Generalized Hodge metrics and BCOV torsion on Calabi-Yau moduli

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Abstract. We establish an unexpected relation among the Weil-Petersson metric, the generalized Hodge metrics and the BCOV torsion. Using this relation, we prove that certain kind of moduli spaces of polarized Calabi-Yau manifolds do not admit complete subvarieties. That is, there is no complete smooth family for certain class of polarized Calabi-Yau manifolds. We also give an estimate of the complex Hessian of the BCOV torsion using the relation. After establishing a degenerate version of the Schwarz Lemma of Yau, we prove that the complex Hessian of the BCOV torsion is bounded by the Poincaré metric.

1. Introduction

A Calabi-Yau manifold is a smooth Kähler manifold with trivial canonical bundle and fundamental group. Moduli space of polarized Calabi-Yau manifolds (Calabi-Yau moduli) is the object to study in Mirror Symmetry, hence the focal point of intensive studies in areas of mathematical physics, algebraic geometry, differential geometry and number theory. For the general reference of Mirror Symmetry and related topics, see the book of Cox and Katz [10] and the recent survey paper of Todorov [26].

Differential geometric objects of Calabi-Yau moduli are usually constructed from global algebro-geometric or analytic properties of the Calabi-Yau manifolds, for example, variation of Hodge structures, or spectral properties of the Ricci-flat metrics. In this paper, we focus on the Weil-Petersson metric, the generalized Hodge metrics, the BCOV torsion and their relations.

Calabi-Yau moduli is amicable from differential geometric point of view. It is smooth (or at worst with quotient singularities, so that it is a smooth Deligne-Mumford stack) and its local uniformization is an integral submanifold of the horizontal distribution of the
classifying space. The curvature of the first Hodge bundle, which is positive definite, defines
the so-called Weil-Petersson metric. The study of this metric initiates the study of differential
gometric aspects of the moduli space. For the properties and applications of the Weil-
Petersson metrics (in more general settings), see the works of Schumacher [22] and Siu [23].

In [16], [17], the second author introduced a new metric, the Hodge metric, on a
Calabi-Yau moduli, mainly inspired from the theory of variation of Hodge structures. Both
the Hodge metric and the Weil-Petersson metric are Kähler orbifold metrics. But the
Hodge metric enjoys better curvature properties: it has non-positive bisectional curvatures.
Furthermore, its holomorphic sectional curvature and Ricci curvature are negative and
bounded away from zero. This is clearly not the case for the Weil-Petersson metric, as
shown in the example of Calabi-Yau quintics [7], page 65. Thus, in terms of differential
gometry, the Hodge metric is more useful than the Weil-Petersson metric.

As natural Kähler metrics on a given Calabi-Yau moduli, the Hodge metric and the
Weil-Petersson metric are closely related. There are explicit relations between the two
metrics for K3 surfaces, Calabi-Yau three and four-folds (see Theorem 2.4). The concept
of Weil-Petersson geometry was introduced in [18] to study these relations for arbitrary
dimensions.

The Hodge metric depends only on the Hodge bundles of the middle dimensional
primitive cohomology groups. It turns out that considering the whole set of Hodge bundles
would be more natural. The universal deformation space of Calabi-Yau manifolds are
horizontal slices of the classifying space of any degree primitive cohomology groups, even
though the period maps fail to be immersive in general. In this paper, we introduce the
pseudo-metric on the classifying space for \( H^k \) for any \( 0 \leq k \leq 2n \). These metrics are called
generalized Hodge metrics. The positive definiteness of the generalized Hodge metrics is
lost due to the possible degeneracy of the corresponding horizontal slices. Nevertheless,
good “curvature” properties of the Hodge metric still hold for these generalized Hodge
metrics. See Appendix A for the precise statements.

We now turn to the other geometric object that will be studied in this paper: the
BCOV torsion. First introduced by and named after Bershadsky-Cecotti-Ooguri-Vafa [1],
[2], the BCOV torsion is a smooth function on the Calabi-Yau moduli. It is defined as:

\[
T = \prod_{1 \leq p, q \leq n} \left( \det \Delta'_{p, q} \right)^{(-1)^{p+q} pq},
\]

where \( \Delta_{p, q} \) is the \( \overline{\partial} \)-Laplace operator on \( (p, q) \) forms with respect to the Ricci-flat metric on
a fiber; \( \Delta'_{p, q} \) represents the non-singular part of \( \Delta_{p, q} \); the determinant is taken in the sense of
zeta function regularization.

In physics literature, \( T \) was first introduced as the stringy genus one partition function
of \( N = 2 \) SCFT. It was computed using Physics Mirror Symmetry and was used to predict
the number of holomorphic elliptic curves embedded in certain Calabi-Yau manifolds.

Due to its central role in the Physics Mirror Symmetry, we are interested in the
analysis of the BCOV torsion mathematically. \( T \) is a spectral invariant of the Ricci-flat
metric. The Weil-Petersson metric and the generalized Hodge metrics, which are defined by
using the variation of Hodge structures, assume no apparent links to the function $T$. Thus the following main result of this paper gives a surprising relation between the metrics and the BCOV torsion.

**Theorem 1.1.** Let $\omega_{WP}$, $\omega_H$ and $\omega_{HI}$ be the Kähler forms of the Weil-Petersson metric, the Hodge metric and the generalized Hodge metrics, respectively (see §2 for the definition). Then

$$\sum_{i=1}^n (-1)^i \omega_{HI} - \frac{\sqrt{-1}}{2\pi} \delta \bar{\delta} \log T = \frac{\chi_Z}{12} \omega_{WP},$$

where $\chi_Z$ is the Euler characteristic number of $Z$. In particular, if the Calabi-Yau manifold is primitive (see Definition 2.5), then

$$\omega_H = (-1)^n \left( \frac{\sqrt{-1}}{2\pi} \delta \bar{\delta} \log T + \frac{\chi_Z}{12} \omega_{WP} \right).$$

As the first application of Theorem 1.1, we have the following

**Corollary 1.2.** If $N \subset \mathcal{M}$ is a $k$-dimensional complete subvariety of $\mathcal{M}$, where $\mathcal{M}$ is the moduli space of a primitive Calabi-Yau manifold, then

$$\text{Vol}_H(N) = \left( \frac{\sqrt{-1}}{12} \chi_Z \right)^k \text{Vol}_{WP}(N),$$

where $\text{Vol}_H(N)$ and $\text{Vol}_{WP}(N)$ are the volumes of $N$ with respect to the Hodge and the Weil-Petersson metrics, respectively.

In Corollary 1.2, the BCOV torsion does not appear explicitly. Even in dimension 3 and 4, where the Hodge metric can be expressed explicitly by the Weil-Petersson metric and its Ricci curvature ([17], [18]), this volume identity is new. One of the notable consequences of the volume identity is the following

**Corollary 1.3.** Assume that a polarized Calabi-Yau manifold $Z$ is primitive, and that $(-1)^{n+1} \chi_Z > -24$. Let $\mathcal{M}$ be the moduli space of $Z$. Then there exists no complete curve in $\mathcal{M}$; hence, there exists no projective subvariety of $\mathcal{M}$ (of positive dimensions). In particular, $\mathcal{M}$ is not compact.

This corollary is purely algebro-geometric. Primitive Calabi-Yau manifolds include interesting examples like Calabi-Yau three-folds and Calabi-Yau hyper-surfaces in projective spaces. It would be interesting to see a direct proof of the result without using differential geometry.

