)}) = 0
\]
with
\[
\int_{C P^n} \left( \rho_i - f_{\rho_i}^*(\varphi_i) \right) f_{\rho_i}^* \theta \omega^n = 0.
\]
By Lemma 7.2, we have
\[
\|\rho_i - f_{\rho_i}^*(\varphi_i)\|_{C^{2n+4}} = \varepsilon_i \to 0.
\]
By Corollary 3.1, we see that there is a constant \(C\) such that
\[
\|\rho_i - f_{\rho_i}^*(\varphi_i)\|_{C^{2n+6}} \leq C \varepsilon_i.
\]
Thus there is a subsequence of $\xi_i = (f_{\rho_i}^*(\phi_i) - \rho_i)/\epsilon_i$, which we still denote as $\xi_i$, which converges to some $\xi \neq 0$ in the $C^{2n+4}\text{-norm}$. Furthermore, using Lemma 7.2 again, we get

$$\int_{\mathbb{CP}^n} \xi \theta \omega_0^n = 0.$$  \hspace{1cm} (7.6)

On the other hand, by (5.1), we have

$$\left| a_{n+1}(x, he^{s\xi}) - a_{n+1}(x, h) \right| \leq C \epsilon_i.$$  \hspace{1cm} (7.7)

Using (5.2), we have

$$\left| a_{n+1}(x, he^{s\rho_i}) - a_{n+1}(x, he^{s\xi}) \right| \leq C \epsilon_i.$$

By assumption, $a_{n+1}(x, he^{\rho_i}) = 0$. Since $h$ is the standard metric of $\mathbb{CP}^n$, $a_{n+1}(x, h) = 0$. Thus, from (7.7) and (7.8), we have

$$\frac{d}{ds} \bigg|_{s=0} a_{n+1}(x, he^{s\xi}) = 0.$$  

By Theorem 6.2, this implies

$$\Delta \xi = -(n + 1)\xi.$$  

From (7.6), $\xi \equiv 0$. This is a contradiction. So for $\|\phi_i\|_{C^{2n+4}}$ small, $a_{n+1}(x, he^{\phi_i}) \neq 0$. This proves the theorem.

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