Trends in
Partial Differential Equations
ALM 10, pp. 241–264

Bounds of Eigenvalues on Riemannian Manifolds

Jun Ling* Zhiqin Lu†

Abstract

In this paper, we first give a short review of the eigenvalue estimates of Laplace operator and Schrödinger operators. Then we discuss the evolution of eigenvalues along the Ricci flow, and two new bounds of the first eigenvalue using gradient estimates.

2000 Mathematics Subject Classification: 58J50, 35P15, 53C21.

Keywords and Phrases: Eigenvalue, lower bounds, upper bounds, Riemannian manifolds.

1 Introduction

In this paper, we discuss the eigenvalue estimates of Laplace operator and Schrödinger operators. Let (M, g) be an n-dimensional compact connected Riemannian manifold with or without boundary. Let Δ be the Laplacian of the metric $g = (g_{ij})_{n\times n}$. In local coordinates $\{x^i\}_{i=1}^n$,

$$\Delta = \frac{1}{\sqrt{\det(g)}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}\, g^{ij}\,\frac{\partial}{\partial x^j}\right),$$

where the matrix (g^{ij}) is the inverse matrix of $g = (g_{ij})$. We consider the following three eigenvalue problems on the manifold (M, g).

*Department of Mathematics, Utah Valley University, Orem, UT 84058, USA. Email: lingju@uvu.edu. Research partially supported by the NSF under the program 'The Geometric Evolution Equations and Related Topics' at the MSRI at Berkeley and by the UVU School of Science and Health.

†Department of Mathematics, University of California, Irvine, CA 92697, USA. Email: zlu@uci.edu. Research partially supported by NSF grant DMS 0347033.

Closed eigenvalue problem for the Laplacian. $\partial M = \emptyset$. Find all real numbers λ for which there are nontrivial solutions $u \in C^2(M)$ to the equations

$$-\Delta u = \lambda u. \tag{1.1}$$

Dirichlet eigenvalue problem for the Laplacian. $\partial M \neq \emptyset$. Find all real numbers λ for which there are nontrivial solutions $u \in C^2(M) \cap C^0(\overline{M})$ to (1.1), subject to the boundary condition

$$u = 0 \qquad \text{on} \quad \partial M.$$

Neumann eigenvalue problem for the Laplacian. $\partial M \neq \emptyset$. Find all real numbers λ for which there are nontrivial solutions $u \in C^2(M) \cap C^1(\overline{M})$ to (1.1), subject to the boundary condition

$$\frac{\partial}{\partial \nu} u = 0 \qquad \text{on} \quad \partial M,$$

where ν is an outward unit normal vector fields of ∂M.

We may consider these three eigenvalue problems for other operators, for instance Schrödinger operators, in similar ways.

The spectrum of the Laplacian of a closed manifold consists of pure point spectrum $\{\lambda_i\}_{i=0}^{\infty}$ that can be arranged in the order

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \longrightarrow \infty.$$

For a manifold with boundary, in the similar way, we can arrange the Dirichlet and Neumann eigenvalues like above (in Dirichlet case, the smallest eigenvalue is always positive).

The eigenvalues can be characterized by the variational principle. For $u \in C^\infty(M)$, let

$$\|u\|^2 = \int_M |u|^2 + \int_M |\nabla u|^2.$$

Let $H^1(M)$, $H_0^1(M)$ be the completion of $C^\infty(M)$, $C_0^\infty(M)$ (C^∞ functions with compact support in M) with respect to the above norm $\|\cdot\|$.

Let $\{u_i\}$ be an orthonormal basis for $L^2(M)$ with $-\Delta u_i = \lambda_i u_i$, $u_i \in C^\infty(M) \cap H$, where $H = H^1(M)$ or $H_0^1(M)$ depending on the boundary conditions. Then we have the following variational principal:

$$\lambda_i = \inf_{\int_M u u_k = 0, \ k<i} \frac{\int_M |\nabla u|^2}{\int_M |u|^2}.$$

It is an interesting problem to study the distribution of eigenvalues and eigenfunctions since they reveal important relations between geometry of the manifold and analysis. Such a problem roots in classical analysis, physics and geometric analysis. Early works in the field include Weyl's asymptotic formula [72], Courant's nodal domain theorem [28], and so on. By study the heat kernel of the Laplacian, Weyl was able to prove that

$$N(\lambda) \sim \omega_n (\mathrm{vol} M) \lambda^{n/2} / (2\pi)^n$$

as $\lambda \to +\infty$, where ω_n is the volume of the unit solid ball in $\mathbb{R}^n$, and $N(\lambda)$ is the number of eigenvalues less than or equal to λ, multiplicity counted. Courant's nodal domain theorem states that the number of nodal domains of the k-th eigenfunction is less than or equal to $k+1$. These theorems are of fundamental importance in understanding the Laplacian. Computing the lower order terms of the asymptotic formula, and finding the limit distribution of the nodal sets are the two ongoing research projects.

In this paper, we only discuss the eigenvalue estimates cotangent to the gradient estimates. We refer to [65] for more details. The paper is organized as follows. In § 2, we discuss the estimates of the first non-zero eigenvalue, λ_1. In § 3, we discuss various bounds of eigenvalues, and eigenvalues on Riemann surfaces, and so on. Finally, in § 4, we give some new lower bounds of the first eigenvalue and the gap of the first two eigenvalues using the gradient estimates.

2 Bounds on the first eigenvalue

2.1 Lichnerowicz bound and the gradient estimates

We first discuss the lower bounds for the first eigenvalue λ_1 of the Laplacian on an n-dimensional compact Riemannian manifold (M, g) with positive Ricci curvature

$$Ric \geq (n-1)Kg \qquad \text{with constant } K > 0. \tag{2.2}$$

If $\partial M = \emptyset$, the classical Lichnerowicz theorem [42] for the first eigenvalue of the Laplacian gives

$$\lambda_1 \geq nK. \tag{2.3}$$

We sketch the proof here. Take a nontrivial solution u to (1.1). By the Schwarz inequality, we have

$$|\nabla^2 u|^2 \geq \frac{1}{n}(\Delta u)^2 = -\frac{\lambda}{n} u \Delta u. \tag{2.4}$$

The Bochner-Lichnerowicz formula gives

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \nabla u \nabla \Delta u + Ric(\nabla u, \nabla u).$$

Using the Ricci lower bound (2.2) and (2.4), we have

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq -\frac{\lambda}{n} u \Delta u - \lambda |\nabla u|^2 + (n-1)K|\nabla u|^2.$$

Integrating the above inequality over M and using the divergence theorem, we have

$$\begin{aligned} 0 &\geq \int_M \left[\frac{\lambda}{n} - \lambda + (n-1)K\right] |\nabla u|^2 \\ &= \frac{n-1}{n}(-\lambda + nK) \int_M |\nabla u|^2, \end{aligned}$$

which gives Inequality (2.3).

In 1962, M. Obata [58] proved that the equality in (2.3) holds if and only if the manifold M is isometric to the n-sphere of constant sectional curvature K. This can be proved easily as follows by S.Y. Cheng's generalized Toponogov theorem [25], which states that if an n-dimensional compact Riemannian manifold has positive Ricci lower bound $(n-1)K > 0$, and maximal diameter $d(M) = \pi/\sqrt{K}$, then it is isometric to the n-sphere. In fact, if $\lambda_1 = nK$, then using the Bochner-Lichnerowicz formula for one more time, we will get that $|\nabla u|^2 + Ku^2$ is a constant. Let $\max u^2 = 1$, then we must have

$$\frac{|\nabla u|}{\sqrt{1-u^2}} = \sqrt{K}.$$

Moreover, we must have $\max u = 1$ and $\min u = -1$. Let p, q be two points on M with $u(p) = 1$ and $u(q) = -1$. Integrating the above equality along the geodesic connecting p, q, we get

$$d(p,q) \geq \pi/\sqrt{K}.$$

Bonnet-Myers Therorem says that the opposite inequality holds. Thus $d(M) = \pi/\sqrt{K}$ and by the theorem of Cheng, we proved that M has to be an n-sphere.

