AN OUTLINE OF THE MASTER THESIS
CHARACTERING THE $q$-COMPLETE MANIFOLDS
AND ESTIMATES OF THE EIGENVALUES OF THE
LAPLACIAN ON COMPACT RIEMANNIAN
MANIFOLDS

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Retyped using $\text{\LaTeX}$ on June 5th, 1998 from the
original version in July, 1988, when a lot of formulas
were hand written.

Date: July, 1988.
1. Introduction

This paper is composed of two parts. The first part is of characterizing $q$-complete manifolds while the second part is on the estimates of the eigenvalues of the Laplacian operator on the compact Riemannian manifolds.

The importance of the concept of the $q$-completeness can be seen in the theorem of Andreotti and Grauert [1] which states that the $q$-th cohomology group of any $q$-complete manifolds with coherent analytic sheaf coefficient must vanish. So it is significant to determine the $q$-complete manifolds via analytic or geometric conditions.

In §3 of the thesis we proved that a $q$-pseudoconvex domain in an $n$-dimensional $r$-complete complex manifold is necessary $\text{Min}(n, r+q-1)$-complete. This result generalized an early result of Giuseppe [12] which states that a $q$-pseudoconvex domain on a Stein manifold is $q$-complete.

In §4 we introduced the concept of the Strongly $(r, q)$-pseudoconvex functions with corners can be uniformly approximated by strongly $\text{Min}(n - \lfloor \frac{n}{r+q-1} \rfloor + 1, n - \lfloor \frac{n}{r} \rfloor + q)$-pseudoconvex functions on Kähler manifolds and thus generalized the theorems of Diederich and Fornæss [11] and Wu [26].

Using the theorem in §4, we generalized a result of Wu [26] in §5 and found a geometric application of the result in Section 4.

The last four sections are devoted to the estimates of the eigenvalues of the Laplacian on Riemannian manifolds. In §6 of the thesis we compared the first eigenvalues of the Laplacian operator with different boundary conditions. By using the Courant’s nodal theorem, we obtained such comparison inequalities.
In §7 we gave an optimal estimates of the first eigenvalue of the compact Riemannian manifold having nonnegative Ricci curvature with convex boundary and Neumannian boundary condition which generalized the result in [21] and [6].

In §8 we sharpened the famous results of Zhong and Yang [28] and Yang [27]. Actually, we gave an estimates of the lower bound of the first eigenvalue on a compact Riemannian manifold without boundary.

In §9, we generalized a result of R. Brooks and P. Waksman [5], where they have given a lower bound of the first eigenvalue of the Laplacian of a convex polygon on a 2-dimensional plane with Dirichlet boundary condition by means of estimating the lower bound of the Cheeger’s isopermetric constant of the polygon. In the section, we resolved the similar problem on a 2-dimensional sphere. It can be seen that the result in [5] turns out to be a special case of our result.

Acknowledgment: The author of this paper thanks Professor Chen Zhi-hua for his guidance and encouragement during the preparation of this paper.

2. q-pseudoconvex Domains in r-complete Manifolds

**Definition 2.1.** Suppose $D$ is an open set in a complex $n$-dimensional manifold. We say $D$ has $C^s$ boundary, if for all $x \in \partial D$, there exists an open neighborhood $U$ of $x$ and a $C^s$ function called the defining function of $\partial D$ at $x$, such that on $M$, we have

$$\partial D \cap U = \{y \in M | \phi(y) = 0\}, D \cap U = \{y \in M | \phi(y) < 0\}$$

and $d\phi(x) \neq 0$.

Suppose $(z^1, \cdots, z^n)$ is the holomorphic coordinates at $x$. We consider the restriction of the Levi form of $\phi$ at $x$

$$L(\phi, x) = \sum_{i,j} \frac{\partial^2 \phi}{\partial z^i \partial \overline{z}^j}(z) dz^i \otimes d\overline{z}^j$$

on

$$T_x(\partial D) = \{v = \sum_{i=1}^n v^i \frac{\partial}{\partial z^i} \in T_x(M) | \sum_{i=1}^n v^i \frac{\partial \phi}{\partial z^i} = 0\}$$

if for all $x \in \partial D$, $L(\phi, x)|_D$ has at least $(n-q)$ nonnegative eigenvalues, we say that $D$ is q-pseudoconvex.