The second application of Theorem 1.1 is on the asymptotic behavior of the complex Hessian of the BCOV torsion.

**Corollary 1.4.** Let $\Delta$ and $\Delta^*$ be the unit disk and the punctured unit disk of $\mathbb{C}$ respectively. Let $(\Delta^*)^l \times \Delta^{m-1}$ be the parameter space of a family of Calabi-Yau manifolds. Then the BOCV torsion $T$ satisfies
\[ -C \omega_p < \frac{\sqrt{-1}}{2\pi} \bar{\partial} \bar{\partial} \log T < C \omega_p, \]

where \( C \) is a constant and \( \omega_p \) is the Poincaré metric, defined as

\[
\omega_p = \sum_{i=1}^{l} \sqrt{-1} \frac{1}{|z_i|^2} \left( \log \frac{1}{|z_i|} \right)^2 dz_i \wedge d\bar{z}_i + \sum_{i=l+1}^{m} \sqrt{-1} dz_i \wedge d\bar{z}_i.
\]

Using Theorem 1.1, the proof of the above corollary is reduced to the fact that the generalized Hodge metrics are bounded by the Poincaré metric. In the Appendix, we establish Theorem A.1, a degenerate version of the Schwarz Lemma of Yau [31], from which Corollary 1.4 is a direct consequence.

The relation among the Weil-Petersson metric, the generalized Hodge metrics, and the BCOV torsion on Calabi-Yau moduli presented in this paper is quite delicate and our understanding of it is far from being complete. Because of the physics background of the BCOV torsion, it is very likely that some deeper relations can be used to explain the current coincidence. It is also expected that these constructions will produce new modular forms on various moduli spaces, as the previous works of Yoshikawa [33], [32] indicated.

We shall proceed to study the asymptotic behavior of the BCOV torsion near the boundary of Calabi-Yau moduli, and the BCOV prediction of counting the rational curves. The results will be the subject of an upcoming paper.

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## 2. Generalized Hodge metrics

Let \( Z \) be a Calabi-Yau manifold and let \( l \) be an ample line bundle over \( Z \). The pair \((Z, l)\) is called a polarized Calabi-Yau manifold. The (coarse) moduli space \( \mathcal{M} \) exists and is constructed as follows: first, choose a large integer \( k \) such that \( l^k \) is very ample. In this way \( Z \) is embedded into a complex projective space \( \mathbb{C}P^N \). Let \( \text{Silb}(Z) \) be the Hilbert scheme of \( Z \). The group \( G = \text{PSL}(N+1, \mathbb{C}) \) acts on \( \text{Silb}(Z) \) and the moduli space \( \mathcal{M} \) is the quotient of the semistable points of \( \text{Silb}(Z) \) by the group \( G \).

By the smoothness theorem of Tian [25] (see also Todorov [27]), the deformation of the complex structures of a Calabi-Yau manifold \( Z \) is unobstructed. That is, the universal deformation space (Kuranishi space) is smooth. Due to the existence of finite automorphism, the moduli space for polarized Calabi-Yau manifolds may have quotient singularities. Thus in general, the moduli space \( \mathcal{M} \) is a complex orbifold, or a smooth Deligne-Mumford stack.

Let \( Z \) be a generic polarized Calabi-Yau manifold. There exists the universal family \( \mathcal{X} \) such that it is parameterized by the moduli space \( \mathcal{M} \):
\[ Z \longrightarrow \mathcal{X} \]

(2.1)

\[ \pi' \]
\[ \mathcal{M} \]

Let \( O \in \mathcal{M} \) such that \( Z \cong \pi^{-1}(O) \). There exists a local universal deformation family. More precisely, there exist a complex space \( \mathcal{X}_Z \), a complex space with marked point \( (\text{Def}_Z, O') \), and a proper, surjective flat holomorphic map \( \pi : \mathcal{X}_Z \rightarrow \text{Def}_Z \), such that \( \pi^{-1}(O') \cong Z \). Furthermore, a neighborhood of \( O \in \mathcal{M} \) can be realized as a finite discrete quotient of a neighborhood of \( O' \in \text{Def}_Z \).

For the local geometry of the moduli space \( \mathcal{M} \), orbifold singularities may never be a problem: We can always pass through a finite covering and work on \( (\mathcal{X}_Z, \text{Def}_Z) \) instead. For simplicity, when no confusion occurs, we write \( (\mathcal{X}_Z, \text{Def}_Z) \) as \( (\mathcal{X}, \text{Def}) \).

By the Kodaira-Spencer deformation theory, there is an isomorphism

\[ \rho : T_t(\text{Def}) \cong H^1(Z_t, \Theta_t), \]

where \( Z_t \) is the fiber of \( \mathcal{X} \rightarrow \text{Def} \) at \( t \), and \( \Theta_t \) is the holomorphic tangent bundle of \( Z_t \).

Let \( (t_1, \ldots, t_m) \) be a local holomorphic coordinate system of \( \text{Def} \). Then \( \rho \left( \frac{\partial}{\partial t_i} \right) \in H^1(Z_t, \Theta_t) \). We define a Hermitian inner product on \( T_t(\text{Def}) \) by

\[ \left( \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{\text{WP}} = \int_{Z_t} A_{ij}^{z} \cdot A_{ij}^{\bar{z}} g^{\bar{b}b} g^{\gamma \bar{\gamma}} dV_{Z_t}, \]

where \( A_i = A_{i\bar{b}} \frac{\partial}{\partial b^i} \otimes d\bar{b} \) (i = 1, ..., m) are the harmonic representation of \( \rho \left( \frac{\partial}{\partial t_i} \right) \). This inner product on each \( T_t(\text{Def}) \) for \( t \in \text{Def} \) gives a Hermitian metric on the deformation space \( \text{Def} \), which descends to a metric on the moduli space \( \mathcal{M} \). We call both metrics the Weil-Petersson metric. Under the Weil-Petersson metric, \( \mathcal{M} \) is a Kähler orbifold. See [21] for details of orbifolds and vector bundles over orbifolds.

Let \( \Omega \) be a (nonzero) holomorphic \((n,0)\)-form on \( Z_t \). Define \( \Omega \cdot \rho \left( \frac{\partial}{\partial t_i} \right) \) to be the contraction of \( \Omega \) and \( \rho \left( \frac{\partial}{\partial t_i} \right) \). The Weil-Petersson metric can be re-written as (cf. [25]):

\[ \left( \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{\text{WP}} = -\frac{\int_{Z_t} \Omega \cdot \rho \left( \frac{\partial}{\partial t_i} \right) \wedge \Omega \cdot \rho \left( \frac{\partial}{\partial t_j} \right)}{\int_{Z_t} \Omega \wedge \overline{\Omega}}. \]

The Weil-Petersson metric is the most natural metric on the moduli space. Unfortunately, it does not have very good curvature properties. In [16], another natural metric called the Hodge metric was defined. In the sequel, we use notations in [18] for the Hodge metric.
Recall that for an $n$-dimensional compact complex manifold $X$ with polarization, for any $0 \leq k \leq n$, we have the decomposition of the cohomology groups

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}).$$

By the Lefschetz decomposition theorem, we can further decompose the groups $H^{p,q}(X, \mathbb{C})$ into their primitive parts as follows.

