ON THE MODULI SPACE OF CALABI-YAU MANIFOLDS

ZHIQIN LU

TALK GIVEN AT ASPEN CENTER OF PHYSICS, JULY 12, 2007

1. Introduction

Let $X$ be a simply connected compact Kähler manifold with zero first Chern class, and let $L$ be an ample line bundle over $X$. The pair $(X, L)$ is called a polarized Calabi-Yau manifold. By a theorem of Mumford, the moduli space of the pair $(X, L)$ (CY moduli) exists and is a complex variety. Locally, up to a finite cover, the moduli space is smooth (see [20, 21]). There is a natural Kähler metric, called the Weil-Petersson metric, on $\mathcal{M}$. In this paper, we summarize and discuss the differential geometry of the couple $(\mathcal{M}, \omega_{WP})$, where $\omega_{WP}$ is the Kähler form of the Weil-Petersson metric.

We will discuss the local, semi-global, and global properties of the moduli space under the Weil-Petersson metric (WP metric). By Freed [11], the four dimensional gauge theories with $N = 2$ supersymmetry are called special Kähler manifold, while the moduli spaces of Calabi-Yau threefolds are called the projective special Kähler manifolds. There are parallel results between the special Kähler manifolds and the projective ones. So we shall also include a discussion of the special Kähler manifolds in the last section.

For the sake of simplicity, we shall only discuss the moduli space of polarized Calabi-Yau threefolds. Of course, most of the results in threefolds extend to the high dimensional cases.

Acknowledgment. The author thanks Professor Ooguri for the invitation to the 2007 Summer workshop String Theory and Quantum Geometry at Aspen Center of Physics. He also thanks Professor Freed and Panetev for the discussions of the topic. An expanded version of this lecture note will be submitted to the Proceedings of the 2007 ICCM (International Congress of Chinese Mathematics) at Hangzhou in December 2007).

2. Local theory

We shall work on the smooth part of the moduli space $\mathcal{M}$. Let $\omega_{WP}$ be the Kähler form of the WP metric, and let a generic point of $\mathcal{M}$ be $t$. We assume that $X_t$ is the CY 3-folds represented by $t$. 
In [3, page 65], it was showed that the curvature of the Weil-Petersson metric is neither positive nor negative on the 1-dimension moduli space. This makes it very difficult to use the tools in differential geometry. In [14, 17], I introduced the Hodge metric as a replacement of the WP metric. The metric, in its simplest form, is defined as

**Definition 1.** Let $m = \dim \mathcal{M}$ and let

$$\omega_H = (m + 3)\omega_{WP} + \text{Ric}(\omega_{WP}).$$

We call the metric $\omega_H$ the Hodge metric ($\omega_H$ is necessarily positive definite).

**Theorem 1** (cf. [14, 17]). Using the above notations, we have

1. The Hodge metric is a Kähler metric, and
   $$2\omega_{WP} \leq \omega_H.$$
2. The bisectional curvatures of the Hodge metric are nonpositive;
3. The holomorphic sectional curvatures of the Hodge metric are bounded from above by a negative constant (with the bound being $((\sqrt{m}+1)^2 + 1)^{-1}$);
4. The Ricci curvature of the Hodge metric is bounded above by a negative constant.

From differential geometric point of view, the Hodge metric is a better metric than the WP metric. The above local result of the WP metric (or the Hodge metric) will be used in the semi-global and global theory of CY moduli.

As the first application of the above result, we have the following *intrinsic* estimate of the derivative of the Yukawa coupling in [16]:

**Theorem 2.** Let $\varphi_1, \ldots, \varphi_m$ be an orthonormal basis of the cohomology group $H^1(X, TX)$ for a generic fiber $X$. Then we have

$$|\nabla F| \leq 12 \sum_{i=1}^m \|\varphi_i\|_{L^4}^4.$$

As a consequence, we have the following estimate of the lower bound of the scalar curvature of the Hodge metric:

$$0 < -\rho \leq 3m^6 + 144m^3 \sum_{i=1}^m \|\varphi_i\|_{L^4}^4.$$

Note that by Theorem 1, the curvature of the Hodge metric is bounded from below by (a multiple of) its scalar curvature.