We proved the following

**Theorem 2.1.** Suppose that $M$ is an $n$-dimensional $r$-complete complex manifold and $D$ is a $q$-pseudoconvex domain in $M$ with $C^3$ boundary, then $D$ is $\text{Min}(n, r + q - 1)$-complete.

Using the same methods we also can prove
Theorem 2.2. Suppose that $M$ is an $n$-dimensional strongly $r$-pseudoconvex manifold which has a $C^\infty$ function $\Psi$ that is strongly $r$-pseudoconvex outside a compact set $K$. $D$ is a $q$-pseudoconvex domain in $M$ with $C^3$ boundary. $D \supset K$. Then $D$ is $\text{Min}(n,r+q-1)$ strongly pseudoconvex.

Remark 2.1. Giuseppe [12] has proved that the $q$-pseudoconvex domain in a Stein manifold is 1-complete. Theorem 2.1 generalized the result of him.

3. Strongly $(r,q)$-pseudoconvex Functions and Strongly $(r,q)$-pseudoconvex Functions with Corners

Definition 3.1. Suppose $M$ is an $n$-dimensional Hermitian manifold. $r, q$ are integers satisfying $r + q \leq n + 1$. $U$ is an open set of $M$. $\phi \in C^\infty(U)$ is called the strongly $(r,q)$-pseudoconvex function on $U$, if for each $m \in U$, there exists positive numbers $\eta, \varepsilon > 0$ and complex linear space $H^r_m(U) \subset T_m(U)$ satisfying $\dim_C H^r_m(U) = n - r + 1$, such that

$$\sum_{j=1}^{q} L(\phi, m)(Z_j) \geq \eta$$

for $q$ unit vectors $\{Z_1, \cdots, Z_q\} \subset H^r_m(U)$ which are $\varepsilon$-orthonormal about the Hermitian metric. We also call $\phi$ is the $q$-positive in $H^r_m(U)$.

Definition 3.2. Suppose $M, U$ are as in Definition 3.1. $\phi \in C^0(U)$ is called strongly $(r,q)$-pseudoconvex function with corners, if for each $m \in U$, there exist an open neighborhood $V \subset U$ of $m$ and $(r,q)$-strongly pseudoconvex functions $\phi_1, \cdots, \phi_l$ on $V$ such that $\phi|_V = \text{Max}(\phi_1, \cdots, \phi_l)$.

In this section we proved the following theorem, which generalized the result of Diederich and Fornæss [11] and Wu [26].

Theorem 3.1. Suppose $\phi$ is a strongly $(r,q)$-pseudoconvex function with corners on an $n$-dimensional Kähler manifold $M$ and $\eta$ is a positive continuous function on $M$. Then there exists a strongly $\text{Min}(n - \lfloor\frac{n}{r+q-1}\rfloor + 1, n - \lfloor\frac{n}{r}\rfloor + q)$-pseudoconvex function $\tilde{\phi}$ such that $|\phi - \tilde{\phi}| < \eta$.

4. On Certain Kähler Manifolds which are $(r,q)$-complete

The main purpose of this section is to generalize a result of H. Wu [26] and gave a geometric application of Theorem of 3.1.

Definition 4.1. Hermitian manifold $D$ is called $(r,q)$-complete with corners, if there exists a continuous exhaustion strongly $(r,q)$-pseudoconvex function on $D$.

We proved:
Theorem 4.1. Suppose $M$ is an $n$-dimensional compact Kähler manifold and $V$ be the complex compact submanifold of $M$ with codimension $r$. The bisectional curvature of $M$ is $q$-positive on $V$ and $q$-nonnegative on $M$. Then $M - V$ is $(r, q)$-complete with corners.

Corollary 4.1. Suppose $M, V$ are as in Theorem 4.1. Then $M - V$ is $\text{Min}(n - \lfloor \frac{n}{r+q-1} \rfloor + 1, n - \lfloor \frac{n}{r} \rfloor + q)$-complete. And $M$ is (up to a homotopy type) obtained from $V$ by successively attaching a finite number of cells, each of dimension $\geq \text{Max}(\lfloor \frac{n}{r+q-1} \rfloor, \lfloor \frac{n}{r} \rfloor - q + 1)$.

