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A complex manifold is a topological space such that one can define holomorphic functions on it without ambiguity.

Let $X$ be a topological space. Then a good way to study $X$ is to study $C(X)$, the vector space of continuous functions of $X$. $C(X)$ is a better object than $X$ because at least $C(X)$ is a vector space. In this sense, $X$ is linearized by $C(X)$.

If $X$ is a complex manifold, then we can define holomorphic functions over $X$. Let $f : X \to \mathbb{C}$ be a function over $X$. Let $U \subset X$ be a coordinate patch, with $\varphi_U : U \to \mathbb{C}^n$ be the local coordinate. Then by definition, $f : U \to \mathbb{C}$ is holomorphic if and only if $f \circ \varphi_U^{-1} : \varphi_U(U) \to \mathbb{C}$ is holomorphic. The question is that, let $x \in u$ and let $(V, \varphi_V)$ be another local holomorphic coordinate system. If $f \circ \varphi_U^{-1}$ is holomorphic on $U$ and if $V, \varphi_V$ is another coordinate system such that $U \cap V \neq \emptyset$, is $f \circ \varphi_V^{-1}$ holomorphic? The answer is yes if $\varphi_U \circ \varphi_V^{-1}$ is holomorphic because

$$f \circ \varphi_V^{-1} = f \circ \varphi_U^{-1} \varphi_U \circ \varphi_V^{-1}.$$ 

But $\varphi_U \circ \varphi_V^{-1}$ being holomorphic is just the complex structure. Thus we say that the complex structure is the one that one can define holomorphic functions without ambiguity.

In Calculus, we define the derivative of a function in the following way.

Let $B$ be the unit ball of $\mathbb{C}^n$ and let $f : B \to \mathbb{C}$ be a holomorphic function. The differential $df$ of $f$ is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i.$$ 

If we want to take the second derivative, we identify

$$df \sim \left( \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right).$$
so that $df$ becomes a $\mathbb{C}^n$-valued function. The second derivative of $f$ is thus defined to be the derivative of the function $(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$.

We must modify the above method in order to take derivative of a function on a complex manifold. For the sake of simplicity, we let $X = U \cup V$

where $(U, \varphi_U)$ and $(V, \varphi_V)$ are two local coordinate system. The first derivative, as above, can be defined by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i = \sum_{j=1}^{n} \frac{\partial f}{\partial w_j} dw_j.$$ 

In order to take the second derivative, we must identify $(\frac{\partial f}{\partial z_i})$ and $(\frac{\partial f}{\partial w_j})$ on $U \cap V$.

The key observation here is the chain rule

$$\frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial w_j} \cdot \frac{\partial w_j}{\partial z_i}$$

where the matrix $(\frac{\partial w_j}{\partial z_i})$ is independent of $f$!

A $\mathbb{C}^n$-valued function on $X$ can be viewed as two functions

$$f_1 : U \to \mathbb{C}^n, f_2 : V \to \mathbb{C}^n,$$

such that

$$f_1 \equiv f_2 \text{ on } U \cap V.$$ 

In this sense, two sets of functions

$$\frac{\partial f}{\partial z_i}, \frac{\partial f}{\partial w_i}$$

are not compatible over $U \cap V$. In fact, they define a section of the vector bundle $T^*X$, the holomorphic cotangent bundle, instead of a function.

In view of the above discussion, we have the following definition of vector bundles.

Let $X$ be a complex manifold and let $X = \bigcup U_\alpha$ where $(U_\alpha, \varphi_\alpha)$ are a coordinate systems of $X$. Given matrix valued-functions

$$g_{\alpha \beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$$

We define a relation $\sim$ on the set

$$\bigcup U_\alpha \times \mathbb{C}^r$$

by

$$(x, V_i) \sim (x, W_j),$$
if and only if

\[(x, V_i) \in U_\alpha \times \mathbb{C}^\gamma \quad \text{and} \quad (x, W_j) \in U_\beta \times \mathbb{C}^\gamma \]

\[V_i = (g_{\alpha\beta})_{ij} W_j, \quad x \in U_\alpha \cap U_\beta \neq \emptyset\]

In order to make \(\sim\) an equivalent relation, we need to assume that

1. \(g_{\alpha\alpha} = I\) on \(U_\alpha\)
2. \(g_{\alpha\beta} \cdot g_{\beta\alpha} = I\) on \(U_\alpha \cap U_\beta \neq \emptyset\)
3. \(g_{\alpha\beta} \cdot g_{\beta\alpha} \cdot g_{\gamma\alpha} = I\) on \(U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset\)

**Definition 1.** \(E = \bigcup_\alpha U_\alpha \times \mathbb{C}^\gamma / \sim\) is the vector bundle defined by \((g_{\alpha\beta})\). The \((g_{\alpha\beta})\) are called transition functions.

1. If \(g_{\alpha\beta}\) are smooth, then \(E\) is a complex vector bundle,
2. If \(g_{\alpha\beta}\) are holomorphic, then \(E\) is a holomorphic vector bundle,
3. If \(\gamma = 1\), then \(E\) is called a line bundle,
4. If \(g_{\alpha\beta} = I\), then \(E\) is a locally trivial bundle.

By definition of \(E\), we see that there is a projection \(x : E \to X\) and for each \(x \in X\), \(\pi^{-1}(x)\) is a linear space isomorphic to \(\mathbb{C}^\gamma\).

A section of \(E\) can be written as

\[S = (S_\alpha)\]

where \(S_\alpha\) is a \(\mathbb{C}^\gamma\)-valued function. Or more invariantly, let \(e_\alpha = (e_{\alpha,1}, \cdots, e_{\alpha,\gamma})^T\) be a frame (that is, \(e_{\alpha,i}\) are smooth and at each point \(x \in U_\alpha\), \((e_{\alpha,1}, \cdots, e_{\alpha,\gamma})\) forms a basis of the fiber). Then

\[S_\alpha = g_{\alpha\beta} S_\beta,\]

where \(S_\alpha\) are considered as a column vectors. \(e_\alpha\) are defined locally, unlike the case of Euclidean space. Thus in order to give the derivative of \(S\), we first need to take the derivative of \(e_\alpha\).

Let’s assume that

\[D e_\alpha = e_\alpha \theta_\alpha\]

where \(\theta_\alpha\) is a matrix-valued 1-form. The matrix \(\theta_\alpha\) is called the connection matrix. Since

\[e_\beta = e_\alpha g_{\alpha\beta}\]

We have

\[g_{\alpha\beta} \theta_\beta g_{\alpha\beta} = \theta_\alpha g_{\alpha\beta} + dg_{\alpha\beta}\]

(1)

The operator \(D\), or the set of matrix valued forms \(\theta_\alpha\) satisfying (1), is called a connection of the vector bundle \(E\).

We can extend \(D\) to the 1-form valued sections of \(E\) by

\[D(w \wedge S) = dw \wedge S - w \wedge DS\]

Using this we compute

\[D^2 e_\alpha = D(e_\alpha \theta_\alpha) = e_\alpha (d \theta_\alpha + \theta_\alpha \wedge \theta_\alpha).\]

Let

\[\Omega_\alpha = d \theta_\alpha - \theta_\alpha \wedge \theta_\alpha\]
Then we have

$$D^2 e_\alpha = e_\alpha \Omega_\alpha.$$  

$\Omega_\alpha$ is called the curvature form of $D$. From the definition, it measures how far away $D^2$ is from zero. In the Euclidean case, $D^2 = d^2 = 0$.