There is another relation between the Hodge metric and the Weil-Petersson metric. Let $T$ be the so-called BCOV torsion (to be defined in the next section). Then we have the following result [9]:

**Theorem 3.** Using the above notations, we have [9]:

$$-\omega_H - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log T = \frac{\chi}{12} \omega_{WP},$$

$\chi$ is the Euler characteristic number of a generic fiber $X_t$. 

As a corollary, if \( \chi > 0 \), we shall see that the moduli space \( \mathcal{M} \) can’t be compact, because otherwise we integrate on both side of the above we will get contradiction. Of course, no body would believe that the moduli space would be compact, but the above argument gives a mathematical proof of the fact.

The formula in Theorem 3 is actually known by Bershadsky, Cecotti, Ooguri, and Vafa [1]. For those who know the Quillen metric of determinant line bundles, the right hand side of the formula is the curvature of the Quillen metric. What we observed in [9] is that \(-\omega_H\) is the curvature of the \(L^2\) metric.

3. Semi-global theory

By the semi-global theory of the CY moduli, we mean the theory of the moduli space in a neighborhood of the infinity. In the theory of topological B-model in the case of \( g = 1 \), Bershadsky, Cecotti, Ooguri, and Vafa [1, 2] introduced the following so-called BCOV torsion: let \( X \) be a Calabi-Yau threefold. Let

\[
T = \prod_{1 \leq p,q \leq 3} (\det \Delta'_{p,q})(-1)^{p+q}pq,
\]

where \( \Delta_{p,q} \) is the \( \bar{\partial} \)-Laplace operator on \((p,q)\) forms on \( X \) with respect to the Ricci-flat metric; \( \Delta'_{p,q} \) represents the non-singular part of \( \Delta_{p,q} \); the determinant is taken in the sense of zeta function regularization.

The BCOV torsion is a real smooth function on the CY moduli. We use the usual convention in the Mirror symmetry: let \( W_\psi \) be the mirror CY 3-folds of the quintic hypersurfaces of \( CP^4 \). Then we have the following result [10]:

**Theorem 4.** Let \( \psi = 1 \) be the conifold point of the moduli space. The following formula holds as \( \psi \to 1 \):

\[
T(\psi) \sim |\psi - 1|^{-1/6}.
\]

In fact, the above result is also true for the degeneration of a family of CY 3 folds to a conifold with one ODP (ordinary rational double point).

With the above result, we are able to answer the Conjecture (B) in the papers of BCOV:

**Conjecture 1.** (A) Let \( N_g(d) \) be the genus-\( g \) Gromov-Witten invariant of degree \( d \) of a general quintic threefold in \( CP^4 \). Then the following identity holds:

\[
q \frac{d}{dq} \log F_{1,A}^{top}(q) = \frac{50}{12} - \sum_{n,d=1}^{\infty} N_1(d) \frac{2ndq^{nd}}{1 - q^{nd}} - \sum_{d=1}^{\infty} N_0(d) \frac{2dq^d}{12(1-q^d)},
\]

or equivalently

\[
F_{1,A}^{top}(q) = \text{Const.} \left\{ q^{25/12} \prod_{d=1}^{\infty} \eta(q^d)^{N_1(d)}(1 - q^d)^{N_0(d)/12} \right\}^2.
\]
The conjecture is

\[ F_{1,A}^{\text{top}}(\psi) = F_{1,B}^{\text{top}}(\psi), \]

where \( F_{1,B}^{\text{top}}(\psi) \) is defined below.

(B) Let \( \| \cdot \| \) be the Hermitian metric on the line bundle

\[ (\pi_\ast K_{W/CP^1})^{\otimes 62} \otimes (T(CP^1))^{\otimes 3}|_{CP^1 \setminus D} \]

induced from the \( L^2 \)-metric on \( \pi_\ast K_{W/CP^1} \) and from the Weil-Petersson metric on \( T(CP^1) \). Then the following identity holds:

\[ \tau_{BCOV}(W_\psi) = \text{Const.} \left\| \psi^{-62}(\psi^5 - 1)^{\frac{3}{2}} (\Omega_\psi)^{62} \otimes \left( \frac{d}{d\psi} \right)^3 \right\|^{\frac{3}{2}} \]

\[ = \text{Const.} \left\| \frac{1}{F_{1,B}^{\text{top}}(\psi)^3} \left( \frac{\Omega_\psi}{y_0(\psi)} \right)^{62} \otimes \left( \frac{d}{dq} \right)^3 \right\|^{\frac{3}{2}}, \]

where \( D \) is the discriminant locus of the fibration \( X \to CP^1 \), and \( \Omega \) is the local holomorphic section of the \((3,0)\) forms.