Define $L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})$ by $[z] \to [z \wedge \omega]$, where $\omega$ is the curvature form of the ample line bundle over $X$. Define the primitive cohomology group $P^k(X, \mathbb{C})$ to be the kernel of $L^{n-k+1}$ on $H^k(X, \mathbb{C})$. Let $PH^{p,q} = P^k(X, \mathbb{C}) \cap H^{p,q}(X)$.

The Lefschetz decomposition theorem states that

$$H^{p,q}(X) = PH^{p,q} \oplus L(PH^{p-1,q-1}) \oplus \cdots \oplus L^r(PH^{r,q-r}),$$

where $r = \text{Min}(p, q)$.

Define

$$Q(\eta_1, \eta_2) = \int \eta_1 \wedge \eta_2 \wedge \omega^{n-2k}$$

for $\eta_1, \eta_2 \in H^k(X, \mathbb{C})$. Then $Q$ extends to a bilinear form on $H^*(X, \mathbb{C})$. The Riemann-Hodge relations are

1. $Q(\eta_1, \eta_2) = 0$, if $\eta_1 \in PH^{p_1,q_1}$, $\eta \in PH^{p_2,q_2}$, but $p_1 + p_2 \neq q_1 + q_2$;

2. $(\sqrt{-1})^{q-p} Q(\eta_1, \bar{\eta}_1) > 0$ if $0 \neq \eta_1 \in PH^{p,q}$.

The second Riemann-Hodge relation defines a Hermitian inner product on the primitive harmonic $(p, q)$ forms as:

$$\langle \eta_1, \eta_2 \rangle = (\sqrt{-1})^{q-p} Q(\eta_1, \bar{\eta}_2).$$

When $X$ is a polarized Calabi-Yau manifold $Z$, the above inner product is equivalent to the inner product induced from the Ricci-flat metric (cf. [29], [11]):

**Theorem 2.1.** Let $\phi \in H^{p,q}(Z)$, $p \geq q$ and let

$$\phi = \phi_0 + L\phi_1 + \cdots + L^q\phi_q$$

be the decomposition corresponding to (2.4). Then we have

$$\langle \phi, \phi \rangle = (-1)^{\frac{1}{2}(p+q)(p+q+1)} \sum_{k=0}^q (-1)^k (n-p-q+2k)! \int_M \|\phi_k\|^2 dV_{CY},$$

where $\| \cdot \|$ is the metric induced from the Ricci-flat metric of $Z$. \qed
If we make the relative version of the above settings, we get the so-called Hodge bundles $PR^qπ_sΩ^p_{/Def} → Def$ in place of the cohomology groups. By functorial property, they descend to vector bundles of the Kähler orbifold $\mathcal{M}$ (cf. [21] for more about vector bundles over Kähler orbifolds).

The Kodaira-Spencer map $T_t(Def) → H^1(Z_t, Θ_t)$ gives a bundle map

$$\frac{∂}{∂t_i} : PR^qπ_sΩ^p_{/Def} → PR^kπ_s(\mathbb{C})/PR^qπ_sΩ^p_{/Def}$$

for $k ≤ n$ by differentiation. In this way, we have a natural bundle map

$$(2.8) \quad T(Def) → \bigoplus_{p+q=k} \text{Hom} \left( PR^qπ_sΩ^p_{/Def}, PR^kπ_s(\mathbb{C})/PR^qπ_sΩ^p_{/Def} \right).$$

**Definition 2.2.** For each $t ∈ Def$ and $Z = Z_t$ with the polarized Ricci flat metric, using Theorem 2.1, we define Hermitian metrics on the bundles $PR^qπ_sΩ^p_{/Def} → Def$. Let $h_{PH^k}$ be the pull back of the natural Hermitian metric on the bundle $\bigoplus_{p+q=k} \text{Hom} \left( PR^qπ_sΩ^p_{/Def}, PR^kπ_s(\mathbb{C})/PR^qπ_sΩ^p_{/Def} \right)$ to $T(Def)$ for $k ≤ n$. Then $h_{PH^k}$ is semi-positive definite. We use $\omega_{PH^k}$ to denote the corresponding Kähler form for $k ≤ n$.\(^1\)

According to (2.4), we define

$$\omega_{H^k} = \omega_{PH^k} + \omega_{PH^{k-2}} + \cdots .$$

We call both $\omega_{H^k}$ and $\omega_{PH^k}$ to be the generalized Hodge metrics. This local construction descends to the moduli space $\mathcal{M}$ by functoriality. We use the same notation when no confusion occurs.

**Remark 2.3.** The above construction is a generalization of the Hodge metric defined by the second author [17]. In fact, it is proved in [18] that

$$\omega_{PH^k} = \omega_H,$$

the latter being the Hodge metric.

Because of the possible degeneration of the action (2.8), the generalized Hodge metric is only semi-positive definite; hence, it is a pseudo-metric. Nevertheless, it enjoys similar “curvature” properties of the Hodge metric [16]. The generalized Hodge metrics are bounded by the Poincaré metric. See Appendix A for more details.

We have the following relations between the Hodge metric and the Weil-Petersson metric:

**Theorem 2.4.** We use the above notation. The Hodge metric $\omega_H$ defines a Kähler metric (i.e. $dω_H = 0$). Furthermore, the bisectional curvature of $h$ is non-positive and the holomorphic sectional curvature and the Ricci curvature are negative away from zero. In particular, we have

\(^1\) That is, if $h_{PH^k} = (h_{PH^k})_i^\bar{j} dt_i \otimes d\bar{t}_j$, then $\omega_{PH^k} = \frac{\sqrt{-1}}{2\pi} (h_{PH^k})_i^\bar{j} dt_i ∧ d\bar{t}_j$. 

(1) if \( n = 2 \), then \( \omega_H = 2\omega_{WP} \);

(2) if \( n = 3 \), then \( \omega_H = (m + 3)\omega_{WP} + \text{Ric}(\omega_{WP}) \) ([17]);

(3) if \( n = 4 \), then \( \omega_H = (2m + 4)\omega_{WP} + 2\text{Ric}(\omega_{WP}) \) ([18]);

where \( n = \text{dim } Z \) and \( m = \text{dim } \mathcal{M} \).

**Definition 2.5.** We call a Calabi-Yau manifold primitive if \( \omega_{H^k} = 0 \) for all \( k < n \).

**Example 2.6.** A Calabi-Yau three-fold is by definition simply connected. It is easy to see that the actions of \( T(\text{Def}) \) on lower degree Hodge bundles are trivial; hence, it is primitive. Similarly, a Calabi-Yau four-fold with vanishing \( h^{1,2} \) is also primitive.

**Example 2.7.** More generally, due to the hard Lefschetz theorem, all the Calabi-Yau hyper-surfaces and complete intersections in projective spaces are primitive.

We give the explicit formulae for the generalized Hodge metrics in the following proposition:

**Proposition 2.8.** Let \( c_1(E) \) be the Ricci form of a vector bundle \( E \). Then we have

\[
\omega_{PH^k} = \sum_{0 \leq p \leq k} pc_1(PR^{k-p}\pi_\ast \Omega^p_{\mathcal{F}/\text{Def}}),
\]

\[
\omega_{H^k} = \sum_{0 \leq p \leq k} pc_1(R^{k-p}\pi_\ast \Omega^p_{\mathcal{F}/\text{Def}}),
\]

for \( k \leq n \).