The results of Lichnerowicz and Obata for closed manifold were generalized to compact manifold with boundary. To state those results, we first make the following definition:

Definition 2.1. *A manifold with boundary is called weakly convex if the mean curvature of the boundary is nonnegative with respect to the outward normal of the boundary, convex if the second fundamental form of the boundary is nonnegative with respect to the outward normal of the boundary, strongly convex if the second fundamental form of the boundary is positive with respect to the outward normal of the boundary.*

R. Reilly [64] proved that the same Lichnerowicz-type lower bound holds for the first Dirichlet eigenvalue of a manifold with weakly convex boundary and that the equality holds if and only if M is isometric to a closed hemisphere of the Euclidean sphere of $S^n(K)$ of radius $1/\sqrt{K}$. J.F. Escobar [30] proved the similar result for the first Neumann eigenvalue of a manifold with convex boundary.

For a closed Riemannian manifold with nonnegative Ricci curvature, Li-Yau [40] introduced the method of gradient estimate and derived a lower bound for the first eigenvalue of the Laplacian in terms of the diameter d of the manifold. They proved

$$\lambda_1 \geq \frac{\pi^2}{2d^2}.$$

Zhong and Yang [80] refined Li-Yau's gradient estimate and obtained the bound

$$\lambda_1 \geq \frac{\pi^2}{d^2}. \tag{2.5}$$

Recently, based on the strong maximum principle, Hang and Wang [32] proved that actually strict inequality holds for dimension ≥ 2. This is a very interesting result.

Peter Li conjectured (cf. [73]) that, when the Ricci curvature of the manifold is positive, then Zhong-Yang's estimate can be further sharpened. In fact, the first eigenvalue of the sphere of dimension n is $n\frac{\pi^2}{d^2}$, which is n-times the Zhong-Yang's estimate. In view of the case on spheres, one version of Li's conjecture is as follows:

Conjecture 2.2 (P. Li)**.** *For a compact manifold with $Ric \geq (n-1)K > 0$, the first eigenvalues λ_1, with respect to the closed, the Neumann, or the Dirichlet Laplacian satisfies*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + (n-1)K.$$

Note that by the theorem of Myers, we always have $\frac{\pi^2}{d^2} \geq K$. Thus the conjecture, if true, will give a common generalization of the result of Lichnerowicz's and the one obtained by gradient estimate.

In this direction, D.G. Yang [73] proved that the first Dirichlet eigenvalue of the Laplacian satisfies

$$\lambda_1 \geq \frac{\pi^2}{\tilde{d}^2} + \frac{1}{4}(n-1)K,$$

if the manifold has weakly convex boundary, where $\tilde{d}$ is the interior diameter of the manifold (the diameter of the largest inscribed ball in the manifold). He also proved that the first closed eigenvalue and the first Neumann eigenvalue of the Laplacian satisfy

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{4}(n-1)K,$$

if the manifold has convex boundary.

By constructing new testing functions, the first author was able to improve the above results. In [45], it was proved that the first Dirichlet eigenvalue satisfies

$$\lambda_1 \geq \frac{\pi^2}{\tilde{d}^2} + \frac{1}{2}(n-1)K,$$

and in [47], it was proved that the first closed and Neumann eigenvalues satisfy

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31}{100}(n-1)K.$$

Moreover, if the first eigenfunction has opposite minimum to its maximum, then the first closed and Neumann eigenvalues satisfy [47]

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K.$$

When K is not necessarily positive, the first author [46, 48, 49] gave the following lower bound for the closed, the Dirichlet and the Neumann eigenvalues

$$\lambda_1 \geq \frac{\pi^2}{\tilde{d}^2} + \frac{1}{2}(n-1)K.$$

In §4 of this paper we will present further improvements of the above results. Our proof is a refinement of Li-Yau's gradient estimate and Zhong-Yang's approach.

2.2 Bounds in terms of Cheeger's isopermetric constants

Using *Cheeger's isopermetric constants*, we can get a lower bound of the first eigenvalue.

Definition 2.3. *Let M be a compact manifold. Define Cheeger's isopermetric constants as follows.*

If $\partial M \neq \emptyset$, define

$$h_D(M) = \inf \left\{ \frac{\mathrm{vol}(\partial\Omega)}{\mathrm{vol}(\Omega)} \Big| \Omega \subset\subset M \right\}.$$

If $\partial M = \emptyset$, define

$$h_C(M) = \inf \left\{ \frac{\mathrm{vol}(H)}{\min\{\mathrm{vol}(M_1), \mathrm{vol}(M_2)\}} \Bigg| \begin{array}{l} H \text{ is a hypersurface that} \\ \text{divides } M \text{ into } M_1, M_2 \\ \text{with } \partial M_1 = \partial M_2 = H \end{array} \right\}.$$

J. Cheeger [22] proved that the first eigenvalue of the Laplacian of a closed manifold satisfies

$$\lambda_1 \geq \frac{1}{4} h_C(M)^2,$$

and the first Dirichlet eigenvalue of the Laplacian satisfies

$$\lambda_1 \geq \frac{1}{4} h_D(M)^2.$$

It is well known that Sobolev constant and the isopermetric constant are mutually bounded. Similarly, we have the Sobolev-type inequality corresponding to Cheerget's constants.

Let's take the h_D as an example. For $u \in C^\infty(M)$, $u|_{\partial M} = 0$, by the Co-Area formula and the definition of h_D, we have

$$\begin{aligned} \int_M |\nabla u| &= \int_{-\infty}^{\infty} (\int_{\{u=t\}} 1) dt \\ &\geq \inf_t \left\{ \frac{\mathrm{area}(\{u=t\})}{\mathrm{vol}(\{u \geq t\})} \right\} \int_{-\infty}^{\infty} \mathrm{vol}(\{u \geq t\}) dt \geq h_D(M) \int_M |u|. \end{aligned}$$

Thus we have proved that following Sobolev type inequality:

$$\int_M |\nabla u| \geq h_D(M) \int_M |u|$$

for any $u \in H^1(M)$.

The above inequality implies the eigenvalue estimate. Take the first Dirichelt eigenfunction f and let $u = f^2$. We have

$$\begin{aligned} h_D(M) \int_M |f|^2 &\leq 2 \int_M |f||\nabla f| \\ &\leq 2 \left(\int_M |f|^2 \right)^{1/2} \left(\int_M |\nabla f|^2 \right)^{1/2} = 2\sqrt{\lambda_1} \int_M |f|^2. \end{aligned}$$

Thus we have

$$\lambda_1 \geq \frac{1}{4} h_D(M)^2.$$

Using the similar method, we can prove the inequality with respect to h_C.

Yau [75] pointed out that the second Cheeger's inequality above implies the following McKean's inequality [56] for the first Dirichlet egenvalue of the Laplacian,

$$\lambda_1(\Omega) \geq -(n-1)^2 K/4 > 0, \tag{2.6}$$

where Ω is a normal domain in a complete and simply connected Riemannian manifold whose sectional curvature $\leq K < 0$. Note that a *normal domain* is a connected domain with compact closure and nonempty piecewise C^∞ boundary. The proof is based on the Bishop's volume comparison theorem. Moreover, using Cheeger's isoperimetric inequality, R. Brooks and P. Waksman [16] gave a lower bound for the first Dirichlet eigenvalue of the Laplacian for a convex polygon in $\mathbb{R}^2$ in terms of the area and the interior angles. The second author [54] obtained similar result on a piece-wise smooth convex domain in S^2.

2.3 Some comparison results

Comparing quantities in a general manifold with the corresponding ones in the space form is an important method in geometric analysis. There are some interesting comparisons theorems for eigenvalues.

The comparison theorem for the first eigenvalue was given by Cheng [25].

Theorem 2.4. *Let M be an n-dimensional complete Riemannian manifold; $B(p, \delta)$ be a ball in M of radius δ with center p; $V_n(K; \delta)$ be a ball of radius δ in the n-dimensional simply connected space form $\mathbb{M}^n_K$ with constant section curvature K; and $\lambda_1(D)$ be the first Dirichlet eigenvalue of the Laplacian on D. Then*

1. *If $Ric \geq (n-1)K$, then*

$$\lambda_1\big(B(p,\delta)\big) \leq \lambda_1\big(V_n(K;\delta)\big),$$

with equality holding if and only if $B(p, \delta)$ is isometric to the ball $V_n(K; \delta)$ in $\mathbb{M}^n_K$.