\( F_{1,B}^{\text{top}}(\psi) \) is defined as

\[ F_{1,B}^{\text{top}}(\psi) := \left( \frac{\psi}{y_0(\psi)} \right)^{62} (\psi^5 - 1)^{-\frac{3}{2}} \frac{d\psi}{dq}. \]

Using Theorem 4, in [10], we proved that

**Theorem 5.** Conjecture (B) holds.

**Remark 1.** Recently Li-Zinger [12] and Zinger [23] made crucial progress in proving Conjecture (A). We wish before long we could verify the equation

\[ F_{1,A}^{\text{top}}(\psi) = F_{1,B}^{\text{top}}(\psi), \]

which will give a very strict test of the mirror symmetry.

4. **Global theory**

The volume and the Chern-Weil forms of the WP metric give important global information in string theory. Let’s start with the following example:

Let the quintic hypersurface in \( CP^4 \) be

\[ X = \{ Z | Z_0^5 + \cdots + Z_4^5 + 5\lambda Z_0 \cdots Z_4 = 0 \} \subset CP^4. \]

It is a smooth hypersurface if \( \lambda \) is not any of the fifth unit roots. To construct the moduli space, we define

\[ V = \{ f | f \text{ is a homogeneous quintic polynomial of } Z_0, \cdots, Z_4 \}. \]

one can verify that \( \dim V = 126 \). Thus for any \( t \in P(V) = CP^{125} \), \( t \) is represented by a hypersurface. However, if two hypersurfaces differ by an element in \( \text{Aut}(CP^4) \),
then they are considered the same. Let $D$ be the divisor in $CP^{125}$ characterizing the singular hypersurfaces in $CP^4$. Then the moduli space of $X$ is

$$M = CP^{125}\setminus D/Aut(CP^4).$$

The dimension of the moduli space is 101.

An interesting and important question is to compute the volume of the moduli space with respect to the WP metric. The physics background of the question is explained in the paper [4] and the references in that paper.

The observation is that the volume can be computed via the computation of the volume in the Hilbert scheme of $M$, which in this case is the $CP^{125}$. However, the difficulty we have to overcome is the control of the WP metric near the discriminant divisor that characterizing the singular quintic hypersurfaces. Partial result in resolving growth near the discriminant divisor was obtained (by Douglas and L).

On the other hand, in the abstract sense, we have the following general result is in [5]

**Theorem 6.** Let $R_{WP}$ be the curvature tensor of $\omega_{WP}$. Let $R = R_{WP} \otimes 1 + 1 \otimes \omega_{WP}$. Let $f$ be any invariant polynomial of $R$ with rational coefficients. Then we have

$$\int_M f(R) \in \mathbb{Q}.$$

The above theorem is equivalent to the following: let $f_1, \cdots, f_s$ be invariant polynomials of $R_{WP}$ of degree $k_1, \cdots, k_s$, respectively, with rational coefficients. Then

$$\int_M \sum_i f_i(R_{WP}) \wedge \omega_{WP}^{m-k_i} \in \mathbb{Q},$$

where $m$ is the complex dimension of $M$.

One can find applications of the above result in [6, 7, 8]. Note that some special cases of the above result was proved in [18, 19, 4]. In particular, in [18, Theorem 5.2], we proved

**Theorem 7.** The volumes of the WP metric and the Hodge metric are finite.

One can generalize the above result to prove that all the 2, 4, \cdots, $2m$ volumes of the moduli space are finite.