**Proof.** Fixing a \( k \leq n \), we define the Hodge bundles \( \mathcal{F}_k^0, \ldots, \mathcal{F}_k^0 \) to be

\[
\mathcal{F}_k^p = PR^0\pi_\ast \Omega^k_{\mathcal{F}/\text{Def}} \oplus \cdots \oplus PR^{k-p}\pi_\ast \Omega^p_{\mathcal{F}/\text{Def}}
\]

for \( p = 0, \ldots, k \). Thus for \( q = k - p \),

\[
PR^q\pi_\ast \Omega^p_{\mathcal{F}/\text{Def}} = \mathcal{F}_k^p / \mathcal{F}_k^{p+1}.
\]

In terms of the curvatures, we have

\[
c_1(PR^q\pi_\ast \Omega^p_{\mathcal{F}/\text{Def}}) = c_1(\mathcal{F}_k^p) - c_1(\mathcal{F}_k^{p+1}).
\]

By the Abel summation formula, we have

\[
\sum_{0 \leq p \leq k} pc_1(PR^{k-p}\pi_\ast \Omega^p_{\mathcal{F}/\text{Def}}) = c_1(\mathcal{F}_k^0) + \cdots + c_1(\mathcal{F}_k^1) + c_1(\mathcal{F}_k^0).
\]

Each \( \mathcal{F}_k^p \) is a sub-bundle of the flat bundle \( \mathcal{F}_k^0 = PR^k\pi_\ast \mathbb{C} \). Let \( t_1, \ldots, t_m \) be the local holomorphic coordinate of \( \text{Def} \) and let the bundle map

\[
\frac{\partial}{\partial t_i} : \mathcal{F}_k^p \to \mathcal{F}_k^0 / \mathcal{F}_k^0, \quad 1 \leq i \leq m,
\]
be represented by the matrix
\[ \frac{\partial \Omega_{s}}{\partial t_k} = b_{k \beta} T_{\mu}, \]
where \( \Omega_{s} \) and \( T_{\mu} \) are the basis of \( \mathcal{F}_k^p \) and \( \mathcal{F}_k^0 / \mathcal{F}_k^p \), respectively. Then the first Chern class can be represented by
\[ c_1(\mathcal{F}_k^p) = \frac{\sqrt{-1}}{2\pi} \sum_{\sigma, \mu} b_{k \sigma} \bar{b}_{k \mu} dt_k \wedge d\bar{t}_l \]
for \( 0 \leq p \leq k \). (2.10) follows from the definition of \( \omega_{PH^s} \). (2.11) follows from (2.10) and (2.4). The proof is completed. \( \square \)

The curvature computation is a natural generalization of the similar result in [13], where only the middle dimensional primitive Hodge structures were considered.

**Remark 2.9.** The Weil-Petersson metric is the curvature of the first Hodge bundle:
\[ \omega_{WP} = c_1(R^0 \pi_s(\Omega_{\mathcal{F}/\text{Def}}^n)). \]
Thus by the above equation, Remark 2.3, and Proposition 2.8, the Weil-Petersson metric, the Hodge metric and the generalized Hodge metrics are the Ricci curvatures of various combination of the Hodge bundles.

With the above interpretation of the metrics, the following is obvious:

**Corollary 2.10.** Using the above notation, for \( n \geq 2 \), we have
\[ \omega_{H} \geq 2\omega_{WP}. \]

*Proof.* By Remark 2.9 and Serre Duality, we get
\[ \omega_{WP} = c_1(R^0 \pi_s(\Omega_{\mathcal{F}/\text{Def}}^n)) = -c_1(R^n \pi_s(\mathcal{F})). \]
By (2.12) and the fact that \( \mathcal{F}_n^0 \) is flat, we have
\[ c_1(\mathcal{F}_n^1) = -c_1(R^n \pi_s(\mathcal{F})) = \omega_{WP}. \]
Also, since \( \mathcal{F}_n^{n+1} = 0 \),
\[ c_1(\mathcal{F}_n^n) = c_1(R^0 \pi_s(\Omega_{\mathcal{F}/\text{Def}}^n)) = \omega_{WP}. \]
According to (2.14), \( c_1(\mathcal{F}_n^p) \geq 0 \) for all \( p \). Hence, when \( n \geq 2 \), by (2.10), (2.13), (2.17) and (2.18), we have
\[ \omega_{H} \geq c_1(\mathcal{F}_n^n) + c_1(\mathcal{F}_n^1) = 2\omega_{WP}. \]
The proof is finished. \( \square \)
3. BCOV torsion

BCOV torsion was first defined by Bershadsky-Ceccotti-Ooguri-Vafa in their study of the Physics Mirror Symmetry. It was constructed as the partition function for the $N=2$ SCFT. In their breakthrough works [1], [2], the torsion was determined using the Mirror Symmetry Conjecture. One amazing consequence of their work is that, given the local expansion of the BCOV torsion, they were able to predict the number of embedded elliptic curves of all degrees in a given Calabi-Yau manifold. The prediction matches all known low degree cases. Furthermore, they also discussed the higher genus cases based on computation of genus one case. Notice that a similar prediction of counting rational curves for quintics, which invoked intensive mathematical research, was first made by Candelas et al. [7]. Through the fundamental works of Kontsevich, Givental, Lian-Liu-Yau and many others (see [10] for a complete reference), the prediction has been mathematically verified. It is thus of crucial interest to understand the BCOV torsion in terms of algebraic and differential geometry.

**Definition 3.1.** The BCOV torsion of a Calabi-Yau manifold is

\[
T = \prod_{1 \leq p, q \leq n} (\det \Delta_{p, q}^t)^{(-1)^{p+q} p q},
\]

where $\Delta_{p, q}$ is the $\tilde{\partial}$-Laplace operator on $(p, q)$ forms with respect to the Ricci-flat metric on a fiber; $\Delta_{p, q}^t$ represents the non-singular part of $\Delta_{p, q}$; the determinant is taken in the sense of zeta function regularization.

The BCOV torsion is an analytic torsion in the sense of Ray-Singer [20]. To see this, we define a holomorphic coefficient vector bundle over the total space $\mathcal{X} \to \text{Def}$,

\[
E = \bigoplus_{p=1}^{n} (-1)^{p} p \Omega_{\mathcal{X}/\text{Def}}^{p}.
\]

$E$ inherits a natural Hermitian metric induced from the metric on the relative tangent bundle. According to [14], there exists a corresponding determinant line bundle over $\mathcal{M}$, which is defined to be

\[
\lambda = \bigwedge_{0 \leq p, q \leq n} \det(H^{p, q}(Z, E, \bar{\partial}))^{(-1)^{p+q} p},
\]

where $H^{p, q}(Z, E, \bar{\partial}) = R^{q} \pi_{*} \Omega_{\mathcal{X}/\text{Def}}^{p}$ are holomorphic vector bundles over $\text{Def}$. We identify the cohomology groups with the corresponding harmonic forms with respect to the naturally induced metrics on various spaces.

There are two natural metrics defined on $\lambda$. The usual $L^2$ metric is defined by the harmonic forms, and the Quillen metric is given by

\[
\| \cdot \|_{L^2}^2 = \| \cdot \|_{L^2}^2 T.
\]

The formulation that we have in (3.2), (3.3) and (3.4) makes it possible to study BCOV torsion in the framework developed in [4], [5], [6]. First, we verify the following which is obvious for polarized families:
Lemma 3.2. There is a Kähler metric $g^X$ on $\mathcal{X}$ such that $g^X|_{\pi^{-1}(t)}$ gives the polarized Ricci flat metric for each $t \in \text{Def}$. \qed

The following lemma of Beshadsky-Cecotti-Ooguri-Vafa [1], [2] characterizes the geometry of the coefficient bundle $E$.