2. *If the sectional curvature of the manifold is $\leq K$, $B(p, \delta)$ is within the cut-locus of p, $V_n(K; \delta)$ is within the cut-locus, then*

$$\lambda_1\big(B(p,\delta)\big) \geq \lambda_1\big(V_n(K;\delta)\big), \tag{2.7}$$

and equality holds if and only if the ball is isometric to the corresponding one in the space form.

As mentioned in § 2.1, a corollary of Cheng's theorem is the generalization of Toponogiv's theorem: If a the sectional curvature of a manifold has a lower bound $K > 0$ and if the diameter $diam(M) = \pi/\sqrt{K}$, the it must be the n sphere. Using Cheng's result, we are able to replace the sectional curvature bound by the Ricci curvature bound.

The first part of Cheng's theorem is more interesting: the comparison is valid regardless of the cut-locus. Because of this, sometimes we refer to the above Cheng's theorem as an upper bound estimate of the first eigenvalue. By refining the argument, Cheng was also able to get the upper bound for the k-th eigenvalue.

By the spectrum expansion of the heat kernel, Cheng's theorem follows from Cheeger-Yau's heat kernel comparison theorem [23].

2.4 Faber-Krahn type bounds

We now discuss the bounds of Faber-Krahn type (cf. [2, 9, 21, 65]). Faber [31] and Krahn [37] proved that for any bounded open set $\Omega \subset \mathbb{R}^n$, its first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplacian, and any ball $D \subset \mathbb{R}^n$,

$$\mathrm{vol}(\Omega) = \mathrm{vol}(D) \qquad \Longrightarrow \qquad \lambda_1(\Omega) \geq \lambda_1(D).$$

The proof is outlined as follows. Let f be the positive first eigenfunction. By the Schwarz spherical rearrangement technique, we can construct a positive radial function u on D that centered at the origin such that $\mathrm{vol}\big(\{f \geq t\}\big) = \mathrm{vol}\big(\{u \geq t\}\big)$, $\forall t \geq 0$. Then

$$\int_\Omega f^2 = \int_0^\infty \mathrm{vol}\big(\{f^2 \geq t\}\big)dt = \int_0^\infty \mathrm{vol}\big(\{u^2 \geq t\}\big)dt = \int_D u^2.$$

By Schwarz inequality,

$$\int_{\{f=t\}} |\nabla f| \int_{\{f=t\}} \frac{1}{|\nabla f|} \geq (\int_{\{f=t\}} 1)^2 = \Big[\mathrm{area}\big(\{f=t\}\big)\Big]^2.$$

Since u is radial,

$$\Big[\mathrm{area}\big(\{u=t\}\big)\Big]^2 = \int_{\{u=t\}} |\nabla u| \int_{\{u=t\}} \frac{1}{|\nabla u|},$$

and $\{u=t\}$ is a sphere, By the isoperimetric inequality in $\mathbb{R}^n$, $\mathrm{vol}\big(\{f \geq t\}\big) = \mathrm{vol}\big(\{u \geq t\}\big)$ implies

$$\mathrm{area}\big(\{f=t\}\big) \geq \mathrm{area}\big(\{u=t\}\big).$$

Therefore, we have

$$\int_{\{f=t\}} |\nabla f| \int_{\{f=t\}} \frac{1}{|\nabla f|} \geq \int_{\{u=t\}} |\nabla u| \int_{\{u=t\}} \frac{1}{|\nabla u|}.$$

By the fact $\mathrm{vol}\big(\{f \geq t\}\big) = \mathrm{vol}\big(\{u \geq t\}\big)$ and the Co-Area formula, we get

$$\int_{\{f=t\}} \frac{1}{|\nabla f|} = -\frac{d\,\mathrm{vol}\big(\{f \geq t\}\big)}{dt} = -\frac{d\,\mathrm{vol}\big(\{u \geq t\}\big)}{dt} = \int_{\{u=t\}} \frac{1}{|\nabla u|}.$$

Thus

$$\int_{\{f=t\}} |\nabla f| \geq \int_{\{u=t\}} |\nabla u|.$$

Consequently, we have

$$\int_\Omega |\nabla f|^2 = \int_0^\infty \left(\int_{\{f=t\}} |\nabla f| \right) dt \geq \left(\int_{\{u=t\}} |\nabla u| \right) = \int_D |\nabla u|^2,$$

which implies the result.

It is worth to mention that using the Steiner symmetrization, Pólya and Szegö [62] proved the following result:

$$\text{area}(\Omega) = \text{area}(D) \qquad \Longrightarrow \qquad \lambda_1(\Omega) \geq \lambda_1(D),$$

where Ω is any triangle (quadrilateral, resp.) in $\mathbb{R}^2$; D the equilateral triangle (square,resp) in $\mathbb{R}^2$; and $\lambda_1(\Omega)$ its first Dirichlet eigenvalue of the Laplacian.

By the theorem of Faber-Krahn and Pólya-Szegö, it is reasonable to believe that the more symmetric the domain, the smaller the first eigenvalue. This observation may be very useful in estimating the first eigenvalue and the eigenvalue gaps.

Let g be any metric on 2-sphere S^2, λ_g its first eigenvalue, and g_0 the standard metric on S^2. J. Hersch [35] showed that

$$\text{area}_g(S^2) = \text{area}_{g_0}(S^2) \qquad \Longrightarrow \qquad \lambda_g \leq \lambda_{g_0},$$

$$\lambda_g \leq \frac{8\pi}{\text{area}_g(S^2)}.$$

The ingredients in the proof of the above Hersch Theorem are as follows. (i) S^2 has only one conformal structure. (ii) The Dirichlet integral is conformal invariant in dimension two. (iii) For a conformal map $\psi : S^2 \to S^2$, one defines a map F open unit ball B^3 in $\mathbb{R}^3$ to itself by

$$F(a) = -\frac{1}{\text{area}_g(S^2)} \int_{S^2} x^i \circ f_a \circ \psi \, dv_g \qquad i = 1, 2, 3,$$

where $\{x^i\}_{i=1}^3$ are the coordinate functions, and where $f_a : B^3 \to B^3$, $f_a : S^2 \to S^2$ is the map

$$f_a(x) = \frac{(1-|a|^2)x - (1 - 2a\cdot x + |x|^2)a}{1 - 2a \cdot x + |a|^2|x|^2}, \quad |a| < 1.$$

It is easy to see that $f_a(x) \to -x_0 \quad \forall x \in S^2 \backslash \{x_0\}$ as $a \to x_0 \in S^2$. So the map F can be extended to $\overline{F} : \overline{B} \to \overline{B}$, the restriction of $\overline{F}$ on S^2 is the identity. Therefore $\overline{F}$ is surjective. (iv) Applying the min-max principle.

P. Li and S.T. Yau [41] derived a non-orientable analogue of the above Hersch's result on the projective plane, which states that for any metric g and canonical metric g_0 on the projective plane $\mathbb{R}\mathrm{P}^2$, the first eigenvalue of the Laplacian λ_g satisfies

$$\text{area}_g(\mathbb{R}\mathrm{P}^2) = \text{area}_{g_0}(\mathbb{R}\mathrm{P}^2) \qquad \Longrightarrow \qquad \lambda_g \leq \lambda_{g_0},$$

$$\lambda_g \leq \frac{12\pi}{\text{area}_g(\mathbb{R}\mathrm{P}^2)}.$$

with equality holding if and only if the metric is the standard one on $\mathbb{R}P^2$. There is an analogue for flat tori. M. Berger [13] showed that in the class of flat tori of the fixed area the eigenvalue λ_1 attains its supremum on the equilateral torus. N. Nadirashvili [57] showed that the product of the first eigenvalue and the area has the maximum $8\pi^2/\sqrt{3}$.

There is a Faber-Frahn type bound by evolution of eigenvalues in § 2.5.

