Normal Scalar Curvature Inequalities

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Suppose that M is an n-dimensional Riemannian manifold immersed into the $(n+m)$ dimensional space form of constant sectional curvature c. Define the normal scalar curvature to be

$$\rho^{\perp} = \frac{1}{n(n-1)}|R^{\perp}|,$$

where $R^{\perp}$ is the curvature tensor of the normal bundle of M. Let ρ be the scalar curvature of M and $\vec{H}$ be the mean curvature tensor.

In 1999, De Smet, Dillen, Verstraelen and Vrancken (DDVV) proposed the following so-called Normal Scalar Curvature Conjecture:

Conjecture

$$\rho + \rho^{\perp} \leq |\vec{H}|^2 + c.$$

The conjecture is equivalent to the following matrix inequality.

Conjecture

Let $A_1, \cdots, A_m$ *be real* $n \times n$ *symmetric matrices. Then*

$$\left(\sum_{r=1}^{m} \|A_r\|^2 \right)^2 \geq 2 \left(\sum_{r<s} \|[A_r, A_s]\|^2 \right),$$

where the norm is the Hilbert-Schmidt norm.

In 2005, Böttcher and Wenzel proposed the following inequality.

Conjecture

Let A_1, A_2 *be* $n \times n$ *matrices. Then*

$$(\|A_1\|^2 + \|A_2\|^2)^2 \geq 2\|[A_1, A_2]\|^2.$$

1. The Normal Scalar Curvature Conjecture was proved by Lu [5, 6] and Ge-Tang [4] independently.
2. The Böttcher-Wenzel Conjecture was proved by Vong-Jin [8] and Lu [6] independently.
3. A generalization of the Simons-type pinching theorem is given in [6, Theorem 6].
4. The submersion version of the Normal Scalar Curvature Conjecture was studied by J. Ge.

1. The matrix inequalities can be generalized to bounded linear operators in a separable Hilbert space by taking the limit.
2. The point-wise equality case of the Normal Scalar Curvature was discussed in [6, Lemma 1, 2] and in [4].
3. Charactering the submanifolds for which the equality is valid at every point is a difficult problem. It is related to the classification of Austere submanifolds. See [2] for details.

[1] A. Böttcher and D. Wenzel, *How big can the commutator of two matrices be and how big is it typically?*, Linear Algebra Appl. **403** (2005), 216–228.

[2] T. Choi and Z. Lu, *On the DDVV conjecture and the comass in calibrated geometry. I*, Math. Z. **260** (2008), no. 2, 409–429.

[3] P. J. De Smet, F. Dillen, L. Verstraelen, and L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) **35** (1999), no. 2, 115–128.

[4] J. Ge and Z. Tang, *A proof of the DDVV conjecture and its equality case*, Pacific J. Math. **237** (2008), no. 1, 87–95.

[5] Z. Lu, *Proof of the normal scalar curvature conjecture*.

[6] ______, *Normal scalar curvature conjecture and its applications*, J. Funct. Anal. **261** (2011), no. 5, 1284–1308.

[7] ______, *Remarks on the Böttcher-Wenzel inequality*, Linear Algebra Appl. **436** (2012), no. 7, 2531–2535.

[8] S.-W. Vong and X.-Q. Jin, *Proof of Böttcher and Wenzel's conjecture*, Oper. Matrices **2** (2008), no. 3, 435–442.