Lemma 3.3. For $E$ associated with the induced metric from $g^X$, as forms on $X$ we have

$$\text{Td}(\mathcal{F}_{\mathcal{X}/\text{Def}}) \text{ch}(E) = -c_{n-1} + \frac{n}{2} c_n - \frac{1}{12} c_1 c_n,$$

where $c_i = c_i(\mathcal{F}_{\mathcal{X}/\text{Def}})$ and $\mathcal{F}_{\mathcal{X}/\text{Def}}$ is the holomorphic relative tangent bundle.

Proof. See [2], page 374. \qed

As a consequence,

Proposition 3.4. The Quillen metric is the potential of $\chi_Z/12$ times the Weil-Petersson metric; in other words,

$$c_1(\lambda, \| \|_Q) = \frac{\chi_Z}{12} \omega_{\text{WP}}.$$

Proof. This also appears in [2]. We include the proof here for completeness. First, notice that when $\mathcal{F}_{\mathcal{X}/\text{Def}}$ is associated with the metric induced from $g^X$, its Ricci form has no vertical component because of definition of $g^X$. In fact, we claim

$$c_1(\mathcal{F}_{\mathcal{X}/\text{Def}}) = -\pi^*(\omega_{\text{WP}}).$$

To see this, we let $\Omega$ be a holomorphic section of the relative canonical bundle $\omega_{\mathcal{X}/\text{Def}} \rightarrow \mathcal{X}$. Then there is a smooth function $f(t)$ on $\text{Def}$ such that

$$\Omega \wedge \bar{\Omega} = \pi^*(f(t)) \omega^n_t;$$

where $\omega_t$ is the polarized Ricci flat metric on $\pi^{-1}(t)$ for $t \in \text{Def}$. Apparently we have

$$\int_{\pi^{-1}(t)} \Omega \wedge \bar{\Omega} = f(t),$$

since the volumes are $c_1(l)^n = 1$. Thus we have

$$\Omega \wedge \bar{\Omega} = \left( \int_{\pi^{-1}(t)} \Omega \wedge \bar{\Omega} \right) \cdot \omega^n_t.$$

Equation (3.7) then follows from (3.9), Remark 2.9 and the Poincaré-Lelong formula.

By Lemma 3.2, a direct application of the family Grothendieck-Riemann-Roch theorem proved by Bismut-Gillet-Soulé [4] gives

$$c_1(\lambda, \| \|_Q) = \left[ \int_{Z} \text{Td}(\mathcal{F}_{\mathcal{X}/\text{Def}}) \text{ch}(E) \right]^{(1,1)}.$$
By Lemma 3.3, (3.7) and the Gauss-Bonnet formula, we have:

\[
(3.11) \quad c_1(\lambda, \| \cdot \|) = -\left( \int_{\mathcal{Z}} \frac{1}{12} c_1(\mathcal{F}_{\mathcal{X}/\text{Def}}) c_n(\mathcal{F}_{\mathcal{X}/\text{Def}}) \right)^{(1,1)} = \frac{\pi}{12} \omega_{\mathcal{WP}}.
\]

**Remark 3.5.** It is worthwhile to point out that Fujiki-Schumacher ([12], Theorem 10.3) have realized the Weil-Petersson metric as the curvature of a determinant line bundle equipped with a Quillen metric. Their construction uses a different coefficient bundle.

The next proposition relates the curvature of \( \lambda \) with respect to the \( L^2 \)-metrics to the generalized Hodge metrics.

**Proposition 3.6.** Using the notations in the above and in the previous section, we have

\[
(3.12) \quad c_1(\lambda, \| \cdot \|_{L^2}) = \sum_{i=1}^{n} (-1)^i \omega_{H^i}.
\]

**Proof.** This is due to Proposition 2.8. It is an easy computation to show that

\[
(3.13) \quad c_1(\lambda, \| \cdot \|_{L^2}) = \sum_{0 \leq p, q \leq n} (-1)^{p+q} pc_1(R^q \pi_* \Omega^n_{\mathcal{X}/\text{Def}}) = \sum_{k=1}^{n} (-1)^k \omega_{H^k}.
\]

The proof is complete. \( \square \)

**Proof of Theorem 1.1.** Because the statements are local, we first prove them on the local deformation space \( \text{Def} \). Using the relation of the \( L^2 \) metric, the Quillen metric and the BCOV torsion, (1.2) follows from (3.4), Proposition 3.4, and Proposition 3.6. The equation (1.3) follows from (1.2) and Remark 2.3. The equations (1.2) and (1.3) are also true on Calabi-Yau moduli \( \mathcal{M} \) because of functoriality. \( \square \)

Theorem 1.1 is an explicit relation between the two canonically defined Kähler metrics, the Weil-Petersson metric and the generalized Hodge metrics, on the deformation space \( \text{Def} \) (as well as the moduli space). The surprising fact is that the bridge is the BCOV torsion, a spectral invariant of Ricci-flat metrics, which is also of its own significance in physics literature.

**Proof of Corollary 1.4.** The global Poincaré metric defined at the beginning of Appendix A is asymptotic to the Poincaré metric in (1.5). So they are equivalent on \( (\Delta^\mathcal{X})^t \times \Delta^{m-t} \). The corollary thus follows from Theorem 1.1, Corollary 2.10 and Theorem A.1. \( \square \)

Asymptotic expansion of \( T \) near points of maximal monodromy degeneration will give the prediction for counting elliptic curves in the Calabi-Yau manifolds. The above result determines the asymptotic behavior of the BCOV torsion up to a (possibly multi-valued) pluri-harmonic function.

**Proof of Corollary 1.2.** If \( N \) is a smooth submanifold in the smooth part of \( \mathcal{M} \), the corollary follows from the ordinary Stokes theorem. In general, we need to generalize the Stokes theorem into the singular case. Let’s first define the Hodge and the Weil-Petersson volumes on \( N \).
Let $m$ be a smooth $2k$ form of the orbifold $\mathcal{M}$, where $k = \dim N$. Then we can define a measure on $N$ as follows:

Let $x \in N \subset \mathcal{M}$ and let $(\tilde{U}, p, U)$ be a local uniformization of $\mathcal{M}$ at $x$; i.e., $p : \tilde{U} \to U = \tilde{U}/G$ for a finite group $G$. Let $V = U \cap N$. Let $\bar{N} \subset \tilde{U}$ be the pre-image of $N$ under $p$. For $y \in V$, define $n(y)$ to be the multiplicity of the map $\bar{N} \to N$. We can define a measure $\mu$ on $V$ by

$$\mu(V) = \int_V \frac{1}{\nu} n(y) \rho^*(\mu)|_N,$$

where $\nu$ is the pre-image of $V$ under $p$. The generalized Hodge metrics, the Weil-Petersson metric, and $\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log T$, through this definition, define the corresponding measures on $N$. In particular, the Hodge and the Weil-Petersson volumes are defined.