The connections on a vector bundle $E$ are not unique. The corresponding curvature is a matrix-valued 2-form. The invariants of the curvature tensor, surprisingly, is independent of the choice of the connection. So they are the properties of the vector bundle $E$ itself. We will come back to this point in our third lecture.

In what follows we shall consider holomorphic vector bundles only. Let $S$ be a section of $E \to X$, a holomorphic vector bundle of rank $r$. We know that

$$S = (S_\alpha)$$

where $(S_\alpha)$ is a set of $\mathbb{C}^r$-valued functions. Then

$$S_\alpha = g_{\alpha\beta} S_\beta.$$  

Since $g_{\alpha\beta}$ is holomorphic, we have

$$\dbar S_\alpha = g_{\alpha\beta} \dbar S_\beta.$$  

Thus $(\dbar S_\alpha)$ is a global section of $E \otimes \Lambda^{0,1}(X)$. Thus we have the following

**Proposition 1.** $\dbar$ is a global operator,

$$\dbar : \Gamma(X,E) \to \Gamma(X,E \otimes \Lambda^{0,1}(X)).$$

Before we define the Dolbeault cohomology, we take the following example:

**Example 1.** The difference between the holomorphic function theories on $\Delta$ and $\Delta^*$.

Let $f$ be a holomorphic function on $\Delta$, the unit disk. Then the Stokes’ theorem gives

$$\oint_{\partial \Delta} f dz = 0.$$  

On the other hand, let $f = \frac{1}{z}$. Then $f$ is holomorphic on $\Delta^*$ but

$$\oint_{\partial \Delta} f dz = 2\pi \sqrt{-1}.$$  

more generally, for any $f$ holomorphic on $\Delta^*$

$$\oint_{\partial \Delta} f dz = 2\pi \sqrt{-1} \text{Res } f_{z=0}.$$  

If we define

$$Z(\Delta^*) = \{ \text{all holomorphic functions on } \Delta^* \};$$

$$B(\Delta^*) = \{ \text{all holomorphic functions with } \text{Res } f_{z=0} = 0 \},$$

then

$$Z(\Delta^*)/B(\Delta^*) = \mathbb{C}.$$  

If $f \in B(\Delta^*)$, then

$$fdz = dg = \partial g.$$
We define
\[ \tilde{Z}(\Delta^*) = \{ f \, d\bar{z} \mid f \text{ holomorphic on } \Delta^* \} \]
\[ \tilde{B}(\Delta^*) = \{ \partial \bar{g} \mid g \text{ holomorphic on } \Delta^* \} \]
Then
\[ \tilde{Z}(\Delta^*) / \tilde{B}_1(\Delta^*) = \mathbb{C} \]
And this is the analytic way to say that the space \( \Delta^* \) has a HOLE!

With the above example, we can define the Dolbecalt cohomology:
Let \( E \to X \) be a holomorphic vector bundle over a compact complex manifold.
Let
\[ Z^{p,q}(E) = \{ \eta \mid \eta \text{ is a } E\text{-valued } (p,q) \text{ form}, \partial \eta = 0 \} \],
\[ B^{p,q}(E) = \{ \eta = \bar{\partial} \xi \mid \xi \text{ is a } E\text{-valued } (p,q-1) \text{ form} \} \].
Then
\[ H^q(X, \Omega^p(E)) = Z^{p,q}(E) / B^{p,q}(E) \].

**Example 2.** Let \( X = T \) be the torus and let \( E = \mathbb{C} \) be the trivial bundle, let \( p = 0, q = 1 \). Then
\[ Z^{0,1}(\mathbb{C}) = \{ \eta \mid \text{any } (0,1) \text{ form} \} \],
\[ B^{0,1}(\mathbb{C}) = \{ \bar{\partial} f \mid f \in C^\infty(T) \} \].

It is easy to check that
\[ H^1(X, \Omega^0(\mathbb{C})) = Z^{0,1}(\mathbb{C}) / B^{0,1}(\mathbb{C}) = \mathbb{C} \].

On a holomorphic vector bundle \( E \to X \), we can define a Hermitian metric \( h_\alpha \) such that
\[ h_\alpha(S_\alpha, S_\alpha) = h_\beta(S_\alpha, S_\beta) \].
A holomorphic vector bundle with a Hermitian metric is called a Hermitian bundle.
A connection \( D \) is called Hermitian connection, if
\[ \begin{cases} 
D = D' + D'' \text{ with } D'' = \bar{\partial}; \\
Dh_\alpha \equiv 0.
\end{cases} \]

**Theorem 1.** On any Hermitian vector bundle there is a unique Hermitian connection.

**Proof.** Let \( \theta_\alpha \) be the connection form defined from
\[ \theta_\alpha = \partial h_\alpha h_\alpha^{-1} \]
Then it is easy to verify that the set of vector-valued functions \( \theta_\alpha \) satisfy the compatibility conditions \([1]\). Thus they define a Hermitian connection. \( \square \)

Let \( X \) be a complex manifold. At each point of \( X \), we can attach a lot of algebraic structures to it. For example, we would can define the \( C^\infty \)-functions, holomorphic functions, constant functions, continuous functions as well as \( C^\infty \), holomorphic, constants, locally constant sections of a vector bundle. A general treatment of the above these falls into the language of sheaf.

**Definition 2.** A sheaf \( S \) of Abelian groups over \( X \) is a triple \((S, \pi, X)\) satisfying
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\( \pi : S \to X \) is local homomorphism of \( S \) onto \( X \);
\( \pi^{-1}(m) \) is an Abelian group for \( m \in X \);
The composition laws are continuous in to topology of \( S \).

We elaborate on 3. Let \( S \circ S \) be the subspace of \( S \times S \) such that \( \pi(S_1) = \pi(S_2) \).
Then 3 requires that the map \( S \circ S \to S \), \((s_1, s_2) \mapsto (s_1 - s_2)\) be continuous.

**Definition 3.** A section \( S : U \to S \) for an open set \( U \subset X \) is a continuous map such that \( \pi \circ S = \text{id} \).

**Example 3.** Let \( R \) be the set of real numbers with the discrete topology. Then the triple \((R \times X, \pi, X)\) is a sheaf, called the constant sheaf.

Let \( U \subset X \) and let \( S : U \to R \times X \) be a section. Since \( R \) is given the discrete topology, let \( V \) be any open set, then \( V \times \{y\} \) is an open set of \( R \times X \). Thus \( \{X \mid S(X) = y\} \) is an open set, it is obviously closed. So \( S(X) \) is locally a constant.

**Example 4.** Sheaves of \( C^\infty \) functions. We assume that \( X \) is a \( C^\infty \) manifold. Let \( p \in X \). Let \( U_p \) be the set of neighborhood of \( p \). Define
\[
\mathcal{F}_p = \lim_{\to U_p} C^\infty(U).
\]

The topology between vector bundles and sheaves are totally different. We can think of a sheaf as a book, whose open sets are made from open sets of each pages. In partial each page of the book is an open set.

Let \( E \to X \) be a holomorphic vector bundle over a compact complex manifold \( X \). We have the following sheaves
1. \( \Omega^p(E) \) sheaf of holomorphic \( E \)-valued \((p, 0)\) forms.
2. \( A^{p,q}(E) \) sheaf of \( C^\infty \) \( E \)-valued \((p, q)\) forms.