5. Estimates of the First Eigenvalue of the Laplacian with Neumannian Boundary Condition

Suppose that $M$ is an $n$-dimensional compact Riemannian manifold with convex boundary and nonnegative Ricci curvature. Let $\eta_1$ be the first eigenvalue of the Laplacian of $M$ with Neumannian boundary condition. There were a lot of work concerning the lower bound estimate of $\eta_1$. Payne and Weinberger [21] claimed they had proved $\eta_1 \geq \frac{\pi^2}{d^2}$ in 1960 where $d$ is the diameter of $M$. But Chavel and Feldman [6] pointed that there was a small gap in their proof which made it invalid. At the same time, they proved $\eta_1 \geq \frac{\pi^2}{d^2}$ in the 2-dimensional case. On the other hand, in [18] and [19], it has been proved $\eta_1 \geq \frac{\pi^2}{4d^2}$ and $\eta_1 \geq \frac{\pi^2}{2d^2}$, respectively. Using the method of Zhong and Yang [28], we obtained the optimal estimates of $\eta_1$, that is

Theorem 5.1. Suppose $M, d, \eta_1$ are as before, then

$$\eta_1 \geq \frac{\pi^2}{d^2}$$


In this section, we proved the following theorems which generalized the result in [28].

Theorem 6.1. Suppose $M$ is an $n$-dimensional orientable compact Riemannian manifold without boundary. Then the first eigenvalue $\lambda_1$ satisfies

$$\lambda_1 \geq \frac{\pi^2}{d^2} + (n - 1)K$$

where $(n - 1)K$ is the lower bound of the Ricci curvature.
Using the auxiliary function

\[
\begin{align*}
\tilde{\phi}(\theta) &= \sec^2 \theta (-\frac{1}{2} \theta^2 - \frac{1}{2} \theta \sin \theta - \frac{1}{4} \cos 2\theta + C \theta + \frac{1}{2} C \sin 2\theta + D) \\
\phi(-\frac{\pi}{2}) &= 0, \quad \theta \in (-\frac{\pi}{2}, 0]
\end{align*}
\]

where \( C, D \) are constants, satisfying

\[
\begin{align*}
-\frac{\pi^2}{8} + \frac{1}{4} - \frac{\pi}{2} C + D &= 0 \\
C &\geq \sqrt{\frac{3}{2}} - \frac{\pi}{6}
\end{align*}
\]

we proved

**Theorem 6.2.** Suppose \( M \) is as in Theorem 6.1, then

\[
\lambda_1 \geq \frac{\pi^2}{d^2} + 0.841(n - 1)K
\]

7. **Estimates of the First Eigenvalue of the Laplacian of the Convex Domain on a 2-dimensional Sphere**

Suppose \( \Omega \) is a convex domain with piecewise smooth boundary on a 2-dimensional sphere with radius \( R \). Let \( \lambda_1(\Omega) \) be the first eigenvalue of the Laplacian with Dirichlet boundary condition on this domain. Then

\[
\lambda_1(\Omega) \geq \frac{1}{4} h(\Omega)^2
\]

where \( h(\Omega) \) is the so-called Cheeger isopermetric constant of \( \Omega \), defined as

\[
h(\Omega) = \inf_{r \subset \Omega} \frac{L(\Gamma)}{A(\Gamma)}
\]

where \( \Gamma \) is a closed curve in \( \Omega \), \( L(\Omega) \) and \( A(\Omega) \) represent the length of \( \Gamma \) and the area bounded by \( \Gamma \), respectively. We proved the following

**Theorem 7.1.** Suppose \( \Omega \) is a convex domain with piecewise smooth boundary the interior angles made by two adjoint smooth boundary curves are \( \sigma_1, \ldots, \sigma_n \), respectively. Let \( S \) be the area of \( \Omega \), where \( \Omega \) is contained in a spherical crown with radius \( \mu \) of the sphere. Set

\[
g = (1 - \frac{\pi^2 \mu^2}{R^2})(1 + \frac{(n + 1)\pi^2 \mu}{2R})^{-1}
\]

Then

\[
\lambda_1(\Omega) \geq \frac{g^2}{4S} \left( \sqrt{\sum_{i=1}^{n} \frac{\sigma_i}{2} - \sum_{i=1}^{n} \left( \frac{\pi}{2} - \frac{\sigma_i}{2} \right) + \pi - \frac{S}{R^2} + \sqrt{\pi}} \right)^2
\]

The result generalized the early result of Brooks and Waksman [5].
References


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