The function $n(y)$ defined above is in fact independent of the choice of the local uniformization so it is a global function on $N$. It is a constant on some Zariski open set $N'$ of $N$. Because of this, it is sufficient to prove that

$$(3.14) \quad \Vol_H(N') = \left[\frac{(-1)^n}{12 \pi \omega}\right]^k \Vol_{WP}(N').$$

Thus the corollary follows from the following version of Stokes theorem:

**Lemma 3.7.** Let $\eta$ be a smooth $2k - 1$ form on the orbifold $\mathcal{M}$. Then $d\eta$ defines a measure $\mu(\eta)$ on $N$. We have

$$\int_{N_{\text{reg}}} \mu(\eta) = 0,$$

where $N_{\text{reg}}$ is the smooth part of $N$.

**Proof.** Let $N'' = N - N_{\text{reg}} \cap N'$. Let $\rho$ be a smooth function such that: (1) $\rho = 0$ if $\text{dist}(x, N'') < \varepsilon$; (2) $\rho = 1$ if $\text{dist}(x, N'') > 2\varepsilon$; and $|\nabla \rho| \leq 3/\varepsilon$.

By the ordinary Stokes Theorem,

$$\int_N d(\rho \eta) = 0.$$

Thus we have

$$\int_{N_{\text{reg}} \cap N'} \rho \, d\eta + \int_N d\rho \wedge \eta = 0.$$

Since $N''$ is compact, its Hausdorff measure is finite. Thus

$$\left| \int_N d\rho \wedge \eta \right| \leq C \cdot \frac{3}{\varepsilon} \cdot \pi \omega^2(H(N'')) \to 0.$$
Let \( n(y) = \text{const} \) on \( N' \). Then
\[
0 \leftarrow \text{const} \cdot \int_{N_{\mu^0 \cap N'}} \rho \, d\eta \rightarrow \int_{N} \mu(\eta).
\]
This completes the proof. \( \square \)

**Remark 3.8.** An alternating way to prove Lemma 3.7 is as follows: since \( d\eta \) is smooth on \( \mathcal{M} \), it defines a current which is equal to measure \( d\eta \) on the regular part of \( N \). The lemma thus follows from the de-Rham theorem for currents.

**Proof of Corollary 1.3.** Suppose \( N \) is a complete curve of \( \mathcal{M} \). Then by Corollary 1.2,
\[
\text{Vol}_H(N) = \frac{(-1)^n}{12} \int_{\mathcal{Z}} \text{Vol}_{\text{WP}}(N).
\]
By the assumption, we thus have
\[
\text{Vol}_H(N) < 2 \text{Vol}_{\text{WP}}(N),
\]
unless \( N \) is of dimension 0. However, the above inequality contradicts Corollary 2.10. \( \square \)

**Remark 3.9.** All Calabi-Yau three-folds are primitive. By the Physics Mirror Symmetry, it is conjectured that all the Calabi-Yau three-folds occur in pairs (mirror pair); and the Euler characteristics of paired Calabi-Yau three-folds are differed only by signs. In Corollary 1.3 we claim that “more than half” of moduli of polarized Calabi-Yau three-folds contains no complete subvariety.

We proceed to discuss the obstruction to the existence of complete curves in the Calabi-Yau moduli, in some more specific situations.

**Remark 3.10.** For a primitive Calabi-Yau four-fold \( Z \), let \( N \) be a complete curve in the moduli space \( \mathcal{M} \). By Theorem 2.4, we have
\[
(3.15) \quad \int_{\mathcal{N}} \omega_H = (2m + 4) \int_{\mathcal{N}} \omega_{\text{WP}} + 2 \int_{\mathcal{N}} \text{Ric}(\omega_{\text{WP}}).
\]
By Corollary 1.2, we have
\[
(3.16) \quad \int_{\mathcal{N}} \omega_H = \frac{\mathcal{X}_Z}{12} \int_{\mathcal{N}} \omega_{\text{WP}}.
\]
If \( \mathcal{X}_Z > 24(m + 2) \), from (3.15) and (3.16), we have
\[
(3.17) \quad \int_{\mathcal{N}} \text{Ric}(\omega_{\text{WP}}) \geq 0.
\]
On the other hand, the Ricci curvature of \( \omega_H \) is negative, so
\[
\int_{\mathcal{N}} \text{Ric}(\omega_H) < 0.
\]
The above inequality contradicts to (3.17) because

$$\text{Ric}(\omega_H) - \text{Ric}(\omega_{WP}) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \frac{\omega_H^n}{\omega_{WP}^m} \right),$$

and thus using Lemma 3.7, the integration of the two Ricci curvatures are the same. This is a contradiction. Thus if $\chi_Z > 24(m+2)$, there is no complete curve in $\mathcal{M}$. The similar result in the case of Calabi-Yau threefold is trivial, while an analogue for high dimensional Calabi-Yau manifolds is still unknown.

**Remark 3.11.** When the dimension of $\mathcal{M}$ is 2, if there exists a complete curve $C \subset \mathcal{M}$, then

(3.18) \[ \int_C \text{Ric}_{WP} = \chi_C + C.C, \]

where $C.C$ is the self-intersection number of $C$. Let $\omega' = \omega_{WP}|_C$. In local coordinate, let

$$\omega' = \frac{\sqrt{-1}}{2\pi} h_{1\bar{1}} dt_1 \wedge d\bar{t}_1,$$

and let

$$\omega_{WP} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^2 h_{ij} dt_i \wedge d\bar{t}_j.$$

Assume that at a point $x$, $h_{ij} = \delta_{ij}$. Then we have

$$\partial_1 \bar{\partial}_1 \log h_{1\bar{1}} = \partial_1^2 h_{1\bar{1}} \partial_1 \bar{\partial}_1 h_{1\bar{1}} - h_{1\bar{1}}^{-1} \left| \partial_1^2 h_{1\bar{1}} \right|^2.$$

Since

$$h_{1\bar{1}}^{-1} \left| \partial_1^2 h_{1\bar{1}} \right|^2 \leq h_{||}^2 \partial_1^2 h_{1\bar{1}} \cdot \partial_1 \bar{\partial}_1 h_{1\bar{1}},$$

we have

$$\partial_1 \bar{\partial}_1 \log h_{1\bar{1}} \geq R_{1\bar{1}1\bar{1}},$$

where $R_{ijkl}$ is the curvature tensor of $\omega_{WP}$. By the (generalized) Strominger formula (cf. [18], Theorem 3.1, [24], and [28]), we have

$$R_{1\bar{1}2\bar{2}} \leq 1.$$

Thus we have the following:

$$\text{Ric}(\omega') \leq \text{Ric}(\omega_{WP}) + 1.$$
Comparing the above equation with (3.18), we have
\[ C.C \geq -1. \]
This would be a new obstruction to the existence of complete curves in these cases.

Notice that the results we proved above are only on the smooth moduli. It is very interesting to see the corresponding results for the compactified moduli. This kind of global results will be achieved by finer local analysis of the BCOV torsion near the boundary of the moduli.

Appendix A. Poincaré metric and generalized Hodge metrics

In this appendix, we prove that the generalized Hodge metrics are bounded by the Poincaré metric. The main result of this appendix, Theorem A.1, is a degenerate version of Yau’s Schwarz Lemma (cf. [31]) from Kähler manifolds to Hermitian manifolds.