We define the Čech cohomology as follows:

Let \( S \) be a sheaf over \( X \). Let
\[
\mathcal{U} = \{U_\alpha\}_{\alpha \in I}
\]
be a covering of \( X \). Let
\[
C^p(\mathcal{U}, S) = \prod \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, S),
\]
where the product is over all sets with \( U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \emptyset \). Then we have
\[
\delta : C^p(\mathcal{U}, S) \to C^{p+1}(\mathcal{U}, S)
\]
by
\[
(\delta \sigma)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0 \cdots \hat{i}_j \cdots i_{p+1}}|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}.\]

It is easy to see that \( \delta \delta = 0 \). Thus the Čech cohomology with respect to the covering \( \mathcal{U} \), is
\[
\check{H}^p(\mathcal{U}, S) = \ker \delta \cap C^p(\mathcal{U}, S)/\text{Im} \delta.
\]
The Čech cohomology is the direct limit of the above cohomology groups depending on the covering.

The Dolbeault Theorem is
Theorem 2. Let $E \to X$ be a holo-vector bundle over $X$ then
$$\hat{H}^q(X, \Omega^p(E)) = H^q(X, \Omega^p(E)).$$
Let $E \to X$ be a Hermitian bundle over a compact Hermitian manifold. The complex Laplace operator is defined as

$$\Delta = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \partial.$$ 

We have the following Hodge theorem.

**Theorem 3** (Complex Hodge Theorem). Define

$$H^{p,q}(X,E) = \{ \varphi \mid \Delta \varphi = 0 \}.$$ 

Then

1. $\dim H^{p,q}(X,E) < +\infty,$
2. $H^q(X,\Omega^p(E)) = H^{p,q}(X,E).$

We also have the real version of Hodge Theorem. In particular, we can define the real Laplacian $\Delta_d.$

Now assume that $X$ is a compact Kähler manifold such that the Kähler form $\omega$ is the curvature of some line bundle $L$ over $X.$ The pair $(X, \omega)$ is called a polarized Kähler manifold. We wish to split the cohomology group $H^k(X, \mathbb{C})$ into pieces. There are two ways to split the group:

1. The Hodge decomposition. On a Kähler manifold.

$$\Delta_d = 2\Delta_\nabla$$

Then we have the following

**Theorem 4** (Hodge decomposition theorem).

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

2. The Lefschetz decomposition. On a polarized Kähler manifold, we can define the map

$$L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C}), \quad [\alpha] \mapsto [\alpha \wedge \omega].$$
Since $H^k(X, \mathbb{C})$, $H^{k+2}(X, \mathbb{C})$ can be identified as harmonic $k$, $(k + 2)$-forms respectively, we can define the dual operator $\Lambda : H^k(X, \mathbb{C}) \to H^{k-2}(\mathbb{C})$. We let

$$B = [\Lambda, L].$$

$B$ is a number operator. In fact

$$B \varphi = (n - p - q) \varphi$$

for any $(p, q)$-form $\varphi$.

**Proposition 2.** We have

\[
\begin{align*}
[B, \Lambda] & = 2\Lambda; \\
[B, L] & = -2L; \\
[\Lambda, L] & = B.
\end{align*}
\]

Thus $B, L, \Lambda$ spanned the Lie algebra $\mathfrak{sl}_2(2, \mathbb{C})$. In this way, $H^*(X, \mathbb{C})$ becomes a $\mathfrak{sl}_2(2, \mathbb{C})$-module.

By purely algebraic consideration, we have the following

**Theorem 5** (Lefschetz Decomposition Theorem). On a polarized algebraic manifold $(X, \omega)$, we have the following

$$H^m(X, \mathbb{C}) = \bigoplus_{k=0} L^k P^{m-2k}(X, \mathbb{C}),$$

where

$$P^k(X, \mathbb{C}) = \{[\alpha] | \Lambda[\alpha] = 0\}.$$

The Hodge decomposition is compatible with the Lefschetz decomposition in the sense that

$$P^k(X, \mathbb{C}) = \bigoplus_{p+q=k} P^p,q(X, \mathbb{C}).$$

Now let

$$H_Z = P^n(X, C) \cap H^n(X, Z).$$

Let

$$H^{p,q} = P^n(X, \mathbb{C}) \cap H^{p,q}(X, \mathbb{C}).$$

Then we have

$$\bigoplus H^{p,q} = H_Z \otimes \mathbb{C}.$$  

We have another way to represent the decomposition of $H_Z \otimes \mathbb{C}$. Define a filtration

$$F^k = H^{n,0} \oplus H^{n-1,1} \oplus \cdots \oplus H^{k,n-k}.$$  

Then

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = H,$$  

and we have

$$H^{p,q} = F^p \cap F^q, F^p \oplus F^{n-p+1} = H_Z \otimes \mathbb{C}.$$  

The set $\{H^{p,q}\}$ and $\{F^p\}$, in determining the decomposition of $H_Z \otimes \mathbb{C}$, are equivalent.

We define a quadratic form $Q$ on $H_Z$ as

$$Q(\varphi, \psi) = (-1)^{n(n-1)/2} \int_X \varphi \wedge \psi.$$
for $\varphi, \psi \in H_Z \otimes C$. $Q$ is a non-degenerate form. Furthermore, $Q$ satisfies the following two Hodge-Riemann relations:

1. $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = n - p, q' = n - q$;
2. $(\sqrt{-1})^{p+q} Q(\varphi, \overline{\varphi}) > 0$ for $\varphi \in H^{p,q}, \varphi \neq 0$.

**Definition 4.** A polarized Hodge structure of weight $n$ is given by $\{H_Z, H^{p,q}, Q\}$ satisfying the above relations.

Next we turn to the deformation of complex structures. We say that the triple $(X, \pi, B)$ is a family of compact complex manifold, if

1. $X, B$ are complex manifolds and $\pi : X \to B$ a is surjective holomorphic map;
2. For arbitrary $p \in B$, $\pi^{-1}(p)$ is a compact complex manifold and $d\pi$ has the full rank.

The implication of deformation of complex structures is that the differentiable structure won’t change. Or in other word, for $p_0 \in B$ and $p \in B$ closed to $p_0$ enough, $\pi^{-1}(p)$ and $\pi^{-1}(p_0)$ are diffeomorphic.

To see this, we consider a vector field $X$ near the point $p_0$ on $B$. If we can find an $\tilde{X}$ on $X$ such that

$$\pi_* \tilde{X} = X,$$

then the flow of $\tilde{X}$ gives the diffeomorphism between fibers. In the $C^\infty$ category, it is not difficult to do so using partition of unity.

