Erdös-Mordell-type inequalities

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The famous Erdös-Mordell inequality states that, if \( P \) is a point in the interior of a triangle \( ABC \) whose distances are \( p, q, r \) from the vertices of the triangle and \( x, y, z \) from its sides, then

\[
p + q + r \geq 2(x + y + z).
\]

In the paper by Satnoianu [1], some generalizations of the above inequality were given. His proof depends heavily on the geometry of the triangle \( ABC \). In this note, we give a more algebraic proof of the Erdös-Mordell inequality.

**Theorem.** Let \( p, q, r \geq 0 \) and let \( \alpha + \beta + \gamma = \pi \). Then we have the inequality

\[
p + q + r \geq 2\sqrt{qr} \cos \alpha + 2\sqrt{rp} \cos \beta + 2\sqrt{pq} \cos \gamma.
\]

(1)

**Proof.** We consider the following quadratic function of \( x \):

\[
x^2 - 2(\sqrt{r} \cos \beta + \sqrt{q} \cos \gamma)x + q + r - 2\sqrt{qr} \cos \alpha.
\]

(2)

Then a quarter of the discriminant is

\[
\frac{1}{4} \Delta = (\sqrt{r} \cos \beta + \sqrt{q} \cos \gamma)^2 - (q + r - 2\sqrt{qr} \cos \alpha).
\]

Since \( \alpha + \beta + \gamma = \pi \), we have

\[
\cos \alpha = -\cos(\beta + \gamma) = -\cos \beta \cos \gamma + \sin \beta \sin \gamma.
\]

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Using the above identity, the discriminant can be simplified as
\[ \Delta = -(\sqrt{r} \sin \beta - \sqrt{q} \sin \gamma)^2 \leq 0. \]
Thus the expression (2) is always nonnegative. Letting \( x = \sqrt{p} \), we get (1).
\[ \square \]

**Corollary.** Let \( x', y', z' \) be the length of the angle bisectors of \( \angle BPC \), \( \angle CPA \), and \( \angle APB \), respectively. Then we have
\[ p + q + r \geq 2(x' + y' + z'). \]

**Proof.** We have
\[ x' = \frac{2qr}{q + r} \cos \gamma \leq \sqrt{qr} \cos \gamma, \]
\[ y' = \frac{2pr}{p + r} \cos \beta \leq \sqrt{pr} \cos \beta, \]
\[ z' = \frac{2pq}{p + q} \cos \alpha \leq \sqrt{pq} \cos \alpha. \]
The corollary follows from the theorem.
\[ \square \]

**Remark.** Since \( x' \geq x, y' \geq y \) and \( z' \geq z \), the corollary implies the Erdős-Mordell inequality
\[ p + q + r \geq 2(x + y + z). \]

**References**