ON THE SPECTRUM OF LAPLACIAN ON MINIMAL HYPER–SURFACE OF A SPHERE

Lu Zhiguo  Chen Zhina
(陆志勤  陈志宏)

Abstract The totally geodesic hypersurface of the \((n + 1)\)–dimensional sphere has been characterized by its spectrum of Laplacian on \(\mathbf{g}\)-forms.

1 Introduction

Let \((M, g)\) be an \(n\)-dimensional compact oriented Riemannian manifold, \(\mathcal{M}(M)\) is the vector space of the \(L^2\)-exterior \(q\)-forms on \(M\), where \(q = 0, 1, \ldots, n\). Recall that \(\text{Spec}(M, g)\) denotes the spectrum of the Laplacian \(\Delta\) on \(M\).

It is an important question in the theory of the spectrum of the Laplacian to characterize Riemannian manifolds by their spectrum\(^{(1)}\), i.e., if isospectrum actually implies isometry. In 1964, J. Milnor gave a negative answer to this problem by using a counterexample of two 16–dimensional torus which are isospectral but not isometric. A large number of counterexamples have been found since then by many authors. But the question is still open for some special class of Riemannian manifolds such as \(n\)-dimensional ball with canonical metric. It is well known if \(\text{Spec}(M, g) = \text{Spec}(M', g')\), the canonical ball with sectional curvature 1, we have the following results:

If one of the following cases have been satisfied:

(a) \(q = 0, n = 1, 2, 3, 4, 5^{(1)}\)
(b) \(q = 1, n = 2, 3, 4, 5, 6^{(1)}\)
(c) \(q = 2, m = 2, 3, 4, 5, 6^{(1)}, 17^{(1)}, 17^{(2)}\)
(d) \(q = 0,\) and \(M\) different from \(S^n\) and \(g\) sufficiently close to \(g^{(0)}\)
(e) \(n\) be an odd number and \(M\) be normally contact\(^{(1)}\)
(f) for each \(n\), there exists \(q = q(n)^{(1)}\)

then \(M\) is isometric to \(S^n\).

All of these results are based on the Minakahisundaram–Pieleg–Gaffney formula of eigenvalues and the result of J. V. K. Patodi first computed the first three coefficients of the MPG formula. In this paper, we consider the minimal hypersurface in \((n + 1)\)–dimensional ball \(S^{n+1}(1)\). By using the second fundamental form of the manifolds in representing its Minakahisundaram–Pieleg–Gaffney coefficients, we

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prove the following:

Theorem: Except for \((s,q) = (1,4)\) and \((s,q) = (5,5)\), if \((M,g)\) is the minimal hypersurface in \(S^{n+1}\) and \(\text{Spec}(M,g) = \text{Spec}(S^{n+1},g_s)\), then \(M\) is totally geodesic.

2 Notes and Notations

Let \((M,g)\) be an \(n\)-dimensional compact Riemannian manifold. \(\mathcal{A}(M)\) is the vector space of the \(L^2\)-exterior \(\mathfrak{d}\)-forms. \(\mathcal{A} = \mathcal{A}^0 + \mathcal{A}^1\) represents the Laplacian on \(M\). The spectra of \(\mathcal{A}\) are all discrete eigenvalues with finite multiplicity. Let \(\text{Spec}^1(M,g)\) represent these eigenvalues:

\[
\Sigma^n \lambda_1 \lambda_2 \cdots \lambda_n > 0 = \Sigma^n (\lambda_1 + \lambda_2 + \cdots + \lambda_n) > 0 = o(\delta^{-n+1})
\]