We assume the base space $B = \Delta$, the unit disk. Since all fibers are diffeomorphic, the different complex structures can be characterized by the $\overline{\partial}$-operator. Assume that for $t \in B$ small, $\overline{\partial}_t$ is the $\overline{\partial}$-operator for the complex manifold $\pi_t^{-1}$. After a diffeomorphism, we may assume that

$$\overline{\partial}_t = \overline{\partial} + \eta,$$

where $\eta = \eta_1(z, t) \frac{\partial}{\partial z_i} d\overline{z_j}$. The integrability condition $\overline{\partial}_t^2 = 0$ gives the equation

$$\overline{\partial}_t \eta + \frac{1}{2} [\eta, \eta] = 0.$$

We assume that

$$\eta = t \eta_1 + t^2 \eta_2 + \cdots.$$

Then

$$\overline{\partial} \eta_1 = 0.$$

Thus $[\eta_1] \in H^1(X, TX)$. $[\eta_1]$ defines the infinitesimal deformation of complex structure. (Kodaira-Spencer map)

**Example 5.** Let $X \to B$ be a family of Kähler-Einstein manifold. That is, for each $t$, there is the unique $g(t)$ such that

$$\text{Ric}(g(t)) = kw_{g(t)}$$

where $k \neq 0$ is a constant and $w_{g(t)}$ is the Kähler form of $g(t)$. The following result is observed by Schumacher [9].
Theorem 6. The form
\[ \xi_t = -\partial \left( \frac{\partial^2}{\partial t \partial \sigma_j} \log \det g(t) \right) \frac{\partial}{\partial z_i} \]
is a harmonic form in \( H^1(\pi^{-1}(0), T_{\pi^{-1}(0)}) \). Furthermore, the Kodaira-Spencer map is given by
\[ \frac{\partial}{\partial t} \mapsto \xi_t \]

Remark 1. Let \( \varphi_t : \pi^{-1}(0) \to \pi^{-1}(t) \) be a harmonic map. Then the infinitesimal of \( \varphi_t \) is the above harmonic \( T_{\pi^{-1}(0)} \)-valued form.

Of course, the biggest problem in deformation theorem is the inverse problem. For any infinitesimal deformation \( \eta \in H^1(X, TX) \), can it be realized as a real deformation?

The above problem, in terms of differential equations, is to solve the following equation
\[
\begin{cases}
\bar{\partial} \eta + \frac{1}{2} [\eta, \eta] = 0 \\
\eta(0) = 0 \\
\eta'(0) = \eta_1
\end{cases}
\]
given \( \eta = t\eta_1 + t^2\eta_2 + t^3\eta_3 + \cdots \).

Theorem 7 (Kodaira-Nirenberg-Spencer). If \( H^2(X, TX) = 0 \), then the above equation is solvable.

Example 6. If \( X \) is Calabi-Yau 3-fold. That is \( K_X \) is trivial. By the Serre duality
\[ H^2(X, TX) = H^1(X, \Omega^1) \neq 0 \]
because \( 0 \neq [\omega] \in H^1(X, \Omega^1) \).

Theorem 8 (Tian). The deformation of \( X \) is unobstructed if \( X \) is a Calabi-Yau manifold.

Proof. We assume that
\[ \eta = t\eta_1 + t^2\eta_2 + \cdots \]
Then the equation becomes
\[ \bar{\partial} \eta_N + \frac{1}{2} \sum_{i=1}^{N-1} [\eta, \eta_{n-i}] = 0, \quad N \geq 2 \]
Since \( K_X = 0 \), Tian observed the following identity
\[ [w_1, w_2] = \partial(w_1 \boxtimes w_2) - \partial w_1 \Delta w_2 + w_1 \Delta \partial w_2 \]
for \( w_1, w_2 \in \Gamma(X, \Omega^{n-1,1}) \). Then the solution of the above equation can be obtained using the standard method.

The following result is from Mumford.

Theorem 9 (Mumford). The coarse moduli space of polarized Calabi-Yau manifolds exists.

Theorem 10. The moduli space of polarized Calabi-Yau manifold is a complex orbifold, or smooth Deligne-Mumford stack.
Moduli spaces of algebraic curves and Calabi-Yau moduli are smooth enough such that we can use differential geometry to study them.

A brief introduction of variation of Hodge structure.

**Definition 5.** The classifying space $D$ for the polarized Hodge structure is the set of all filtrations

$$0 \subset F^m \subset \cdots \subset F^1 \subset F^0 = H$$

with $F^p \oplus F^{n-p+1} = H$ such that they satisfy the Hodge-Riemann relation.

**Remark 2.** $D$ is a homogeneous complex manifold, but the invariant Hermitian metric is in general not Kähler.

Griffith defined the following period map

$$f: \mathcal{M} \to \Gamma \backslash D$$

where $\mathcal{M}$ is the Calabi-Yau moduli, and $\Gamma$ is the monodromy group. The map essentially sends a Calabi-Yau manifold to its Hodge structures.

We have the following property.

**Proposition 3.**

1. If $p \in \mu$ is a smooth point, then $f$ is an immersion,
2. $f$ is holomorphic.

We have Hodge bundles $F^k$ over the classifying space. The pullback of these Hodge bundles give a set of Hodge bundles over $\mathcal{M}$. Let $c_1(F^n)$ be the curvature of the first Hodge bundle. Then it was observed by Tian that the classical Weil-Petersson metric can be defined via the period map:

$$\omega_{WP} = p^* c_1(F^n).$$

For Calabi-Yau 3-fold moduli space, the curvature of $w_{wp}$ is

$$R_{i\bar{j}k\bar{l}} = \delta_{ij} \delta_{k\bar{l}} + \delta_{i\bar{l}} \delta_{j\bar{k}} - F \ast F,$$

where $F$ is the Yukawa coupling.

In order to seek a better metric on $\mu$, we pull back the invariant Hermitian metric of $D$ to $\mu$. Define $\omega_H = p \ast \omega_D$, then $[6]$

**Theorem 11** (Lu). $\omega_H$ is a Kähler metric. Furthermore, the bi-sectional curvature of $\omega_H$ is non-positive and the holomorphic sectional curvature and Ricci curvature are all bounded away from zero.

The most important global property of $\mathcal{M}$ is that it is quasi-projective. Viehweg was the first one who proved this fact. Recently, Donaldson proved the following result.

**Theorem 12.** (Donaldson) If $\text{Aut}(\mathcal{M}, L)$ is discrete, then $\mathcal{M}$ has a metric of constant scalar curvature $\Rightarrow \mathcal{M}$ is Hilbert-Mumford stable.

To see Donaldson’s result implies the second proof of the quasi-projectivity of $\mu$, we need the following lemma.

**Lemma 1.** If $\mathcal{M}$ is Calabi-Yau, then $\text{Aut}(\mathcal{M})$ is discrete.
**Proof.** $\text{Aut}(M)$ is a Lie group. Let $X$ be a holomorphic vector field, then
$$\Delta |X|^2 \geq 0.$$ Thus $|X|^2 = \text{const}$. In fact, we can prove that $X \equiv 0$. Thus $\text{Aut}(M)$ is discrete. □

Using the above lemma, $(M, L)$ is always stable. Thus using GIT, the moduli space is quasi-projective.
Chapter 3

Chern Classes and Donaldson’s functionals, 8/13/2003

Let \( E \rightarrow X \) be a complex vector bundle over a compact Hermitian manifold \( X \). Then for any connection \( D \), let \( \Omega = D^2 \) be the curvature. Then \( \Omega \) is a matrix-valued 2-form. Define
\[
c(E) = \det\left( \frac{-1}{2\pi i} (I + \Omega) \right)
\]
to be the total Chern class.

Definition 6 (Grothendieck Ring \( G(X) \)). Let \( \xi \) be the set of all vector bundles over \( X \). Then under the binary operator \( \oplus \), \( \xi \) is a semi-group. Let \( G(X) \) be the Abelian group generated by \( \xi \). Then under the tensor product, \( \otimes \) \( G(X) \) is a commutative ring, which is called the Grothendieck ring of \( X \).

An element of \( G(X) \) can be written as \( E_1 - E_2 \) formally, where \( E_1, E_2 \in \xi, E_1 - E_2 = E_3 - E_4 \), if there is an \( E_5 \) such that \( E_1 + E_4 + E_5 = E_2 + E_3 + E_5 \).

