FROM HOLOMORPHIC FUNCTIONS TO HOLOMORPHIC SECTIONS

ZHIQIN LU

1. Introduction

It is a pleasure to have the opportunity in the graduate colloquium to introduce my research field. I am a differential geometer. To be more precise, I am a complex differential geometer, although I am equally interested in real differential geometry.

To many people, geometry is a kind of mathematics that is related to length, area, volume, etc. For the Euclidean geometry, this is indeed the case. Differential geometry is a kind of geometry on curved space. The basic geometric objects like length, area and volume are also very important in differential geometry. However, I prefer to think differential geometry is a kind of calculus that takes the underlying curved space into account. What we mean here is that in calculus, we study very complicated functions on relatively simple spaces: 1-d or n-d Euclidean spaces; on the other hand, in topology, we study sophisticated spaces but with relative simple function theory on it. In this sense, differential geometry is a balanced mathematics that pays equal attention to the function theory (calculus) and the topology of the space where the functions are defined. Thus it is not surprising that there is an interaction between calculus and topology in differential geometry. Or in other words, the deep connections between analysis and topology are expected to be found in differential geometry.

One of the perfect example of this kind is called the Atiyah-Singer index theorem. But today, I will not discuss this topic, as it is too lengthy. The main point I will make today is that even one is a purely analyst, working on the function theory on simple space like the complex plane, then to some level, topology is naturally introduced, and it will interacts with the analysis.

I begin with the simplest complex analysis everybody knows. Since I am a complex geometer, I will begin with complex function of one variable.

There are three different ways to define a holomorphic function. Namely the definition of Cauchy-Riemann, the definition of Cauchy and the definition of Weierstrass. Let's briefly go over the three definitions.

1. **The point of view of Cauchy-Riemann.** A holomorphic function f on a domain Ω is a C^1 complex valued function whose real and imaginary part u and v satisfy the so-called Cauchy-Riemann equations
$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}.$$
From this point of view, a holomorphic function is a function satisfying the above partial differential equations (PDE). Thus we can use the PDE methods to study it. In particular, we have
$$\begin{cases} \Delta u = 0 \\ \Delta v = 0, \end{cases}$$
where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. Thus we have the whole theory of elliptic PDE to study it.

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2. **The point of view of Cauchy.** A holomorphic function f defined on a domain Ω is a continuous complex valued function such that for any contour C, we have

$$\oint_C f(z)dz = 0.$$

Among the three definitions, this one assumes the least smoothness. However, the conclusion is that f is actually analytic.

3. **The point of view of Weierstrass.** A holomorphic function f defined on a domain Ω is an object such that for any $z_0 \in \Omega$, there is a small disk U centered at z_0 and is contained in Ω such that $f(z)$ can be represented as a convergent power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

on U. A holomorphic function f is the set of pairs of the form $(U, \sum_{k=0}^{\infty} a_k(z-z_0)^k)$, where U is an open disk of radius r and is contained in Ω and $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is a convergent power series. The compatibility condition is that if we have two pairs $(U, \sum_{k=0}^{\infty} a_k(z-z_0)^k)$ and $(V, \sum_{k=0}^{\infty} b_k(z - z_1)^k)$, then if $U \cap V \neq \emptyset$, we have

$$\sum_{k=0}^{\infty} a_k(z - z_0)^k = \sum_{k=0}^{\infty} b_k(z - z_1)^k.$$

Although the above three definitions of holomorphic function are equivalent and are of equal importance, I would like to make a further comment on the third definition.

Recall that a function is a triple (A, f, B), where A is a set, called the domain, B is a set, called the target and $f : A \to B$ is the assignment such that for any $x \in A$, there is the unique $f(x) \in B$.

In complex analysis, we must cope with functions like $\sqrt[m]{z}$. We call $\sqrt[m]{z}$ a multi-valued function. However, according to the definition, it is not a function. If we use the definition of Weierstrass, then the above question can easily be solved.

Let's use $\sqrt{z}$ as an example. In complex analysis, $\sqrt{z}$ is a multivalued holomorphic function defined on $C - \{0\}$. Let U be a ball in $C - \{0\}$, then on U, let $z = re^{i\theta}$, where θ is a continuous function on U, then $\sqrt{z}$ takes two values $f_1 = \sqrt{r}e^{\frac{1}{2}i\theta}$ and $f_2 = \sqrt{r}e^{-\frac{1}{2}i\theta}$. According to Weierstrass, $\sqrt{z}$ is defined as the set of pairs $(U, \check{f}_1)$ and $(U, \check{f}_2)$, where $\check{f}_1$ and $\check{f}_2$ are power series of f_1 and f_2, respectively. In complex analysis, we know that we can suitably glue U's together to form a topological space, X such that on X, $\sqrt{z}$ is single valued holomorphic function.