2.5 Evolution of eigenvalues along the Ricci flow

The evolution of eigenvalues along the Ricci flow is important for studying geometry and topology of manifolds. In the breakthrough paper, G. Perelman [61] proved that on a Riemannian manifold, the first eigenvalue of the operator $-\Delta + R/4$ is nondecreasing under the Ricci flow [33], where Δ is the Laplacian and R is the scalar curvature. By using this estimate, he was able to rule out nontrivial steady breathers on compact manifolds. Since Perelman's, there has been a lot of work on the monotonicity of quantities related eigenvalues under the Ricci flow. To list a few, L. Ma [55] studied monotonicity of the first eigenvalue on a domain D in the manifold M with Dirichlet boundary condition, along the unnormalized Ricci flow. X. Cao [18] proved that on a Riemannian manifold with nonnegative curvature operator, the eigenvalues of the operator $-\Delta + R/2$ are nondecreasing under the unnormalized Ricci flow, where Δ is the Laplacian and R is the scalar curvature. J.F. Li [39] showed that the same result holds for the first eigenvalue without assuming the nonnegativity of the curvature operator. The first author [52] constructed a class of monotonic quantities from the first eigenvalue of the Laplacian. X. Cao [19] studied monotonicity of the eigenvalues of operator $-\Delta + cR (c \geq 1/4)$ for unnormalized Ricci flow and for normalized Ricci flow for $c = 1/4$. X. Cao, S. Hou and the first author [20] constructed a class of monotonic quantities from the first eigenvalue of the geometric operator, both along the normalized Ricci flow, and etc.

Through the evolution of the Ricci flow, the first author obtained a comparison result of eigenvalues of Faber-Krahn type in a recent preprint [51]. The result is stated as follows

$$\mathrm{vol}_g(M) = \mathrm{vol}_{\tilde{g}}(M) \qquad \Longrightarrow \qquad \frac{\lambda_g}{\kappa_g} \geq \frac{\lambda_{\tilde{g}}}{\kappa_{\tilde{g}}},$$

where M is a compact surface with Euler Characteristic $\chi < 0$; g is any Riemannian metric on M; λ_g is its first eigenvalue of the Laplacian; κ_g is the minimum of its Gauss curvature; and $\tilde{g}$ is a Riemannian metric on M that has constant Gauss curvature which is in the same conformal class of g; $\lambda_{\tilde{g}}$ is its first eigenvalue of the Laplacian; $\kappa_{\tilde{g}}$ is the (minimum of) constant Gauss curvature.

The result was proved by evoluting the metric g by the normalized Ricci flow

$$\frac{\partial}{\partial t} g(t) = (r - R) g(t), \qquad g(0) = g,$$

where R is the the scalar curvature of the metric $g(t)$, and r is the average of the scalar curvature $r = \int_M R d\mu / \int_M d\mu$. Along the flow the volume is preserved. The first eigenvalue is evolved by

$$\lambda(t) = \lambda(0) \exp\Big\{ \int_0^t \int_M (R - r) u^2 d\mu d\tau \Big\}.$$

Use the maximum principle, we get

$$\lambda(t) \geq \lambda(0) \frac{\frac{r}{2\kappa_g}}{1 - \left(1 - \frac{r}{2\kappa_g}\right) e^{rt}}.$$

Letting $t \to \infty$, we get the result.

3 Various eigenvalue bounds

3.1 Bounds of all eigenvalues

For the Dirichlet eigenvalues of the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$, H. Weyl proved the asymptotic formula in 1912:

$$\lambda_k \sim c_n \left(\frac{k}{V}\right)^{2/n}, \qquad \text{as} \quad k \to \infty,$$

where

$$c_n = (2\pi)^2 \Big/ \Big(\frac{\omega_{n-1}}{n}\Big)^{2/n}, \quad \omega_{n-1} = \text{area}(S^{n-1}), \quad V = \text{vol}(\Omega).$$

Based on this, G. Pólya conjecture the following.

Conjecture 3.1 (Pólya)**.**

$$\lambda_k \geq c_n \left(\frac{k}{V}\right)^{2/n}, \qquad \forall\ k.$$

E. Lieb [43] proved that the inequality holds for a smaller number c_n' than c_n. Li-Yau [40] showed that

$$\lambda_k \geq \frac{n}{n+2} c_n \left(\frac{k}{V}\right)^{2/n}, \qquad \forall\ k.$$

The proof involves the Fourier transformation and the Planchel formula.

3.2 Bounds of eigenvalues on surfaces

There are many classical results on eigenvalues on surfaces. These results provided solid examples for further study in high dimensions. G. Szegö [68] gave an upper bound for the first Neumann eigenvalue λ_1 of the Laplacian on a simply connected bounded domain $\Omega \subset \mathbb{R}^2$

$$\lambda_1 \leq \frac{\xi^2 \pi}{\text{area}(\Omega)},$$

where $\xi \approx 1.8412$ is a constant related to the first zero of some Bessel function. The equality holds if and only if Ω is a disk. This inequality naturally caused the following conjecture.

Conjecture 3.2 (G. Pólya)**.**
For Dirichlet condition,

$$\lambda_i \geq \frac{4\pi i}{\mathrm{area}(\Omega)}.$$

For Neumann condition,

$$\lambda_i \leq \frac{4\pi i}{\mathrm{area}(\Omega)}.$$

There are two kinds of generations of the Szegö inequality. H. Weinberger [71] generalized Szegö' result to higher dimensions, that is, for a simply connected bounded domain $\Omega \subset \mathbb{R}^n$, the first Neumann eigenvalue λ_1 satisfies

$$\lambda_1 \leq \frac{c}{\mathrm{vol}(\Omega)^{2/n}},$$

where c is a constant related to the volume of unit ball, and the equality holds if and only if Ω is a ball. The proof of the this Szegö-Weinberger inequality is based on the min-max principle and classical analysis. There are more discussions on the Szegö-Weinberger inequality in [2] by M. Ashbaugh and [9] by M. Ashbaugh and R. Benguria. J. Hersch generalized Szegö's result to the compact surface S^2 (cf. § 2.4). For a surface with genus $\boldsymbol{g} \neq 0$, the result corresponding to Hersch's is the P. Yang and S.T. Yau's theorem [74] that states that for a compact Riemannian surface $\Lambda_{\boldsymbol{g}}$ of genus $\boldsymbol{g}$ and any metric on $\Lambda_{\boldsymbol{g}}$, the first eigenvalue of the Laplacian satisfies

$$\lambda_1 \leq \frac{8\pi(1+\boldsymbol{g})}{\mathrm{area}(\Lambda_{\boldsymbol{g}})}.$$

The proof of the above inequality involves branched conformal maps and is similar to that of Hersch's. One can also prove it as follows by the notion of conformal volumes introduced by Li-Yau [41].

Let M be an m-dimensional compact Riemannian manifold that admits a branched conformal immersion $\psi : M \to S^n$, ds_0^2 the standard metric on S^n, g in the group G of conformal diffeomorphism of S^n, dv_g the volume form of $(g\circ\psi)^* ds_0^2$ on M. The conformal n-volume of ψ is defined to be

$$V_c(n,\psi) = \sup_{g\in G} \int_M dv_g.$$

The conformal n-volume of M is defined to be

$$V_c(n,M) = \inf_{\psi} V_c(n,\psi).$$

Li-Yau [41] proved that for a compact surface M,

$$\lambda_1(M)\mathrm{area}(M) \leq 2V_c(n,M)$$

for all n such that $V_c(n, M)$ is defined. Moreover, equality holds if and only if M is a minimal surface in S^n, and the isometric immersion $M \to S^n$ is induced by the first eigenfunction of S^n. Note that $V_c(2, S^2) = 4\pi$.

Now one derives Yang-Yau theorem from the above Li-Yau's. By Riemann-Roch theorem, one can take a branched conformal covering $\psi : M \to S^2$, $\deg \psi \leq 1 + g$. Thus

$$\lambda_1 \text{area}(M) \leq 2V_c(2, M) \leq 2V_c(2, S^2)(1 + g) = 8\pi(1 + g).$$

Let us mention that in general the volume alone is not enough to bound λ_1. H. Urakawa [69] gave a family of metrics with unit volume on S^3 whose first eigenvalues go to infinity. The following are some conjectures on the bounds of eigenvalues of Riemannian surfaces (cf. [65, 76]).