By the Whitney formula for Chern classes, we have
\[
c(E_1)c(E_4)c(E_5) = c(E_2)c(E_3)c(E_5).
\]
Thus
\[
c(E_1)/c(E_2) = c(E_3)/c(E_4).
\]

From the above, we have

**Theorem 13.** The total Chern class is a ring homomorphism between the Grothendieck ring to the ring of differential forms.

Let’s define the following characteristic classes: let \( \Omega \) be the curvature matrix of the bundle \( E \).

- Chern character

We consider the invariant polynomial (in this case, it is an invariant analytic function)
\[
Ch(X_1, \ldots, X_r) = \sum e^{X_i}.
\]

Define
\[
Ch(E) = Ch(\Omega).
\]
to be the Chern character.

2. **Chern Class**

Define

\[
Ch_n(X_1, \cdots, X_r) = (X_1^n + \cdots + X_r^n)/n!.
\]

Then let

\[
Ch_n(E) = Ch_n(\Omega).
\]

3. **Todd Class**

Define

\[
Td(X_1, \cdots, X_r) = \prod_{i} \frac{X_i}{1 - e^{-X_i}}
\]

Then the Todd class is defined as

\[
Td(E) = Td(\Omega).
\]

We have the following results.

1. \(c(E \oplus F) = c(E) \cdot c(F)\);
2. \(Ch(E \oplus F) = Ch(E) + Ch(F); Ch(E \otimes F) = Ch(E) \cdot Ch(F)\);
3. \(Td(E \oplus F) = Td(E) \cdot Td(F)\).

**Example 7.** Let \(X\) be a smooth 4th order hyper-surface of \(\mathbb{C}P^3\), \(X\) is a K3 surface. Compute the dimension \(\dim H^1(X, TX)\) of universal deformation space.

**Solution.** Using Serre duality we have

\[
H^1(X, TX) = H^{1,1}(X).
\]

By the Lefschetz Hyperplane Section Theorem

\[
\dim H^1(X) = 0.
\]

The Euler characteristic number of \(X\) is

\[
\chi = 1 + \dim H^2(X) + 1.
\]

By the Hodge decomposition theorem

\[
\dim H^2(X) = 2 + \dim H^{1,1}(X).
\]

Thus

\[
\chi = 4 + \dim H^1(X, TX).
\]

We claim that \(\chi = 24\). By the Gauss-Bonnet Theorem

\[
\chi = \int_X c_2(X).
\]

On the other hand, we have the exact sequence

\[
0 \to TX \to T_{\mathbb{C}P^n}|_X \to N \to 0.
\]

Thus we have

(3) \(c(T_{\mathbb{C}P^n}|_X) = c(N)c(TX)\).

Since \(N\) is a line bundle, we have

\[
c(N) = 1 + c_1(N).
\]

Since \(X\) is a Calabi-Yau manifold, we have

\[
c(X) = 1 + c_2(X).
\]
We have
\[ c(T_{\mathbb{C}P^n}|_X) = (1 + \omega)^4 = \cdots + 6\omega^2 + \cdots. \]
Using (3), we have
\[ c_2(X) = 6\omega^2. \]
Thus
\[ \chi = \int C_2(X) = 6 \int w^2 = 24. \]

hence
\[ \dim H^1(X,TX) = 20. \]

Another Solution. Suppose \( f = \sum a_{i_0...i_3}Z_0^{a_{i_0}}...Z_3^{a_{i_3}} \) be a 4th order equation. The space \( f \) is of dimension \( N \) where \( N \) is the number of solutions of the equation \( \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 4, \alpha_i \geq 0 \)
A straightforward computation gives \( N = 35 \). Thus the space of Hilbert scheme is of dimension 34. Since \( \dim Aut(\mathbb{C}P^3) = 15 \). Thus the dimension of the deformation space is \( 35 - 15 = 19 \).
The result is not surprising because \( w \) determine a deformation that is not with respect to the same polarization.

Now let’s turn to the Bott-Chern class. Assume now that \( E \) is a Hermitian bundle over \( X \). That is, \( E \) is a holomorphic vector bundle with a Hermitian metric \( h \). Let \( h_0 \) be a fixed Hermitian metric \( h_0 \). Let \( \varphi \) be any invariant polynomial. Then by the \( \partial \bar{\partial} \)-Poincaré Lemma, we know that
\[ \varphi(R(h)) - \varphi(R(h_0)) = \sqrt{-1} \partial \bar{\partial} \xi \]
for some \((k - 1, k - 1)\) form. In what follows, we define the form \( \xi \).

Let \((h_0, h_1)\) be two metrics on the Hermitian vector bundle \( E \). Let \( h_t \) be a smooth curve connecting \( h_0 \) and \( h_1 \). For example, \( h_t = (1 - t)h_0 + th_1 \). Let \( \varphi \) be an invariant polynomial. Then

Lemma 2 (Donaldson).
\[ \int_0^1 \varphi(h_t h^{-1}_t, R(h_t), \cdots, R(h_t)) dt \in I_m \partial \oplus I_m \bar{\partial}. \]

In particular, up to the subspace of \( Im \partial \oplus Im \bar{\partial} \), the integral is independent of the choice of the path connecting \( h_0 \) and \( h_1 \).

Definition 7. Let \( K \) be the space of Hermitian metrics of \( E \). Define a functional
\[ BC(\varphi, \cdot, \cdot) : K \times K \to \Omega^{k-1,k-1}/Im(\partial) \oplus Im(\bar{\partial}) \]
by
\[ BC(\varphi, h_0, h_1) = \int_0^1 k \tilde{\varphi}(h_t h^{-1}_t, R(h_t), \cdots, R(h_t)) dt. \]
Then \( BC(\varphi, h_0, h_1) \) is called the Bott-Chern classes or the second characteristic class.

The following properties of Bott-Chern classes are essential:
\( \Box \) \( BC(\varphi, h, h) = 0 = BC(\varphi, h_0, h_1) + BC(\varphi, h_1, h_2) = BC(\varphi, h_0, h_2). \)
\( \Box \) \( BC(\varphi_1 + \varphi_2, h_1, h_2) = BC(\varphi_1, h_1, h_2) + BC(\varphi_2, h_1, h_2). \)
3. CHERN CLASSES

\[ \sqrt{-1}\partial\bar{\partial}BC(\varphi, h_1, h_2) = \varphi(R(h_2)) - \varphi(R(h_1)). \]

As in the case of Chern classes, we can extend the Bott-Chern classes into virtual bundles. In view of the fact that

\[ Ch(E \oplus F) = Ch(E) + Ch(F); \]
\[ Ch_n(E \oplus F) = Ch_n(E) + Ch_n(F). \]

We make the following definition:

**Definition 8.** Suppose \( h_0 = h_{0,1} - h_{0,2}, h_1 = h_{1,1} - h_{1,2} \) are two virtual metrics.

We define

\[ BC(Ch_k, h_0, h_1) = BC(Ch_k, h_{0,1}, h_{1,1}) - BC(Ch_k, h_{1,1}, h_{1,2}). \]

In general, if \( \varphi \) is an invariant polynomial, then there is a polynomial \( f \) such that

\[ \varphi = f(Ch_1, \cdots, Ch_r) \]

We define

\[ BC(\varphi, h_0, h_1) = \sum_i \int_0^1 \frac{\partial f}{\partial Ch_i}(Ch_1(R(h_i)), \cdots, Ch_n(R(h_i))) \times i[Ch_i(h_{i,1}^{-1}, \cdots, R(h_{i,1})), - Ch_i(h_{i,2}, R(h_{i,2}))]. \]

**Definition 9** (Donaldson’s functionals). Let \( \varphi_1, \cdots, \varphi_r \) be \( k_1, \cdots, k_r \) homogeneous invariant polynomial. Let \( h_0, h_1 \) be two metrics on the vector bundle \( E \). Then

\[ D(\varphi_1, \cdots, \varphi_r, h_0, h_1) = \sum_{\alpha=1}^r \int_X BC(\varphi_\alpha, h_0, h_1) \wedge \omega^{n-k_\alpha+1} \]

where \( \omega \) is a fixed Kähler form on \( X \).