X is a two branched covering space of $C - \{0\}$. The above result shows that even in the simple cases, topology is naturally introduced. X is called the Riemann surface of the function $\sqrt{z}$ and it is not surprising that understanding more of the function $\sqrt{z}$ helps to understand more of the Riemann surface.

However, in this case, although X is not just a domain of C, the function is still not twisted. In the next section, we shall show that how the twisted function, or sections, are naturally introduced.

2. The problem

We consider the function theory on C. The function space we consider should not be too big, nor too small. As a simple example, consider the space A of polynomial of degree k on C. In general, $f \in A$ can be represented by

$$f = a_0 + a_1 z + \cdots + a_k z^k.$$

As a finite dimensional vector space, the only invariant of it is its dimension. We have the following

Theorem 1 (Folk Theorem)**.**

$$\dim A = k + 1.$$

In order to see the important relation behind this theorem, we consider little bit more complicated example. Assume that $p_1, \cdots, p_r, q_1 \cdots, q_s$ be points on C. Let A be the space of meromorphic functions f such that

1. $\lim_{z\to\infty} \frac{f}{z^k}$ exists;
2. f has poles of order n_i at p_i;
3. f has zeros of order m_i at q_i.

Then we have

Theorem 2. *If*

$$\sum n_i - \sum m_i + k + 1 \geq 0,$$

then

$$\dim A = \sum n_i - \sum m_i + k + 1.$$

Otherwise,

$$\dim A = 0.$$

Proof. Let A' be the space of meromorphic functions that have poles of order n_i at p_i and $\lim_{z\to\infty} \frac{f}{z^k}$ exists. Then a generic element f' in A' can be written as

$$f = \sum_{i=0}^{k} a_i z^i + \sum_{i=1}^{r} \sum_{l=1}^{n_i} \frac{a_{i,l}}{(z - p_i)^l}$$

Prescribing a zero point of order m_i gives m_i restrictions, thus

$$\dim A = \dim A' - \sum m_i,$$

if $\sum n_i - \sum m_i + k + 1$ is nonnegative.

□

Let B be the space of functions such that

1. $\lim_{z\to\infty} z^{k+2} f$ exists;
2. f has poles of order m_i at q_i;
3. f has zeros of order n_i at p_i.

Then we have

Theorem 3. *If* $\sum n_i - \sum m_i + k + 1 < 0$, *then*

$$\dim B = -(\sum n_i - \sum m_i + k + 1).$$

Otherwise, $\dim B = 0$.

Theorem 4 (Riemann-Roch Theorem on the sphere)**.** *Using the above notations,*

$$\dim A - \dim B = \sum n_i - \sum m_i + k + 1.$$

If we think the above results more, we could find that in the statement, poles and zeros are symmetric. That is, one can think zeros are those "poles" with negative order. The only non-symmetric notation is the order of the function at infinity is not symmetric for the spaces A and B. To solve the problem, we introduce the following idea.

Consider a nonzero meromorphic function f in A. Near the points p_i, f can be written as

$$f(z) = \frac{c}{(z-p_i)^{n_i}} + \cdots .$$

Or in order word, the main part of f near p_i is $\frac{c}{(z-p_i)^{n_i}}$. In order to study the local behavior of f near p_i, we rescale the function f to be $(z-p_i)^{n_i} f(z)$. In this way, locally f can be represented by a holomorphic function. For zeros, one can use $f(z)/(z-q_i)^{m_i}$ to rescale $f(z)$. Using the same idea, we can treat f at infinity. Since $f \sim z^k$ at infinity, we rescale f to be

$$\frac{f(z)}{z^k} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots .$$

The only difference is that in order to make the left hand side of the above holomorphic, we have to introduce a new variable $w = \frac{1}{z}$. In such a new variable, the original point of it corresponding to the infinity of C. Topologically, introducing the infinity into the complex plane is the process of compactification. Namely, let $D = \{w \| w| < 1\}$, then we can glue D to C using the the identification $w = 1/z$. We get a sphere S^2, at least topologically. On the sphere, we can redefine $f(z)$ as follows: Let $U_1, \cdots, U_{r+s+1}$ be open neighborhoods of p_i and q_i and ∞, respectively. Assume that there don't intersect each other. Define the following

$$f_i(z) = \begin{cases} (z-p_i)^{n_i} f(z) & 1 \leq i \leq r \\ (z-q_i)^{-m_i} f(z) & r < i \leq r+s \\ z^{-k} f(z) & i = r+s+1 \\ f(z) & i > r+s+1 \end{cases} .$$

The key observation here is that on each $U_i \cap U_j \neq \emptyset$,

$$g_{ij} = \frac{f_i(z)}{f_j(z)}$$

is independent of the choice of $f(z)$! Thus we can define a twisted space, and define the "functions" on the twisted space. In differential geometry, the twisted space is called a holomorphic line bundle and the twisted "functions" is called holomorphic sections.