Conjecture 3.3 (Yau).
I. *For a Riemannian Surface Λ_g of genus g, there is an absolute constant c, such that for any metric on Λ_g,*

$$\frac{\lambda_k}{k} \leq \frac{c(1 + g)}{\text{area}(\Lambda_g)}.$$

II. *For an embedded compact minimal surface in S^3, $\lambda_1 = 2$.*

Yang-Yau's theorem shows that in Conjecture I is true for $k = 1$. H.I. Choi and A.N. Wang [27] has a partial result for II, which says that $\lambda_1 \geq n/2$ for any embedded compact minimal surface in S^{n+1} and hence $\lambda_1 \geq 1$ for $n = 2$.

We now state some results Riemannian surfaces with *constant negatives curvature*. Let us take the constant negative Gauss curvature $\equiv -1$. McKean inequality says for a normal domain Ω in such surface that is simply connected, the first Dirichlet eigenvalue $\lambda_1(\Omega) \geq 1/4$. On the other hand, B. Randol [63] proved that given any such compact surface, it has a compact covering for which there are eigenvalues arbitrarily close to 0. P. Buser [17] gave the following result. Given $\varepsilon > 0$ and integer $g \geq 2$, there is a compact Riemannian surface with constant negative curvature -1 and genus g satisfying

$$\lambda_{2g-3} < \varepsilon.$$

The proof consists of constructing a collection of Löbell surfaces, first constructed by Löbell [53].

There are some other results related to eigenvalues on surfaces stated in § 2.4 and § 2.5.

3.3 On the ratios of eigenvalues for the Laplacian

Let us consider the Dirichlet eigenvalues of the Laplacian on a bounded domain in Euclidean space $\mathbb{R}^n$. The extended PPW conjecture states the following.

Conjecture 3.4. *For a bounded domain $\Omega \subset \mathbb{R}^n$, we have*

$$\left.\frac{\lambda_{m+1}}{\lambda_m}\right|_{\Omega} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{n\text{-ball}}.$$

The conjecture for the case $m = 1$ was referred to as the Payne-Pólya-Weinberger (PPW) conjecture [1, 2, 76, 65]). It was proved by M. Ashbaugh and R. Benguria [4, 6] 35 years later. Moreover, the equality holds if and only if Ω is a ball. Using the symmetric rearrangement technique, Ashbaugh and Benguria [8] later generalized the above bound to the one for domains in a hemisphere in S^n.

$$\mathrm{vol}(\Omega) = \mathrm{vol}(D) \qquad \Longrightarrow \qquad \frac{\lambda_2}{\lambda_1}(\Omega) \leq \frac{\lambda_2}{\lambda_1}(D),$$

where Ω is any domain and D any ball in the hemisphere of S^n, with the quality if and only if is a geodesic ball in S^n. Partial results were obtained by M. Ashbaugh and R. Benguria [5, 7] for the case $m = 2$ and $m = 3$ of the extended PPW.

Related to the original conjecture of Payne-Pólya-Weinberger is the following

Conjecture 3.5. *For bounded domain $\Omega \subset \mathbb{R}^n$,*

$$\frac{\lambda_2 + \cdots + \lambda_{n+1}}{\lambda_1}\Big|_{\Omega} \leq \frac{\lambda_2 + \cdots + \lambda_{n+1}}{\lambda_1}\Big|_{n\text{-ball}}.$$

Similar open problems for other ratios can be proposed:

Open Problems 3.1. *Find a domain that maximize,*

$$\frac{\lambda_3}{\lambda_1}, \ \frac{\lambda_{m+1}}{\lambda_m}, \ldots, \text{ and etc.}$$

We refer Ashbaugh [2] and Ashbaugh-Benguria [9] for more details in this direction.

3.4 On the gap for Schrödinger operators

Let us consider the gap between the first two Dirichlet eigenvalues λ_1 and λ_2 for the Schrödinger operator $-\Delta + V$ on piecewise smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, where V is a nonnegative function defined on $\overline{\Omega}$. The gap is called the fundamental gap of the Schrödinger operator.

Singer, Wong, Yau and Yau [67] established the first general lower bound estimate: if the potential function V is convex, then $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4d^2}$, where d is the diameter of the domain. Yu and Zhong [79] later refined the argument in [67] and obtained the bound $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2}$. The method of proofs is essentially a version of gradient estimate of Li-Yau type. One of the important result used in the proof is the log concavity of the first eigenfunction, which is a theorem of Brascamp and Lieb [15]. In [67], there is a simple proof of the log concavity.

We make the following definition:

Definition 3.6. *Let*

$$\alpha(x) = \inf_{\tau \in T_x\Omega, |\tau|=1} \left[\nabla^2 \left(-\ln f\right)\right](\tau, \tau)(x),$$

$$\alpha = \inf_{x \in \Omega} \alpha(x), \tag{3.8}$$

where f is a positive first eigenfunction.

By the result of Brascamp and Lieb [15], $\alpha \geq 0$. Moreover, if the domain is strictly convex, then in fact $\alpha > 0$ (cf. [44]). Using this, we get

$$\lambda_2 - \lambda_1 \geq \frac{4}{d^2}\Big(\int_0^{\pi/2} \frac{dt}{\sqrt{1-\sigma \sin^2 t}}\Big)^2 > \frac{\pi^2}{d^2},$$

where $\sigma > 0$ can be estimated from below by the global log-convexity α. This actually shows that strict inequality in Yu-Zhong's result holds.

R. Smits [66] gave an alternative derivation of the above result by the methods in [36] and [60]. There are many other estimates, e.g., F.Y. Wang [70].

Note that the ratio u of a second eigenfunction and a first eigenfunction satisfies the equation

$$\Delta u = -\lambda u - 2\nabla \ln f \, \nabla u,$$

where f is a positive first eigenfunction and $\lambda = \lambda_2 - \lambda_1$.

S.T. Yau [65, 76] conjectured the following sharpened version of gap estimate.

Conjecture 3.7. *Assume that the potential function V is convex. Then we have*

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2},$$

where d is the diameter of the domain.

The equality of the conjecture can asymptotically be reached by a thin rectangular. The case when $n = 1$, and the potential $V = 0$ is, of course, trivial. For dimension $n = 1$ the conjecture has been proved by R. Lavine [38]. For $n > 1$, the conjecture was proved under some kind of symmetry assumptions. B. Davis [29] proved the conjecture for the Laplacian on a bounded domain $\Omega \in \mathbb{R}^2$ that is symmetric and convex with respect to both x and y axes. Bañuelos and Méndez-Hernández [11] proved the conjecture for the domain $\Omega \in \mathbb{R}^2$ that is symmetric with respect to one axis x and convex with respect to the other axis and for the potential being symmetric and single-well in one variable. Bañuelos and Kröger [10] showed that if $\Omega \in \mathbb{R}^2$ is a smooth convex domain included in an infinite slab $\{(x, y)||x| < b\}$ and Ω is symmetric with respect to both coordinate axes, then the conjecture holds. See also [1, 3, 9] for some other results on the conjecture.

On the other hand, motivated by geometry, S.T. Yau [77] gave another kind of low bound for the gap between the first two Dirichlet eigenvalues of Schrödinger operators $-\Delta + V$ with convex potential V,

$$\lambda_2 - \lambda_1 \geq \theta \frac{\pi^2}{d^2} + 2\big(\cos \pi\sqrt{\theta}\big)^2 \alpha,$$

where θ is any constant with $0 \leq \theta \leq 1/4$, d is the diameter, and $\alpha > 0$ is the global log-convexity in (3.8). The first author [50] improved the above bound by constructing new test functions,

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + 0.62\alpha,$$

and

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + \alpha.$$

if the ratio of first two eigenfunctions has opposite maximum and minimum.

The case when the potential function is not convex is very interesting and has physics applications (for example, the double well potential). S.T. Yau [77, Theorem 3.2] proved the following gap theorem for a bounded convex domain Ω and a non-convex potential V.