Calabi-Yau moduli \( \mathcal{M} \) are quasi-projective. The smooth part \( \mathcal{M}_{\text{reg}} \) of \( \mathcal{M} \) allows a compactification \( \tilde{\mathcal{M}} \), where \( \mathcal{M} \) is a compact smooth manifold such that \( \tilde{\mathcal{M}} \backslash \mathcal{M}_{\text{reg}} \) is a divisor of normal crossing. For the pair of manifolds \( (\tilde{\mathcal{M}}, \mathcal{M}_{\text{reg}}) \), we define a Kähler metric \( \omega_{GP} \) called the global Poincaré metric as follows: in a neighborhood \( (\Delta^e)^I \times \Delta^{m-I} \) of \( x \in \tilde{\mathcal{M}} \setminus \mathcal{M} \), the global Poincaré metric \( \omega_{GP} \) is asymptotically the Poincaré metric \( \omega_P \) defined in (1.5):

\[
\omega_{GP} \sim \omega_P = \sum_{i=1}^j \sqrt{-1} \frac{1}{|z_i|^2 \log \frac{1}{|z_i|}} dz_i \wedge d\bar{z}_i + \sum_{i=1}^m \sqrt{-1} dz_i \wedge d\bar{z}_i.
\]

The global Poincaré metric \( \omega_{GP} \) is a complete Kähler metric on \( \tilde{\mathcal{M}} \) with lowerly bounded Ricci curvature and finite volume. For the detailed construction of \( \omega_{GP} \), see [9], [15].

The main result of this section is the following:

**Theorem A.1.** Using the above notations, we have

\[ \omega_{PH} \leq C \omega_{GP} \]

for \( 0 \leq k \leq n \) and a constant \( C \) depending on the lower bound of Ric\( (\omega_{GP}) \), dimension of the Calabi-Yau manifolds and dimension of the moduli space \( \mathcal{M} \).

We use \( H^{p,q} \) to denote the bundle \( PR^{p,q} \Omega_{\mathcal{M}/\mathcal{X}}^p \) for \( 0 \leq p, q \leq n \). Let

\[ F_k^p = H^{k,0} \oplus \cdots \oplus H^{p,k-p} \quad \text{for } p = 0, \ldots, k. \]

We also assume that \( H^{k+2, -2} = H^{k+1, -1} = H^{-1, k+1} = H^{-2, k+2} = 0 \) and \( F_{k+1}^k = 0 \), \( F_{k-1}^k = F_0^k \), for the sake of simplicity.

First we try to express the generalized Hodge metrics in local coordinates. Fix \( k \leq n \), \( p \leq k \) and \( q = k - p \). Let \( \{ \Omega_{p,i} \}, i = 1, \ldots, h^{p,q} \) be a local holomorphic frame of \( H^{p,q} \).
Definition A.2. Let \((t_1, \ldots, t_m)\) be a holomorphic local coordinate at a point of \(\mathcal{M}_{reg}\). We define \(D_\alpha \Omega_{p,i} \in H^{p-1,q+1}\) to be the projection of \(\partial \Omega_{p,i} = \partial \Omega_{p,i} / C_{01}^1 \) to \(H^{p-1,q+1}\) with respect to the bilinear form \(Q(\ , \) \()\) in (2.5).

For simplicity, we shall use \((\ , \) \()\) instead of the bilinear form \(Q\) in (2.5). With the above notation,

\[
(A.1) \quad (g_p)_{ij} = \langle \Omega_{p,i}, \overline{\Omega}_{p,j} \rangle = (-1)^{q-p}(\Omega_{p,i}, \overline{\Omega}_{p,j})
\]

is the Hermitian metric matrix of \(H^{p,q}\) for \(p = 0, \ldots, k\). It is thus easy to see

Proposition A.3. For fixed \(k\), the generalized Hodge metric matrix for the local coordinate system \((t_1, \ldots, t_m)\) with respect to \(PH^k\), defined in Definition 2.2, is

\[
(A.2) \quad h_{\alpha\beta} = \sum_{p=0}^{k} (-1)^{q-p+1} g_p^{ij} (D_{\alpha} \overline{\Omega}_{p,i}, D_{\beta} \overline{\Omega}_{p,j}),
\]

where \((g_p^{ij})\) is the inverse of \((g_p)_{ij}\).  

We proceed with two technical lemmas.

Lemma A.4.

\[
(A.3) \quad \overline{\partial}_\beta D_\alpha \Omega_{p,i} = g_p^{ij} \langle \overline{\partial}_\beta D_\alpha \Omega_{p,i}, \overline{\Omega}_{p,j} \rangle \Omega_{p,j}.
\]

Proof. We first claim that \(\overline{\partial} D_\alpha \Omega_{p,i} \in H^{p,q}\). To see this, let \(\Omega_1 \in \mathcal{F}_{k+1}^p\). Then

\[
(A.4) \quad \langle \overline{\partial} D_\alpha \Omega_{p,i}, \overline{\Omega}_1 \rangle = -(D_\alpha \Omega_{p,i}, \overline{\Omega}_1) = 0.
\]

On the other hand, if \(\Omega_2 \in H^{p-1,q+1}\), we have the decomposition

\[
\overline{\partial} \Omega_{p,i} = D_\alpha \Omega_{p,i} + B
\]

for \(B \in H^{p,q}\). Furthermore,

\[
\overline{\partial} B \in \mathcal{F}_k^p.
\]

Thus,

\[
(A.5) \quad \overline{\partial} D_\alpha \Omega_{p,i} \in \mathcal{F}_k^p.
\]

Combining (A.4), (A.5), we have \(\overline{\partial} D_\alpha \Omega_{p,i} \in H^{p,q}\). Writing \(\overline{\partial} D_\alpha \Omega_{p,i}\) as the linear combination of \(\Omega_{p,i}\)'s, we get (A.3).  

Lemma A.5. If \(A\) is a local section of \(H^{p,q}\) and \(B\) is a local section of \(H^{p-1,q+1}\), then

\[
\langle D_\alpha A, B \rangle = \langle A, \overline{\partial} B \rangle.
\]
Proof. This follows from a straightforward computation:

\[
\langle D_x A, B \rangle = (\sqrt{-1})^{g-p+2}(\bar{\partial}_x A, B)
\]

\[
= -(\sqrt{-1})^{g-p+2}(A, \bar{\partial}_x B) = \langle A, \bar{\partial}_x B \rangle.
\]

Proof of Theorem A.1. The generalized Hodge metrics are only semi-positive definite but not positive definite. If they were positive definite, then using the similar method as in [16], we should have been able to prove that the holomorphic sectional curvatures of the metrics were negative and bounded away from zero, and the holomorphic bisectional curvatures of the metrics were nonpositive. Thus we could have used Yau’s Schwarz Lemma [31] to get the conclusion. The contribution of this appendix is that we prove the same result even if the generalized Hodge metrics fail to be positive definite.

We assume all the notations in the previous sections. Write the global Poincaré metric in local coordinates as

\[
\omega_{GP} = \frac{\sqrt{-1}}{2\pi} \tau^\beta \sigma^\gamma dt^\gamma \wedge d\bar{t}^\beta.
\]

Let \(-C_1\) be the lower bound of the Ricci curvature of \(\omega_{GP}\) for some constant \(C_1 > 0\). We define a smooth function

\[
f = \sum \tau^{a\bar{a}} h_{a\bar{a}}
\]

on \(\mathcal{M}\). \(f\) is nonnegative. If \(f\) is bounded, then \(\omega_{PH}\) is bounded by \(\omega_{GP}\).

For the rest of the appendix, we assume that at the given point \(x\), \(\tau_{ij} = \delta_{ij}\) and \(d\tau_{ij} = 0\). Then at \(x\),

\[
(A.6) \quad \Delta f \geq -C_1 f + \frac{\partial^2}{\partial t^i \partial \bar{t}^j} h_{i\bar{j}},
\]

where \(\Delta\) is the Laplacian on \(\mathcal{M}\) with respect to \(\omega_{GP}\).