Why we should make a meromorphic function to a set of functions which seems more complicated? First of all, f_i's are not more complicated than f, because once you set up the line bundle, a section is just a mapping. Second, we have the following benefits:

1. All f_i's are holomorphic;
2. We are working on S^2, which is compact and is better than C, which is not compact;
3. We shall see that using the language of line bundles, the Riemann-Roch theorem is can be explained in a much simpler way.

From now on we shall rewrite the Riemann-Roch theorem. Let $p_0 = \infty$ We use

$$D = \sum n_i p_i - \sum m_i q_i + k p_0$$

represent the configuration. Obviously, the line bundle purely depends on D. With abusing the notations, we use D to denote the line bundle as well. Then we have

Lemma 1. *A can be represented as the space of sections of the bundle D.*

We use the notation $H^0(S^2, D)$ to denote the space of holomorphic sections of D.

Using the same notations, we have the following

Lemma 2.

$$B = H^0(S^2, -D - 2p_0).$$

Theorem 5 (Riemann-Roch theorem on the sphere)**.** *Using the above notations, we have*

$$\dim H^0(S^2, D) - \dim H^0(S^2, -D - 2p_0) = \deg D + 1.$$

3. Riemann-Roch theorem

Let X be a compact Riemann surface. Let $D = \sum n_i p_i$ be a divisor. We use D also to denote the line bundle defined by the divisor D. Then topologically, X can be represented by C plus 'something' at infinity. In the case that $X = S^2$, the 'something' at infinity can be represented by the divisor $-2p_0$,where p_0 is the infinity of C. In general case, we use the notation K_X to denote the divisor at infinity. Thus we have

Theorem 6 (Riemann-Roch Theorem)**.** *Using the notations as above, we have*

$$\dim\, H^0(X, D) - \dim\, H^0(X, -D - K_X) = \deg D - g + 1,$$

where $\deg D = \sum n_i$ *and* g *is the genus of the Riemann surface.*

In the last section, we prove the case where $X = S^2$. Now let's prove the following partial result on T^2:

Theorem 7. *Let* $D = \sum n_i p_i$ *on* T^2 *with* $n_i > 0$*. Then*

$$\dim\, H^0(T^2, D) = \deg D.$$

Proof. We use the elliptic function theory. Let $\Lambda = \{ma + nb | m, n \in \mathbb{Z}\}$ be the lattice defining T^2. Then for each n_i, if $n_i > 2$, we can define

$$f_{ij}(z) = \sum_{\omega \in \Lambda} \frac{1}{(z - z_i - w)^j}$$

for $j = 3, \cdots, n_i$, and

$$f_{i2}(z) = \frac{1}{(z - z_i)^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z - z_i - \omega)^2} - \frac{1}{\omega^2} \right).$$

We also construct

$$g_i(z) = (z_i - z_{i+1}) \left[\frac{1}{(z - z_i)(z - z_{i+1})} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z - z_i - \omega)(z - z_{i+1} - \omega)} - \frac{1}{\omega^2} \right) \right]$$

for $i = 1, \cdots, k - 1$. Thus the functions f_{ij} and g_i, together with the constant function 1, gives a basis of $H^0(T^2, D)$. Its dimension is

$$\sum (n_i - 1) + k - 1 + 1 = \deg D.$$

In general, we have to use some sheaf and cohomology to prove Theorem 5. In algebraic topology, we shall see that

$$H^0(X, -D - K_X) = H^1(X, D).$$

Then we have the following version of the Riemann-Roch theorem:

Theorem 8 (Riemann-Roch)**.** *Using the above notations, we have*

$$\dim\, H^0(X, D) - \dim H^1(X, D) = \deg D - g + 1.$$

In the higher dimensional case, we have the following Riemann-Roch-Hirzebruch Theorem:

Theorem 9 (Riemann-Roch-Hirzebruch)**.** *Let* E *be a vector bundle over a compact Kähler manifold* X*. Then*

$$\sum_{k=0}^{n} (-1)^k \dim H^k(X, E) = \int_X ch(E) td(X).$$