Theorem 3.8. *The gap of the first two Dirichlet eigenvalues of the Schrödinger operator* $-\Delta + V$ *satisfies*

$$\lambda_2 - \lambda_1 \geq \frac{2}{d^2} \exp(-ad^2),$$

where d *is the diameter of the domain* Ω *and* $-a < 0$ *is a lower bound of the Hessian of* $-\log f$ *for the first eigenfunction* f *of the operator* $-\Delta + V$.

After the Hessian of the log of the first eigenfunction was estimated, the result is the counterpart of the Li-Yau estimate [40] on the first eigenvalue with Ricci curvature being bounded below. Note that studying the Schrödinger operators rather than the Laplacian is not only a generalization but is also necessary. For example, a simple proof of the log concavity of the first eigenfunction was obtained in [67] by viewing the Laplacian as the limiting operator of a series of Schrödinger operators. With more careful analysis, the above result can be refined to include the case when the potential function is convex [78].

Back to the case of strictly convex potential, a new bound (Theorem 4.4) on the fundamental gap is given in § 4.

4 New bounds

In this section, we sharpened the method of gradient estimates and improved the eigenvalue estimates.

Theorem 4.1. *Let* M *be a compact Riemannian manifold with convex boundary. Suppose that* $Ric \geq (n-1)Kg$ *with constant* $K > 0$. *Then the first closed eigenvalue and first Neumann eigenvalue of the Laplacian satisfy*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{34}{100}(n-1)K,$$

where d *is the diameter of the manifold.*

Our method is based on Lemma 4.2, which sharpened the estimates on test functions. The lemma itself is interesting as a stand-alone result in analysis and may have other applications.

Sketch of the proof. We follow the lines in the proof of Theorem 1 in [47] and use the same notations there. Note that in this case $\alpha = (n-1)K/2$.

Arguments before (II-a) are the same as in the proof of Theorem 1 in [47]. Now in that proof replace 0.765 by $a_0 = 0.843$ and 1.53 by 1.686.

For Case (II-a): $a_0 \leq a < \frac{\pi^2}{4}\delta$, we can apply Theorem 8 in [47] and we get

$$\lambda \geq \frac{1}{1 - \frac{4a_0}{\pi^2}} \frac{\pi^2}{d^2}.$$

For Case (II-b-1): $0 < a < a_0$, $a < \frac{\pi^2}{4}\delta$ and $a \geq 1.686\delta$, we apply Theorem 8 in [47] and we get

$$\lambda \geq \frac{\pi^2}{d^2} + \frac{4}{\pi^2} 1.686\alpha > \frac{\pi^2}{d^2} + \frac{34}{50}\alpha.$$

For the remaining Case (II-b-2): $0 < a < a_0$, $a < \frac{\pi^2}{4}\delta$ and $a < 1.686\delta$, we give new values for σ and τ.

$$\sigma = \{\tau\} \Bigg/ \left\{ \left(\frac{\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{\frac{12-\pi^2}{1.686} - 1} \right) c \right\} \tag{4.9}$$

and

$$\tau = \frac{2}{3\pi^2} \left(\frac{12 - \pi^2}{2(4-\pi)} + \frac{2(4-\pi)}{12-\pi^2} - 2 \right) \approx 0.003158975652. \tag{4.10}$$

By Lemma 13 and Lemma 14, both in [47], and Lemma 4.2 in this paper, we have

$$\frac{\pi}{2} = \frac{\xi'(t)}{t\eta'(t)}\Big|_{\frac{\pi}{2}} \leq \frac{\xi'(t)}{t\eta'(t)} \leq \lim_{t\to 0} \frac{\xi'(t)}{t\eta'(t)} = \frac{\xi''(0)}{\eta(0)} = \frac{\pi(12-\pi^2)}{4(4-\pi)}$$
$$\approx 1.94920.$$

So

$$1 \leq \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \leq \frac{12-\pi^2}{2(4-\pi)} \approx 1.240900,$$

$$\left| -\left(1 + \frac{m\xi'(t_0)}{c\eta'(t_0)}\right)\left(1 + \frac{\pi m}{2c} t_0\right) \right| \leq \frac{1}{4}\left(\frac{12-\pi^2}{2(4-\pi)} + \frac{2(4-\pi)}{12-\pi^2} - 2 \right)$$
$$\approx 0.01169169.$$

Using $|\eta'| \leq 8/(3\pi)$ and the above estimate, we have

$$\sigma + P \leq \frac{\tau}{w(t_0)}. \tag{4.11}$$

$$\tau = \frac{2}{3\pi^2} \left(\frac{12 - \pi^2}{2(4-\pi)} + \frac{2(4-\pi)}{12-\pi^2} - 2 \right) \approx 0.003158975652.$$

On the other hand, by Lemma 4.3,

$$z(t_0) \geq \left(\frac{\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{\frac{12-\pi^2}{1.686} - 1} \right) c = \frac{\tau}{\sigma} > 0. \tag{4.12}$$

Since $-P\xi(t_0) \geq 0$, we have $w(t_0) \geq z(t_0)$. This fact, (4.11) and (4.12) imply that for $P > 0$

$$\sigma + P < \sigma,$$

which is impossible.

We proceed further as in the proof of Theorem 8 in [47]. We get the following

$$\lambda d^2 \geq \frac{\pi^3}{\pi[1-(\delta-\sigma c^2)]}.$$

Since $\delta - \sigma c^2 > 0.68\delta$ by Lemma 4.3, we have

$$\lambda \geq \frac{1}{[1-(\delta-\sigma c^2)]}\frac{\pi^2}{d^2} \geq \frac{1}{[1-0.68\delta]}\frac{\pi^2}{d^2}$$

and

$$\lambda \geq \frac{\pi^2}{d^2} + 0.68\alpha > \frac{\pi^2}{d^2} + \frac{34}{50}\alpha.$$

In [47], the first author studied the function

$$\xi(t) = \frac{\cos^2 t + 2t\sin t\cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \qquad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}], \tag{4.13}$$

and the Zhong-Yang's function

$$\eta(t) = \frac{\frac{4}{\pi}t + \frac{4}{\pi}\cos t\sin t - 2\sin t}{\cos^2 t} \qquad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{4.14}$$

Many properties of functions ξ and η are in Lemmas 13, 14 and 15 in [47]. We give a new result for ratio $r(t) = \frac{\xi'(t)}{\eta'(t)}$, which is needed in the proofs of Theorems 4.1 and 4.4.

Lemma 4.2. *Let ξ and η be two functions defined in* (4.13) *and in* (4.14) *and let $r(t) = \xi'(t)/\eta'(t)$. Then we have $r'' < 0$ in $(0, \pi/2)$.*

Proof. Let $A = \frac{1}{2}p\cos t$, $B = \frac{3}{2}p'\cos t - \frac{5}{2}p\sin t$, $C = p'\sin t + 2p\cos t - 3$. Thus

$$Ar''' + Br'' + Cr' = -4\cos t,$$

$$Ar^{(4)} + (A' + B)r''' + (B' + C)r'' = 4\sin t - C'r'.$$

Suppose that at $t_0 \in (0, \pi/2)$ r'' achieves it non positive maximum on $[0, \pi/2]$. Then at t_0 we have $r''' = 0$, and $r^(4) \leq 0$. Thus at t_0

$$(B' + C)r'' \geq 4\sin t - C'r' > -C'r'.$$

If $B' + C < 0$ and $C' < 0$, then by Lemma 15 in [47] there is a contradiction to the above inequality.