We assume that at the point \(x\), the frames \(\Omega_{p,i}\) are chosen so that \((g_p)_{ij} = \delta_{ij}\), and \(\frac{\partial}{\partial t_a} (g_p)_{ij} = 0\) for \(p = 0, \ldots, k\) and \(a = 1, \ldots, m\). Then a straightforward computation gives

\[
(A.7) \quad \frac{\partial^2}{\partial t^i \partial \bar{t}^j} h_{i\bar{j}} = \sum_{p}(\sqrt{-1})^{g-p+2} (- (R_{p})_{i\bar{j}k\bar{l}}) (D_x \Omega_{p,i}, \bar{D}_x \Omega_{p,j})
\]

\[
+ \sum_{p}(\sqrt{-1})^{g-p+2} (\bar{\partial}_x D_x \Omega_{p,i}, \bar{\partial}_x D_x \Omega_{p,j})
\]

\[
+ \sum_{p}(\sqrt{-1})^{g-p+2} (\bar{\partial}_x D_x \Omega_{p,i}, \bar{\partial}_x D_x \Omega_{p,i})
\]

\[
+ \sum_{p}(\sqrt{-1})^{g-p+2} (\bar{\partial}_x D_x \Omega_{p,i}, \bar{\partial}_x D_x \Omega_{p,j})
\]

\[
+ \sum_{p}(\sqrt{-1})^{g-p+2} (D_x \Omega_{p,i}, \bar{\partial}_x D_x \Omega_{p,j}),
\]
where \((R_p)_{ij}^l\) is the curvature tensor of \(g_p\) for \(p = 0, \ldots, k\). By Lemma A.4, we have

(A.8) \[ (\partial_j \partial_i D_{\omega p, i}, D_{\omega p, i}) = - (\partial_j \partial_i D_{\omega p, i}, \overline{\partial_j \partial_i D_{\omega p, i}}); \]

(A.9) \[ (D_{\omega p, i}, \bar{\partial}_j \partial_i D_{\omega p, i}) = - (\partial_j \partial_i D_{\omega p, i}, \bar{\partial}_j \partial_i D_{\omega p, i}). \]

Inserting the above two equations into (A.7), we have

(A.10) \[
\frac{\partial^2}{\partial t_i \partial t_j} \hat{h}_{ij} = \sum_p (\sqrt{-1})^{q_p+2} (R_p)_{ij}^l (D_{\omega p, i}, D_{\omega p, j}) + \sum_p (\sqrt{-1})^{q_p+2} (\partial_j \partial_i D_{\omega p, i}, \bar{\partial}_j \partial_i D_{\omega p, i}) - \sum_p (\sqrt{-1})^{q_p+2} (\bar{\partial}_j \partial_i D_{\omega p, i}, \partial_j \partial_i D_{\omega p, i}).
\]

By [13], page 33, Proposition 4, the curvature of \((g_p)_{ij}\) is

(A.11) \[ (R_p)_{ij}^l = (\sqrt{-1})^{q_p} (D_{\omega p, i}, D_{\omega p, j}) - (\sqrt{-1})^{q_p} (\partial_j \partial_i D_{\omega p, i}, \bar{\partial}_j \partial_i D_{\omega p, i}). \]

Let

(A.12) \[ \partial_j D_{\omega p, i} = A_{p+2, i} + B_{p+2, i}, \]

where \(A_{p+2, i} \in H^{p-2, q+2}\) and \(B_{p+2, i} \in H^{p-1, q+1}\). Then

(A.13) \[
\frac{\partial^2}{\partial t_i \partial t_j} \hat{h}_{ij} = -\sum_p (\sqrt{-1})^{q_p+2} (R_p)_{ij}^l (D_{\omega p, i}, D_{\omega p, j}) - |A_{p+2, i}|^2 + |B_{p+2, i}|^2 - \sum_p (\sqrt{-1})^{q_p+2} (\partial_j \partial_i D_{\omega p, i}, \bar{\partial}_j \partial_i D_{\omega p, i}).
\]

By Lemma A.4 and Lemma A.5, let

\[ D_{\omega p, i} = (A^p_{2})_{i} \Omega_{p+1, i}, \]

for matrices \(A^p_{2} = (A^p_{2})_{il}\). Then we have

\[ \partial_j D_{\omega p, i} = (A^p_{2} - 1)_{il} \Omega_{p-1, i}. \]

Thus from (A.11), (A.13), in terms of the matrices \(A^p_{2}\), we have

(A.14) \[
\frac{\partial^2}{\partial t_i \partial t_j} \hat{h}_{ij} = \sum (A^p_{2})_{il} (\bar{A}^p_{2})_{lj} (A^p_{2})_{il} (\bar{A}^p_{2})_{lj} - \sum (A^p_{2} - 1)_{il} (\bar{A}^p_{2} - 1)_{lj} (A^p_{2})_{il} (\bar{A}^p_{2})_{lj} - \sum |\sum (A^p_{2} - 1)_{il} (A^p_{2} - 1)_{lj}|^2 + \sum |\sum (A^p_{2})_{il} (\bar{A}^p_{2})_{lj}|^2 + |B_{p+2, i}|^2.
\]
The $H^{p-2,q+2}$ part $A_{p;i}$ of $\partial_z D_i \Omega_{p;i}$ is the same as the $H^{p-2,q+2}$ part of $\partial_z \partial_j \Omega_{p;i}$. Thus

$$A_{p;i} = A_{p;i}$$

for $1 \leq \alpha, \gamma \leq m$ and $p = 0, \ldots, k$. In terms of the matrices $A^p_z$, we have


for $1 \leq \alpha, \gamma \leq m$ and $p = 0, \ldots, k$. Using these commutative relations of the matrices $A^p_z$, from (A.14), we have

$$\sum_{\alpha, \beta, \gamma \in \mathcal{L}_z \mathcal{L}_\gamma} \frac{\partial^2}{\partial \alpha \partial \gamma} h_{zz} = \sum_{\alpha, \beta, \gamma \in \mathcal{L}_z \mathcal{L}_\gamma} |B_{p;i}|^2 + \sum_{\alpha, \beta, \gamma} \text{Tr} \left[ \left( \left( A^p_A \right)^T A^p_{A+1} - A^p_A \left( A^p_A \right)^T \right) \right.
\times \left. \left( \left( A^p_A \right)^T A^p_{A+1} - A^p_A \left( A^p_A \right)^T \right)^T \right].$$

Since the last term of the above expression is zero if and only if $A^p_A \equiv 0$ for any $\alpha$ and $p$, there exists an $\varepsilon > 0$, such that

$$\sum_{\alpha, \beta, \gamma} \frac{\partial^2}{\partial \alpha \partial \gamma} h_{zz} \geq \varepsilon \sum_{p} \text{Tr} \left( A^p_A \left( A^p_A \right)^T \right) \geq \varepsilon |h_{zz}|^2.$$  

Finally, from (A.6), and (A.16), we have

$$\Delta f \geq \frac{\varepsilon}{m} f^2 - C_1 f.$$ 

Since the Ricci curvature of $\omega_{WP}$ is lower bounded and since $f$ is nonnegative, the generalized maximum principal (cf. [8]) gives

$$f \leq mC_1 / \varepsilon$$

and the theorem is thus proved. \qed

References

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