For these eigenvalues we have the famous Minkowski-Ignat'tov-Pleijel-Gaffney formula:

\[
\sum_{\lambda \in \Sigma^n} e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \sigma_{n-1} \sigma_{n-2} \cdots \sigma_{n-\lambda_1} \sigma_{n-\lambda_2} \cdots \sigma_{n-\lambda_n} \sigma_0 (\delta^{-\lambda_1})^{(n-\lambda)} = C_0 \delta^n + C_1 \delta^{n-1} + C_2 \delta^{n-2} + \cdots + C_n \delta^{n-n}
\]

where \(\sigma_1, \sigma_2, \cdots, \sigma_n\) are constants. In \([9,1]\), the first three \(\sigma_1, \cdots, \sigma_3\) are computed:

\[
\begin{align*}
\alpha_0 &= \left( \frac{n}{q} \right) \frac{\text{vol}(M)}{q} \\
\alpha_1 &= \int_\mathcal{A} C(s,q) \text{ vol}(M) \\
\alpha_2 &= \int_\mathcal{A} C(S,q) \text{ vol}(M)
\end{align*}
\]

where

\[
\begin{align*}
C(s,q) &= \frac{1}{6}(n-2) - \frac{1}{6}(q-1) \\
C(s,q) &= \frac{1}{72}(n-2) - \frac{1}{6}(q-1) \\
C(s,q) &= - \frac{1}{108}(n-2) - \frac{1}{6}(q-1) \\
C(s,q) &= - \frac{1}{108}(n-2) - \frac{1}{6}(q-1)
\end{align*}
\]

and \(R, S, \rho\) represent the curvature tensor, Ricci curvature and scalar curvature of \(M\) respectively. \(|R|\), \(|S|\) are the norms related to the Riemannian metric.

Now we confine the manifold \(M\) to minimal hypersurface of \(S^{n+1}\) with the first fundamental form \(g = \sum_{i=1}^{n+1} \mathfrak{e}_i \otimes \mathfrak{e}_i\) and the second fundamental form \(\mathfrak{h}_i = \mathfrak{h}_i \otimes \mathfrak{h}_i\). Moreover, Suppose that at a fixed point \(p\), we have \(\mathfrak{h}_i = \chi_1 \mathfrak{e}_i\), then we have \(\sum_{i=1}^{n+1} \chi_1 = 0\) since the hypersurface is minimal. Then we have the following Gauss equation concerning the minimal hypersurface:

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Finally, let $\rho'$, $S'$, $R'$ represent the scalar curvature, Ricci curvature and sectional curvature respectively and $\theta$ represents the Hodge star operator.

3 Proof of the Theorem

First, we will need the following elementary:

**Lemma 1** The theorem is valid if
\[ \frac{1}{2} \left( \frac{n}{q} \right) \frac{n}{n-2} \]

**Proof** We know that
\[ \rho' = n(n-1) \]
(11)

From identity (4) and under the assumption of the lemma we have $C(n,q) \neq 0$, thus by $\text{Spec}^e(M,\rho) = \text{Spec}^e(S(1),\rho')$, we know
\[ \int_M C(n,q) \rho' \equiv 1 = C(n,q) \cdot n(n-1) \text{vol}(M) \]
(12)

By (10) we have $\int_M \sum_i \pi_i^2 = 0$, and $M$ is totally geodesic.

It is somewhat difficult in the case of $C(n,q) = 0$, i.e. $n(n-1)/q = q(n-q)$. We have the following result:

**Lemma 2** There exist infinite many pairs of $(n,q)$ with $n>0$, $q>0$, $q<n$ satisfying the equation $n(n-1)/q = q(n-q)$. If $n>25$, there are six solutions: $(n,q) = (1,0), (1,1), (6,6), (25,20), (25,25)$.

**Proof** Let $r = n+1$, $s = 2n-q$, then
\[ r^2 - 2s^2 = 1 \]
(13)

This is a Pell equation which has infinite solutions with $r+s$ odd numbers. Thus $n,q$ are always integers. This proves the first of the lemma, while the second part of the lemma is followed by a tedious calculation.