In $(-\frac{\pi}{2}, \frac{\pi}{2})$ $p = \eta'$ satisfies the equation

$$\frac{1}{2}p'' \cos t - 2p' \sin t - 2p \cos t = -1. \tag{4.15}$$

We have

$$\frac{1}{2}p'' \cos t = -1 + 2(p' \sin t + p \cos t) \leq -1 + 2\sqrt{p^2 + (p')^2}.$$

Since $|p| \leq 8/(3\pi)$ and $|p'| \leq 1/2$ we have $|\frac{1}{2}p'' \cos t| < 1$. Thus by Lemma 14 in [47]

$$B' + C = \frac{3}{2}p'' \cos t - 3p' \sin t - \frac{1}{2}p \cos t - 3 < 0.$$

Direct computations show that

$$C' = -\frac{4 \sin t}{\cos^5 t}\left(2\cos^3 t - 3\pi \cos^2 t - 24 \cos t - 6\frac{t}{\sin t} - 18t \sin t + 12\pi\right).$$

To prove $C' < 0$, we need only prove that

$$f(t) = 2\cos^3 t - 3\pi \cos^2 t - 24 \cos t - 6\frac{t}{\sin t} - 18t \sin t + 12\pi > 0$$

Since $f(\pi/2) = 0$ we need only show

$$f'(t) = 6 \cos t(-\cos t \sin t + \pi \sin t - \frac{\cos t}{\sin t} + \frac{t}{\sin^2 t} - 3t) = \frac{6 \cos t}{\sin^2 t} g(t) < 0,$$

or equivalently $g(t) < 0$ in $(0, \pi/2)$, where

$$g(t) = t - \cos t \sin t - \cos t \sin^3 t + (\pi \sin t - 3t) \sin^2 t.$$

Now $g'(t) = h(t) \cos t \sin t$, where $h(t) = -4 \cos t \sin t + 3\pi \sin t - 6t$. The function

$$h'(t) = -8 \cos^2 t + 3\pi \cos t - 2$$

has at most two zeros on $[0, \pi/2]$. Therefore the function h has at most three zeros on $[0, \pi/2]$. Since 0 and $\pi/2$ are two zeros of h, h has at most one zero in $(0, \pi/2)$. Since $h'(0) = 3\pi - 10 < 0$ and $h'(\pi/2) = -2 < 0$, h has the unique zero $t_1 \in (0, \pi/2)$. Therefore

$$g'(t) < 0 \text{ in } (0, t_1) \quad \text{and} \quad g'(t) > 0 \text{ in } (t_1, \pi/2).$$

Since $g(0) = 0 = g(\pi/2)$, we have $g(t) < 0$ in $(0, \pi/2)$.

Lemma 4.3. *If $a < 1.686\delta$ and $0 < a < 0.843$ then*

$$z(t) = 1 + c\eta(t) + \delta\xi(t) > 0$$

for $t \in [-\pi/2, \pi/2]$ and

$$\delta - \sigma c^2 \approx 0.68880\delta > 0.68\delta,$$

where $c = a/b$ and $b > 1$ is any constant and σ is the constant in (4.9).

Proof. Proceed the same way as in the proof of Lemma 10 in [47], except that we let $\nu = 1.686$ instead of 1.53, and $a_0 = 0.843$ instead of 0.765. We get

$$\begin{aligned} &1 + c\eta(t) + \delta\xi(t) \\ &\geq \left(\frac{\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{\frac{12-\pi^2}{1.686} - 1} \right) c > 0, \end{aligned}$$

Let τ be the constant in (4.10). Then

$$\sigma c^2 \leq \tau\nu\delta \Big/ \left(\frac{\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{\frac{12-\pi^2}{1.686} - 1} \right)$$

and

$$\delta - \sigma c^2 > 0.68\delta.$$

Similarly, we have the following new result on the fundamental gap.

Theorem 4.4. *If Ω is a smooth bounded strictly convex domain in $\mathbb{R}^n$, V is a smooth nonnegative convex function on Ω, then the difference of the first two Dirichlet eigenvalues of Schrödinger operator $-\Delta + V$ satisfies*

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + 0.68\alpha$$

where d is the diameter, and $\alpha > 0$ is the global log-convexity in (3.8).

Sketch of the proof. Similar to the proof as the above, except that this time we follow the lines in the proof of Theorem 1 in [50] instead.

References

[1] M. Ashbaugh, *Some eigenvalue comparison results for domains in S^n and for annular domains in $\mathbb{R}^{n+1}$*, `The Erwin Schödinger International Institute for Mathematical Physics` - Workshop on Geometrical Aspects of Spectral Theory, Editors Leonid Friedlander and Thomas Hoffmann-Ostenhof, ftp://ftp.esi.ac.at/pub/Preprints/esi768.pdf, East Tyola, 1999.

[2] M. Ashbaugh, *Isoperimetric and universal inequalities for eigenvalues*, Spectral theory and geometry (Edinburgh, 1998), 95–139, London Math. Soc. Lecture Note Ser., 273, Cambridge Univ. Press, Cambridge, 1999.

[3] M. Ashbaugh, *The fundamental gap*, Workshop on Low Eigenvalues of Laplace and Schrödinger Operators, American Institute of Mathmatics, Palo Alto, California, 2006.

[4] M. Ashbaugh and R. Benguria, *Proof of the Payne-Pólya-Weinberger conjecture*, Bull. Amer. Math. Soc., **25**(1991), no. 1, 19–29.

[5] M. Ashbaugh and R. Benguria, *Isoperimetric bound for λ_3/λ_2 for the membrane problem*, Duke Math. J., **63**(1991), 333–341.

[6] M. Ashbaugh and R. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. Math., **135**(1992), 601–628.

[7] M. Ashbaugh and R. Benguria, *Isoperimetric bounds for higher eigenvalue ratios for n-dimensional fixed membrane problem*, Proc. Roy. Soc. Edinburgh, **123A**(1993), 977–985.

[8] M. Ashbaugh and R. Benguria, *A sharp bound for the ratio of the first two Dirichlet eigenvalues of a doamin in a hemisphere of S^n*, Trans. Amer. Math. Soc., **353**(2001), No. 3, 1055–1087.

[9] M. Ashbaugh and R. Benguria, *Isoperimetric inequalities for eigenvalues of the Laplacian*, 105–139, Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, Proc. of Symp. in Pure Math. 76, Part 1, F. Gesztesy, P. Deift, C. Galvez, P. Perry, and W. Schlag, eds., Amer. Math. Soc., Providence, RI, 2007.

[10] R. Bañuelos and P. Kröger, *Gradient estimates for ground state Schrodinger operators and applications*, Comm. Math. Phy., **224**(2002), 545–550.

[11] R. Bañuelos and P.J. Méndez-Hernández, *Sharp inequalities for rations of heat kernels of Schrödinger operators and applications to spectral gaps*, J. Funct. Anal., **176**(2000), 368–399.

[12] M. van den Berg, *On condensation in the free-boson gas and the spectrum of Laplacian*, J. Statist. Phys., **31**(1983), 623–637.

[13] M. Berger, *Sur les premières valeurs propres des variétés Riemanniennes*, Composito. Math., **26**(1973), 129–149.

[14] R. Bishop and R. Crittenden, Geometry of Manifolds, Academic Press, New York, 1964.

[15] H. Brascamp and E. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Func. Anal., **22**(1976), 366–389.

[16] R. Brooks and P. Waksman, *The first eigenvalue of a scalene triangle*, Proc. Amer. Math. Soc., **100**(1987), no. 1, 175–182.

[17] P. Buser, *Riemannsche Flachen mit Eigenwerte in* $(0, \frac{1}{4})$, Comment. Math. Helv., **52**(1977), no. 2, 25–34.

[18] X. Cao, *Eigenvalues of* $(-\Delta+R/2)$ *on manifolds with nonnegative curvature operator*, Math. Ann., **337**(2007), no. 2, 435–441.

[19] X. Cao, *First eigenvalues of geometric operators under the Ricci flow*, Proc. Amer. Math. Soc., **136**(2008), 4075–4078.

[20] X. Cao, S. Hou and J. Ling, *Estimate and monotonicity of the first eigenvalue under Ricci Flow*, Preprint.

[21] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, Orlando, Fla, 1984.

[22] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis, Princeton Univ. Press, Princeton, NJ, 1970.

[23] J. Cheeger and S.-T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math., **34**(1981), 465–480.

[24] S.Y. Cheng, *Eigenfunctions and eigenvalues of the Laplacian*, Amer. Math.

Sco. Proc. Symp. Pure Math., **27**(1975), Part II, 185–193.

[25] S.Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z., **143**(1975), 289–297.

[26] S.Y. Cheng, *Eigenvalues and nodal sets*, Comment. Math. Helv., **51**(1976), 43–55.