We can also get some Chern number inequalities on the moduli space. Let

$$c_{k_1, \cdots, k_r} = \int_M \bigwedge_{k=1}^r c_k(\omega_{WP}),$$

where $\sum k_j = m$, and $c_k(\omega_{WP})$ is the Chern-Weil form of the curvature tensor of the Weil-Petersson metric. Let

$$\tilde{c}_j = \int_M \text{Ric}(\omega_{WP})^j \wedge \omega_{WP}^{m-j}.$$

Then we have [5],

$$\int_M \sum \tilde{c}_j \in \mathbb{Q},$$
Theorem 8. Using the above notations, we have
\[ c_{k_1, \ldots, k_r} \leq \sum_{j=0}^{m} \frac{2^m m!}{j!(m-j)!} (m+3)^{m-j} \tilde{c}_j. \]

Let’s now turn to the question of global completeness (or incompleteness) of the Weil-Petersson metric. In [15].

Theorem 9. Let \( m = \dim \mathcal{M} \). Assume that the Weil-Petersson metric is complete. Then there is a constant \( C_1(m,n) \), depending only on \( m, n \), such that
\[ |\nabla^m F|^2 \leq C_1(m,n), \]
for any nonnegative integer \( m \).

In fact, we have proved that
\[ |F| \leq \sqrt{\frac{m(m+3)}{3}}, \]
if the WP metric is complete. Thus, in order to check the completeness, we just need to find a point at which the norm of the Yukawa coupling is bigger than \( \sqrt{\frac{m(m+3)}{3}} \).

For all known examples, the CY moduli are WP incomplete. However, mathematically prove the incompleteness of the metric is very important because it is related to the following

Conjecture 2. Let \( X \) be a (non-rigid) Calabi-Yau threefold. Then it can be degenerated to a singular Calabi-Yau variety with one ODP.

To see the relations between the above conjecture to the incompleteness of the WP metric, we recall the following fact (from the private communication with C.L. Wang\(^1\))

Theorem 10 (Wang). Let \( \mathcal{X} \to \Delta^* \) be a one-parameter family of smooth Calabi-Yau manifold. Suppose that the center fiber is a variety with isolated terminal singularities. Then the WP metric, restricted to \( \Delta^* \), is incomplete.

By the above result, we know that Conjecture 2 implies the incompleteness of the WP metric. Thus proving the incompleteness will be a convincing evidence that Conjecture 2 is true.

5. Special Kähler manifold

All the methods and tools used in the previous sections apply to the case of special Kähler manifolds, defined by Freed [11]. In [13], using the same technique as in the proof of Theorem 1, Theorem 9, we proved

---

\(^1\)We are also informed by him that finding the necessary and sufficient conditions of the WP completeness on a one-parameter family is related to the minimal model conjecture.
**Theorem 11.** Let $M$ be a special Kähler manifold. If $M$ is complete with respect to the special Kähler metric, then $M$ is totally flat.

In the other words, a non-trivial special Kähler manifold is not complete. The similar problem in projective special Kähler manifold is discussed in the previous sections.

Using the same method as in Theorem 6, we have the following:

**Theorem 12.** Let $M$ be a quasi-projective special Kähler manifold and let $R$ be the curvature tensor of the manifold. Let $f$ be invariant polynomial of the curvature tensor. Then

$$\int_M f(R) < +\infty.$$ 

Of course, hinted by Theorem 6, we can expect the above integration is even a rational number (if the coefficients of $f$ are rational). However, at this moment, the assumption of $M$ being quasi-projective is quite artificial and we want to solve this problem first before going to the rationality of the integral (note that any CY moduli is quasi-projective by a theorem of Viehweg [22]).

We end this talk by giving the following table:

<table>
<thead>
<tr>
<th>Completeness</th>
<th>Special Kähler manifold</th>
<th>Proj Special Kähler manifold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
<td>$Unknown$ (may be infinite)</td>
<td>finite</td>
</tr>
</tbody>
</table>

The methods in proving the incompleteness of special Kähler manifold and in proving the volume finiteness in the projective special Kähler manifold are similar. Is there any duality there?

**References**


E-mail address, Zhiqin Lu, Department of Mathematics, UC Irvine, Irvine, CA 92697, USA: zlu@math.uci.edu