By lemma 2 and the assumption of the theorem, in order to prove the theorem, we can suppose $n>25$. In this case, we have $q=2$, By (3) and $\text{Spec}^e(M,\rho) = \text{Spec}^e(S(1),\rho')$, we know $\alpha_{n,q} = \alpha'_{n,q}$, where $\alpha'_{n,q}$ is the corresponding Minkowskian-Darboux-Pleijhö-Gaffney coefficient, thus
\[ \int_M \left[ C_1(n,q) \rho' + C_2(n,q) S' + C_3(n,q) R' \right] \]
(14)

where $\rho'$, $S'$, $R'$ represent the scalar curvature, Ricci curvature and sectional curvature of $S^1(1)$ respectively. By (8), (9), (10) we have
\[
|R|^4 = 2(\sum \lambda_i)^4 - 2 \sum \lambda_i^4 + 4 \sum \lambda_i^2 + 2 n(n-1) \quad (15)
\]
\[
|S|^4 = \sum \lambda_i^4 - 2(n-1) \sum \lambda_i^2 + n(n-1)^2 \quad (16)
\]
\[
\rho^* = (\sum \lambda_i)^3 - 2 n(n-1) \sum \lambda_i^2 + n^2(n-1)^2 \quad (17)
\]

Substituting them into (14), we have:

\[
\int [C_1(n, q) + \sum \lambda_i^3 - 2 n(n-1) \sum \lambda_i^2 + n^2(n-1)^2]
\]
\[+ C_2(n, q) \left( \sum \lambda_i^2 - 2(n-1) \sum \lambda_i + n(n-1)^2 \right)
\]
\[+ C_3(n, q) \left( \sum \lambda_i^2 - 2 \sum \lambda_i + 2 n(n-1) \right) \right] \geq 1
\]
\[
\int \left[ C_1(n, q) + 2 C_2(n, q) \sum \lambda_i^3 \right] + 1 = \int \left[ C_1(n, q) + 2 C_2(n, q) \sum \lambda_i^3 \right] 
\]
\[
+ \int \left[ -2 n(n-1) C_3(n, q) - 2(n-1) C_4(n, q) - 4 C_5(n, q) \sum \lambda_i^2 \right] \geq 1 = 0 \quad (18)
\]

By the fact that \(S^n(1)\) is the totally geodesic hypersurface of \(S^{n+1}(1)\) we have
\[\rho^* = n(n-1), \quad |S|^4 = n(n-1)^2, \quad |R|^4 = 2 n(n-1), \quad \text{Substituting them into (18) and using } vol(M) = vol(S^n(1)) \text{ we have}
\]
\[
\int \left[ C_1(n, q) - 2 C_2(n, q) \sum \lambda_i^3 \right] = 1 + \int \left[ C_1(n, q) + 2 C_2(n, q) \sum \lambda_i^3 \right] = 1 = 0 \quad (19)
\]

Since \(\frac{1}{15} \left( \sum \lambda_i^2 \right) \geq n-2\), we have
\[
C_1(n, q) - 2 C_2(n, q) \sum \lambda_i^3 = - \left( \frac{n-4}{(n-1)(n^2-n-1)} \right) \left( \frac{1}{15} \left( \sum \lambda_i^2 \right) + \frac{n^2 + 2}{2} \right) \quad (20)
\]
\[
C_1(n, q) + 2 C_2(n, q) \sum \lambda_i^3 = \frac{(n-4)}{(n-1)(n^2-n-1)} \left( \frac{1}{15} \left( \sum \lambda_i^2 \right) + \frac{n^2 + 2}{2} \right) \quad (21)
\]
\[
- 2 n(n-1) C_3(n, q) - 2(n-1) C_4(n, q) - 4 C_5(n, q)
\]
\[= (n-2)(n+2)/15(n-1)(n-3) \leq \frac{1}{n-1} \quad (22)
\]
All of the above three terms are positive if \(n = 5,6\), so in order to make (19) valid, it must be \(\sum \lambda_i^2 = 0\), i.e., \(M\) is totally geodesic.

References:

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