By the properties of the Bott-Chern classes, we know that Donaldson functional is well-defined and its critical point is independent of the choice of the initial metric \( h_0 \).

**Remark 3.** Donaldson’s functional can be naturally extended to virtual bundles.

Let \( Q_i = Ch_i \), then the Euler-Lagrange equation of \( h \) is

\[ \sum_{\alpha=1}^r \int_X \frac{\partial f}{\partial Q_\alpha}(Q_1, \cdots, Q_r)[Q_1(u_1 h_1^{-1}, R(h_1)) - Q_1(u_2 h_2^{-1}, R(h_2)) \wedge \omega^{n-k_\alpha+1} = 0. \]

In what follows we shall see the restricted Donaldson’s functional and its relations to Kähler-Einstein geometry.

**Example 8.** Let \( X \) be a compact Kähler manifold with positive first Chern class. Such a manifold is called a Fano manifold. It is polarized by the anti-canonical bundle \( K_X^{-1} \rightarrow X \). Let

\[ E = (K_X^{-1} - K_X)^{n+1}. \]

Let \( K \) be the set of all Kähler metrics in the cohomological class \( C_1(X) \). Then for any \( \omega \in K \), there is a Hermitian metric \( h \) such that \( -\sqrt{-1}\partial\bar{\partial}\log h = \omega \). \( h \) is unique up to a constant. Since \( h \), as a metric on the anti-canonical bundle, is just a volume form, we can fix \( h_\omega \) by assuming

\[ \int_X h_\omega \omega^n = \int_X \omega^n. \]
We define the Donaldson’s functional on the set of Hermitian metrics $h_\omega$ such that

1. $\int_X h_\omega \omega^n = \int_X \omega^n$;
2. $\text{Ric}(h_\omega) > 0$

by

$$G(h) = D(Ch_{n+1}, h_0, h_1).$$

Recall that $G(h)$ is defined on a subset of all Hermitian metrics and thus it is more nonlinear. In fact, it is more nonlinear in order to be able to use in the Kähler-Einstein geometry. This observation was made by Tian.

We need to elaborate the notations. $h_i = (h - h^{-1})^{n+1}$ is the virtual Hermitian metric on the virtual bundle $(K_{X}^{-1} - K_X)^{n+1}$.

Since

$$E = \sum_{k=0}^{n+1} C_{n+1}^k (-1)^k K_X^{n+1-2k},$$

So

$$R(h_E) = \sum_{k=0}^{n+1} C_{n+1}^k (-1)^k (-n + 1 - 2k) R(h).$$

Thus the Euler-Lagrange equation is given by

$$\int_X (-1)^k C_{n+1}^k (n + 1 - 2k)^{n+1} w h^{-1} w^n - \lambda \int_X u = 0,$$

where the $\lambda$ is the Lagrange multiplier, this is because the restriction is

$$\int_X u = 0.$$

Thus

$$h^{-1} w^n = \text{constant}.$$ But since

$$\int_X h_\omega \omega^n = \int_X \omega^n,$$

we have

$$\omega^n = h.$$

Therefore

(4) $\text{Ric}(\omega) = -\partial \bar{\partial} \log \omega^n = -\partial \bar{\partial} \log h = \omega.$$

The functional $G$ on $K$ can be expressed explicitly as follows. Let $g_0$ be a reference Kähler metric. Then for any $g \in K$, we have

$$\omega_g = \omega_{g_0} + \sqrt{-1} \partial \bar{\partial} \varphi.$$

The normalization condition is

$$\int_X e^{-\varphi} h_0 = \int_X w_{g_0}^n = \int_X c_1(X)^n.$$

Define a family of Kähler metrics $g_t$ by

$$\omega_{g_t} = \omega_{g_0} + \partial \bar{\partial} (t \varphi).$$
Then
\[ G(g) = a_n \int_0^1 \int_X (\varphi(\omega_{g_0} - \omega_{g_t})) dt - \int_X \varphi \omega_{g_0}^n \]
\[ = a_n (\tau_{g_0}(\varphi) - \int_X \varphi \omega_{g_0}^n). \]

where \( \tau(\varphi) \) is the Aubin’s \( \tau \)-functional.

**Example 9.** Let \( L \rightarrow X \) be an ample line bundle over \( X \). Let \( h_0, h_1 \) be two Hermitian metrics on \( L \).

(5)
\[ A = \prod_i \frac{x_i}{1 - e^{-x_i}} \]
be the Todd polynomial and let
\[ Q = \sum_k \sum_i x_i^k. \]

We define
\[ K(h) = D((AQ)_n, (h_0 - h_0^{-1})^n), (g_1, (h_1 - h_1^{-1})^n)). \]

We wish to express \( K(h) \) in terms of \( h_0, h_1 \). Since by Donaldson’s lemma, the functional is independent of the choice of the path connecting the two virtual metrics, we choose the path
\[ g_0, (ht - h_1^{-1})^n \]
and then followed by
\[ g_t, (h_1 - h_1^{-1})^n. \]

In this way, up to \( \text{Im} \, \partial + \text{Im} \, \partial \), we have
\[ BC((AQ)_{n+1}, (h_0 - h_0^{-1})^n), (g_1, (h_1 - h_1^{-1})^n)) \]
\[ = \sum_{i=0}^{n+1} BC(A_i, g_0, g_1) Q_{n+1-i}(R((h_1 - h_1^{-1})^n)) \]
\[ + \sum_{i=0}^{n+1} A_i(R(g_0)) BC(Q_{n+1-i}, (h_0 - h_0^{-1})^n, (h_1 - h_1^{-1})^n) \]

For \( BC(Q_{n+1-i}, \cdots) \), we have
\[ BC(Q_{n+1-i}, (h_0 - h_0^{-1})^n, (h_1 - h_1^{-1})^n) \]
\[ = \sum_{j=0}^{n} (-1)^j C_i^n BC(Q_{n+1-i}, h_0^{n-2j}, h_1^{n-2i}) \]
\[ = \sum_{j=0}^{n} (-1)^j C_i^n (n-2j)^{n+1-i} BC(B_{n+1-i}, h_0, h_1) \]
\[ = \begin{cases} 0, & i \neq 1 \\ 2^n n! BC(Q_n, h_0, h_1). & i \neq 1 \end{cases} \]