[27] H.I. Choi and A. N. Wang, *A first eigenvalue estimate for minimal surfaces*, J. Diff. Geom., **18**(1983), 559–562.

[28] R. Courant and D. Hilbert, Method of Mathematical Physics, Vol. I., Wiley (interscience), New York, 1953; Vol. II. 1967.

[29] B. Davis, *On the spectral gap for fixed membranes*, Ark. Mat., **39**(2001), 65–74.

[30] J.F. Escobar, *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math., **43**(1990), no. 7, 857–883.

[31] G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitz. Ber. Bayer. Akad. Wiss., 169–172, 1923.

[32] F.B. Hang and X.D. Wang, *A remark on Zhong-Yang's eigenvalue estimate*, Int. Math. Res. Notices, **2007**(2007), article ID rnm064.

[33] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom., **17**(1982), no. 2, 255–306.

[34] A. Henot, *Minimization problems for eigenvalues of the Laplacian*, J. Evolution Equation, **3**(2003), 443–461.

[35] J. Hersch, *Sur la frequence fondementale d'une membrane vibrante: evaluations par defaut et principe de maximum*, Z. Angw. Math. Phys., **XI** Fasc. 5(1960), 387–413.

[36] W. Kirsch and B. Simon, *Comparison theorems for the gap of the schrödinger operators*, J. Func. Anal., **73**(1987), 396–410.

[37] E. Krahn, *Über Minimaleigenschaften der Kugel in drei un mehr Dimensionen*, Acta Comm. Univ. Dorpat. A9, 144, 1926.

[38] R. Lavine, *The eigenvalue gap for one-dimensional convex potentials*, Proc. Amer. Math. Soc., **73**(1987), 396–410.

[39] J.F. Li, *Eigenvalues and energy functionals with monotonicity formulae under Ricci flow*, Math. Ann., **338**(2007), no. 24, 927–946.

[40] P. Li and S.T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, AMS Proc. Symp. Pure Math., **36**(1980), 205–239.

[41] P. Li and S.T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math., **69**(1982), 269–291.

[42] A. Lichnerowicz, Geometrie des groupes de tansformations, Dunod, Paris, 1958.

[43] E.H. Lieb, *The number of bound states of one-body Schroedinger operators and the Weyl problem*, Ameri. Math. Soc. Proc. Sympos. Pure Math., **36** (1980), 241–252.

[44] J. Ling, *A lower bound for the gap between the first two eigenvalues of Schrödinger operators on the convex domains in S^n or $\mathbb{R}^n$*, Michigan Math-

ematical Journal, **40** (1993), no. 2 , 259–270.

[45] J. Ling, *A lower bound of the first Dirichlet eigenvalue of a compact manifold with positive Ricci curvature*, International J. Math. , **17**(2006), no. 5, 605–617.

[46] J. Ling, *The first eigenvalue of a closed manifold with positive Ricci curvature*, Proc. Amer. Math. Soc., **134**(2006), no. 10, 3071–3079.

[47] J. Ling, *Lower bounds of the eigenvalues of compact manifolds w ith positive Ricci curvature*, Ann. of Global Anal. and Geo., **31**(2007), no. 4, 385–408.

[48] J. Ling, *The first Dirichlet eigenvalue of a compact manifold and the Yang conjecture*, Math. Nach., **280**(2007), no. 12, 1354–1362.

[49] J. Ling, *An exact solution to an equation and the first eigenvalue of a compact manifold*, Illinois J. Math., **51**(2007), no. 3, 853–860.

[50] J. Ling, *Estimates on the lower bound of the first gap*, Comm. Anal. Geometry, **16**(2008), no. 3, 539–563.

[51] J. Ling, *A comparison theorem and a sharp bound via the Ricci flow*, Preprint, `arxiv:math.DG/07102574`.

[52] J. Ling, *A class of monotonic quantities along the Ricci flow*, Preprint, `arxiv:math.DG/07104291`.

[53] F. Löbell, Die überall regulären begrenzten Flächen fdester Krümmung, Dissertation, Tübingen Universität, 1927.

[54] Z. Lu, *On estimates on the first eigenvalue of the Laplacian of the convex domain on a 2-dimensional shere*, Chinese Science Bulletin, **31**(1991), No. 7, 529–533.

[55] L. Ma, *Eigenvalue monotonicity for the Ricci-Hamilton Flow*, Annals of Global Analysis and Geometry, **337**(2006), no. 2, 435–441.

[56] H.P. McKean, *An upper bound for the spectrum of* Δ *on a manifold of negative curvature*, J. Differ. Geom., **4**(1970), 359–366.

[57] N. Nadirashvili, *Berger's isoperimetric problem and minimal immersions of surfaces*, Geo. Func. Anal, **6**(1996), no. 5, 876–897.

[58] M. Obata, C*ertain conditionbs for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan, **14**(1962), 333–340.

[59] L.E. Payne, G. Pólya and H.F. Weinberger, *On the ratio of consective eigenvalues*, J. Math. Phys., **35**(1956), 289–298.

[60] L.E. Payne and H.F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Rat. Mech. Anal., **5**(1960), 286–292.

[61] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, arXiv:math.DG/0211159.

[62] G. Pólya and G. Szegö, Isoperimetric inequalities in mathematical physics, Ann. Math. Studies, **27**, Priceton Univ. Press, 1951.

[63] B. Randol, *Small eigenvalues of the Laplace operator on compact Riemann surfaces*, Bull. Amer. Math. Soc., **80**(1974), 996–1000.

[64] R. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J., **26**(1977), 459–472.

[65] R. Schoen and S.T. Yau, Lecture Notes on Differential Geometry, Conference Procedings and Lecture Notes in Geometry and Topology, Vol. **1**, International Press, 1994.

[66] R.G. Smits, *Spectral gaps and rates to equilibrium for diffusions in convex domains*, Michigan Math. J., **43**(1996), no. 1, 141–157.

[67] I.M. Singer, B. Wong, S.T. Yau, and S.S.T. Yau, *An estimate of the gap of the first two eigenvalues*, Ann. Scuola Norm. Sup. Pisa, CI. Sci., **12**(1985), 319–333.

[68] G. Szegö, *Inequalities for certain eigenvalues of membrane of given area*, J. Rat. Mech. Anal., **3**(1954), 343–356.

[69] H. Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan, **31**(1979), no.6, 209–226.

[70] F.Y. Wang, *On estimation of the Dirichlet spectral gap*, Arch. Mat., **75**(2000), no.6, 450–455.

[71] H.F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Rat. Mech. Anal., **5**(1956), no. 4, 633–636.

[72] H. Weyl, *Der Asymptotische Verteilungsgesetz der Eigenwerte Linearer Partieller Differentialgleichungen*, Math. Ann., **71**(1912), 441–469.

[73] D.G. Yang, *Lower bound estimates on the first eigenvalue for compact manifolds with positive Ricci curvature*, Pacific J. of Math., **190**(1999), 383–398.

[74] P. Yang and S.-T. Yau, *Eigenvalue of the Laplacian of compact Riemannian surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa, **7**(1980), 55–63.

[75] S.-T. Yau, *Isoperimetric constants and the first eigenvalue of a compact manifold*, Ann. Sci. Ec. Norm. Super., Paris **8**(1975), 487–507.

[76] S.-T. Yau, editor., Problem Section, Seminar on Diff. Geom., Princeton University Press, Princeton, 1982.

[77] S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, In SunYung Alice Chang, Chang-Shou Lin and Horng-Tzer Yau, editors, Lectures on Partial Differential Equations: Proceedings in honor of Louis Nirenberg's 75th Birthday, pages 223–235, International Press, 2003.

[78] S.T. Yau, preprint.

[79] Q.H. Yu and J.Q. Zhong, *Lower bounds of the gap between the first and second eigenvalues in the Schrödinger operators*, Trans. Amer. Math. Sco., **294**(1986), 341–349.

[80] J.Q. Zhong and H.C. Yang, *On the estimate of the first eigenvalue of a compact Riemannian manifod*, Sci. Sinica, **Ser. A 27**, (1984), 1265–1273.