On the other hand
\[ Q_{n+1-i}(R((h_1 - h_1^{-1})^n)) = \begin{cases} 0, & i \neq 1 \\ 2^n n! R(h_1)^n, & i \neq 1 \end{cases}. \]
Therefore, we have
\[
BC((AQ)_{n+1}, (h_0 - h_0^{-1})^{n}, (h_1 - h_1^{-1})^{n})
\]
\[= BC(A_1, g_0, g_1)2^n!R(h_1)^n
\]
\[+ A_1(R(g_0))2^n!Q, h_0, h_1)
\]
We have
\[
BC(A_1, R(h_0), R(h_1)) = -\frac{1}{2} \log \frac{\det g_1}{\det g_0}
\]
\[
BC(Q_n, h_0, h_1) = n \varphi \int_0^1 R(h_t)^{n-1} dt.
\]
Thus
\[
F(\varphi) = K(h)
\]
\[= 2^n n! (-\frac{1}{2} \int_X \frac{\det(R(e^{-\varphi} h_0))}{\det(R(h_0))} R(h_1)^n
\]
\[+ n \int_X \varphi \text{Ric}(R(h_0)) \wedge \int_0^1 R(e^{-t \varphi} h_0)^{n-1} dt
\]
The K-energy is defined to be
\[
\gamma(\omega_0, \omega_1) = -\int_X \int_0^1 \varphi (\text{Ric}(\omega_s) - \omega_s) \wedge \omega_s^{n-1}
\]
where
\[
\omega_1 = \omega_0 + \partial \bar{\partial} \varphi, \omega_s = \omega_0 + s \partial \bar{\partial} \varphi.
\]

**Example 10.** Let
\[
E = (n + 1)(K^{-1} - K) \otimes (L - L^{-1})^{n} - n(L - L^{-1})^{n+1},
\]
where \(L\) is the polarization of \(X\) with \(c_1(L) = c_1(X)\) and \(K = K_X\) is the canonical line bundle. Let
\[
D(Ch_{n+1}, h_0, h_1) = \gamma(\omega_0, \omega_1),
\]
where \(\omega_1 = -\partial \bar{\partial} \log h_i\).

Now assume that \(V\) is a holomorphic vector field on \(X\). Let \(w_0 \in c_1(X)\). Let \(w_t = \delta_t^* (w_0)\) where \(\delta_t\) is the flow on \(X\) defined by \(X\) (or \(2ReX\), to be precise). Then we see that
\[
D(\varphi_1, \cdots, \varphi, h_0, h_t) = D(\varphi_1, \cdots, \varphi, \omega_0, \omega_t)
\]
is of the form \(k \cdot t + l\). The constant \(k\) is the Futaki invariant. In particular, the Futaki invariant is independent of the choice of the representative of the cohomological class. If the Donaldson’s functional has a lower bound, then the Futaki invariant is automatically non-negative.

**Remark 4.** The above discussions are also true in the case of extreme metrics.

**Generalized Futaki invariant.**

Let \((X, \omega)\) be a compact Kähler manifold and let \(E \to X\) be a virtual holomorphic bundle. Let \(G\) be a subgroup of \(Aut(X)\), the holomorphic automorphism group. Assume that \(G\) can be lifted to an automorphism of \(E\) preserving the fibers. Assume that \(X\) preserves the cohomological class \([\omega]\). Then
Proposition 4.

\[
\int_X \tilde{\varphi}(Q_h(X^*), R(h), \ldots, R(h)) \land \omega^{n-k+1} - \frac{n - k + 1}{k} \int_X f_X^* \varphi(R(h), \ldots, R(h)) \land \omega^{n-k}
\]

is independent of the choice of \( h \), where \( f_X \) is the Hermitian function of \( X \) with respect to \( \omega : i_X \omega = \partial \bar{\partial} f_X \).

Proposition 5. Let \( \varphi = c_1^{n+1} \), \( h \) be the metric of the line bundle \( L \to X \). Then the above is the Futaki invariant.
A line bundle $L$ over a compact complex manifold $X$ is ample, if there is a Hermitian metric $h$ on $L$ such that

$$-\sqrt{-1} \partial \bar{\partial} \log h > 0$$

A line bundle $L$ over a compact complex manifold $X$ is very ample, if $\dim H^0(X, L) > 1$ and if we can choose a basis $S_0, \cdots, S_{d-1}$ of $H^0(X, L)$ such that the map

$$X \to \mathbb{C} \mathbb{P}^{d-1}, x \mapsto [S_0(x), \cdots, S_{d-1}(x)]$$

is an embedding of $X$ into $\mathbb{C} \mathbb{P}^{d-1}$. If $X$ can be embedded into some $\mathbb{C} \mathbb{P}^N$, then $X$ is called an algebraic manifold.

The following theorem of Kodaira is classical:

**Theorem 14.** If $L$ is an ample line bundle, then for some $k \gg 0$, $L^k$ is very ample.

In order to prove the theorem, we use the $L^2$-estimates. The following result is due to Demailly [2].

**Theorem 15.** Suppose $(X, g)$ is a compact Kähler manifold of complex dimension $n$. $L$ is a line bundle on $X$ with Hermitian metric $h$. Let $\psi$ be a function on $X$. Assume that

$$\langle \sqrt{-1} \partial \bar{\partial} \psi + \text{Ric}(h) + \text{Ric}(g), v \wedge \bar{v} \rangle \geq C \|v\|^2_g$$

for any tangent vector $v$ of type $(1, 0)$. Then for any smooth $L$-valued $(0, 1)$-form $\eta$ on $X$ with

$$\bar{\partial} \eta = 0$$

and

$$\int_X \|\eta\|^2 e^{-\psi} dV_g < +\infty,$$

there exists a smooth section $u$ of $L$ such that

$$\bar{\partial} u = \eta$$

and

$$\int_X \|u\|^2 e^{-\psi} dV_g \leq \frac{1}{C} \int \|\eta\|^2 e^{-\psi} dV_g.$$
where \( dV_g \) is the volume form of \( g \) and \( \| \cdot \| \) is the point-wise norm induced by both \( h \) and \( g \).

One remark we would like to make here is that we allow \( \psi \) to have singularities. The typical choice of \( \psi \) would be \( \psi = C(\log r) \) near the original point 0 of the local coordinate system. If \( C \) is large enough, the function \( e^{-\psi} \) is not integrable, which will force \( u \) to vanish at 0 up to certain order.

The Kodaira embedding theorem is implied by the following:

**Proposition 6.** Let \( k \gg 0 \), then for any \( x, y \in X \), and a neighborhood \( \{ U, \varphi_U \} \) of \( x \), and any vector \( v \), we can find a holomorphic section \( S \) such that

\[
S(x) = 0, S(y) \neq 0; \\
dS \circ \varphi_U^{-1} \big|_{\varphi_U(x)} = v.
\]

**Proof of the Kodaira theorem from the above proposition:** Let \( \varphi : X \to \mathbb{C} P^N, x \mapsto [S_0(x), \cdots, S_{d-1}(x)] \) be the embedding. Then if \( x \neq y \),

\[(S_0(x), \cdots, S_{d-1}(x)) \neq \lambda(S_0(y), \cdots, S_{d-1}(y)) \]

for any non-zero complex number \( \lambda \). Otherwise write

\[S = \sum C_i S_i.\]

Then we have

\[0 = \lambda S(y),\]

a contradiction. So \( \varphi \) is 1-1. On the other hand, \( \varphi \) is an immersion because essentially we can prescribing all the first order derivatives by the above proposition. □

In order to prove Proposition 6 we use \( L^2 \)-estimate. For the sake of simplicity, we assume that \( y \in U \) where \( U \) is a geodesic neighborhood of \( x \). We notice that the size of \( U \) depends on the injectivity radius of the manifold \( X \).

On the neighborhood \( U \), assume that \( L \) is trivial. Let \((z_1, \cdots, z_n)\) be the local holomorphic coordinates. A section of \( L \) or \( L^k \) \((k \gg 0)\) is a holomorphic function. At least on the neighborhood \( U \), we can define a holomorphic function \( f \) such that

\[f(x) = 0, df(x) = v, f(y) \neq 0.\]

Using cut-off function, we can extend \( f \) to be a global \( C^\infty \) section of \( L^k \) \((k \text{ to be determined})\). Let \( \eta = \overline{\partial} f \). Let \( \psi = C_1 \log(\sum |z_i|^2) \) where \( C \) is a constant to be determined. We choose \( C_1 \) large enough such that

\[
\int_B e^{-\psi + n} dV = +\infty,
\]

and we then choose \( k \) large enough such that

\[
\langle \sqrt{-1} \partial \overline{\partial} \psi + k \text{Ric}(h) + \text{Ric}(g), v \wedge v \rangle \geq C\|v\|_{g'}^2.
\]

By \( \overline{\partial} \)-estimate, there exists \( u \) such that \( \overline{\partial} u = \eta \) and

\[
\int_X \|u\|^2 e^{-\psi} dV_g \leq \frac{1}{C} \int \|\eta\|^2 e^{-\psi} dV_g < +\infty.
\]
Since $\int e^{-\psi} = +\infty$, we have $u(x) = 0$, $du(x) = 0$. Thus $S = f - u$, a holomorphic section satisfying the required conditions.

□

A naturally question about the Kodaira’s embedding theorem is that how large $k$ we should choose in order that $L^k$ is very ample. From the above proof, we know that $k$ depends on the injectivity radius and possibly the bound of the curvature, etc. However, we have the following important:

**Conjecture 1** (Fujita Conjecture). $(n + 2)L + K_X$ is very ample if $L \to X$ is ample. $(n + 1)L + K_X$ is free if $L$ is ample.

Note: $L \to X$ is called free, if $\forall x \exists S \in H^0(X, L)$ such that $S(x) \neq 0$.

Demailly [3] used the method of solving degenerated Monge-Ampère equation and proved that $12n^2L + 2K_X$ is very ample. After his work, many people, Ein-Lazarsfeld-Nakamaye [5] improved his result. The following result is due to Siu.

**Theorem 16.** $mL + K_X$ is very ample if

$$m > 2 \left( n + 2 + n \left( \frac{3n + 1}{n} \right) \right)$$

$mL + K_X$ is free if $m = O(n^2)$.

Siu’s method is basically algebraic-geometric, which is very beautiful. At the risk of making it more confusing, I try to explain his proof in a naive way (Let me state one more time that Siu’s method is very deep and beautiful).

**Key Step 1.** Both Siu and Demailly used the Hilbert polynomial. Let $F$ be a line bundle and $\mathcal{F}$ be a coherent sheaf. Then

$$\dim \sum (-1)^r \dim H^r(X, (mL + F) \otimes \mathcal{F})$$

is a polynomial of degree at most $n$ in the variable $m$.

Since it is a polynomial, for $m$ fixed but large

$$\dim H^0(X, (mL + F) \otimes \mathcal{F})$$

will be large. So we would have enough section to play with.

**Key Step 2.** Assume that we have already found some $m$ depending only on $n$ such that other than a fixed point $x_0 \in X$, $\exists S \in H^0(X, L^m)$ such that $S(x) \neq 0, \forall x \neq x_0$. Then we can form a singular metric $\log(\sum |S_i(X)|^\xi)$ of $L$ at $x_0$. Using this and the $L^2$-estimate, we can find the required section.

Next we discuss the recent result of Donaldson [4].

**Theorem 17.** Let $(X, L)$ be a polarized Kähler manifold. Let $\text{Aut}(X, L)$ be discrete. If $X$ admits a Kähler metric of constant scalar curvature, then $X$ is Hilbert-Mumford stable.

There are two main steps in Donaldson’s proof.
Step 1. Donaldson proved that if $\text{Aut}(X,L)$ is discrete and if $X$ admits a Kähler metric of constant scalar curvature, then $X$ is balanced.

Step 2. By the theorem of Luo [8] and S. Zhang [13], balanced manifolds are Hilbert-Mumford stable.

Definition of Stability. We assume that $X \subset \mathbb{C}P^N$ is already embedded into certain complex projective space. The Hilbert polynomial

$$P(m) = \dim H^0(X, L^m)$$

is a polynomial of $m$ of degree $n$. Grothendieck proved the existence of a compact variety, called the Hilbert scheme with fixed Hilbert polynomial, which parametrized all the subscheme of $\mathbb{C}P^N$ with the given Hilbert polynomial. The Hilbert scheme is denoted as $\text{Hilb}_h$. On $\text{Hilb}_h$, there is a canonical ample line bundle $L \rightarrow \text{Hilb}_h$.

Assume that $X \subset \mathbb{C}P^N$, then $\text{Aut}(\mathbb{C}P^N)$ acting on $\text{Hilb}_h$ and has a linearization on the line bundle $L$. We have the following definition of stability:

**Definition 10.** A point $x \in H = \text{Hilb}_h$ is called stable with respect to $G = \text{Aut}(\mathbb{C}P^N)$, $L$ and the given linearization, if $x$ has finite stabilizer and for some $m \geq 1$, there exists a section $t \in \Gamma(\text{Hilb}_h, L^m) G$ such that

1. $H_t = H - V(t)$ is affine, where $V(t)$ is the zero locus of $t$,
2. $x \in H_t$ or in other terms, $t(x) \neq 0$,
3. $G$ acting on $H_t$ is a closed action.

**Definition 11** (Definition of “Balanced” metric). Let $(X,L)$ be a polarized Kähler manifold. Assume that $X \subset \mathbb{C}P^N$. If $S_0, \ldots, S_N$ be the standard section of the hyperplane bundle $\xi \rightarrow \mathbb{C}P^N$ and if

$$\int_X \langle S_i, S_j \rangle \sum |S_i|^2 = \delta_{ij},$$

Then we say that $X$ is balanced.

The concept “balanced” is equivalent to the following:

**Proposition 7.** If $X$ is a Kähler manifold such that for $S_0, \ldots, S_{N-1} \in H^0(X, L^m)$ an orthonormal basis of the Hermitian vector space with respect to the $L^2$-inner product, the point-wise sum

$$\sum ||S_i||^2 = \text{const},$$

then $X$ is balanced.

A sketch of the proof that constant scalar curvature implies

$$\sum ||S_i||^2 = \text{const}.$$

First, if $m$ is large enough, then we have the Tian-Yau-Zelditch expansion [1, 7, 11, 12]

$$\sum ||S_i||^2 \sim m^n (a_0 + \frac{a_1}{m} + \frac{a_2}{m^2} + \cdots),$$

where $a_0 = 1/\text{vol}(X)$. In [6] proved that $a_1 = \frac{1}{2}\rho$, where $\rho$ is the scalar curvature. Therefore, by assumption, $a_1$ is a constant. If $m$ is large,

$$\sum ||S_i||^2$$

is almost constant to some accuracy. Donaldson proved
1. After a change of the metric, one can make
\[ \sum \| S_i \|^2 = const + O\left( \frac{1}{m^e} \right) \]
for any \( e \);
2. After a change of the frame of \( \mathbb{CP}^N \), one can make
\[ \sum \| S_i \|^2 = const. \]
Bibliography


