Erdös-Mordell inequality and beyond

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In 1935, the following problem proposal appeared in the "Advanced Problems" section of the American Mathematical Monthly:

Advanced section, Problem 3740
American Math. Monthly, 42, 1935.

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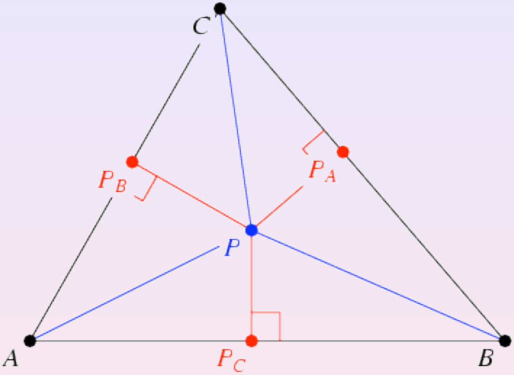


From a point P inside a given triangle ABC the perpendiculars PP_A, PP_B, PP_C are drawn to its sides. Prove that

$$PA + PB + PC \geq 2(PP_A + PP_B + PP_C).$$

From *MathWorld*

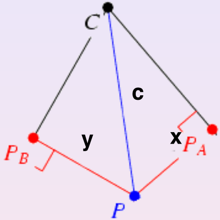
This inequality was proposed by Erdös (1935), and solved by Mordell and Barrow (1937) two years later. Elementary proofs were subsequently found by Kazarinoff in 1945 (Kazarinoff 1962, p. 78) and Bankoff (1958).
Oppenheim (1961) and Mordell (1962) also showed that

$$PA \times PB \times PC \geq (PP_B + PP_C)(PP_C + PP_A)(PP_A + PP_B).$$

(These results are all in *Amer. Math. Monthly*.)
There are many different proofs, simplifications, and generalizations of the result.

Proof given by Mordell, 1937

The Mordell's proof was given two years later and was regarded as a "simple proof". However, a good knowledge in trigonometry and algebra is needed.



We let $\angle C = \gamma$, $\angle A = \alpha$, and $\angle B = \beta$. Law of sine:

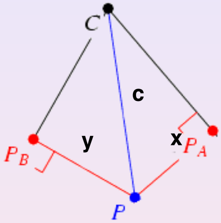
$$P_AP_B = c\sin\gamma.$$

Law of cosine

$$P_AP_B^2 = y^2 + x^2 - 2yx\cos(\pi - \gamma).$$

Thus we have

$$c^2\sin^2\gamma = y^2 + x^2 + 2yx\cos\gamma.$$



We let $\angle C = \gamma$, $\angle A = \alpha$, and $\angle B = \beta$. Law of sine:

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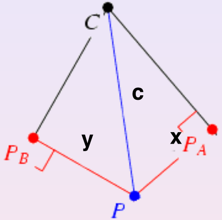

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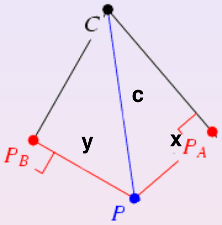

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Lagrange's method of complete square

Since

$$\alpha + \beta + \gamma = \pi,$$

we have

$$\cos\gamma = -\cos(\alpha+\beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Then we have (The Lagrange's complete square method)

$$y^2 + x^2 + 2yx\cos\gamma = (y\cos\alpha - x\cos\beta)^2 + (y\sin\alpha + x\sin\beta)^2$$

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Remeber, by the laws of sine and cosine, we have

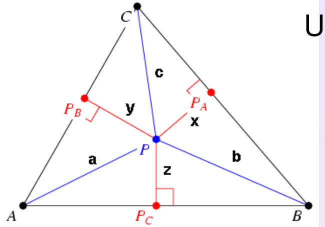
$$c^2 \sin^2 \gamma = (y \cos \alpha - x \cos \beta)^2 + (y \sin \alpha + x \sin \beta)^2.$$

Thus we have

$$c \sin \gamma \geq y \sin \alpha + x \sin \beta.$$

Or, in a more symmetric way

$$c \geq y \frac{\sin \alpha}{\sin \gamma} + x \frac{\sin \beta}{\sin \gamma}$$



Using the same method, we have

$$b \geq z\frac{\sin\alpha}{\sin\beta} + x\frac{\sin\gamma}{\sin\beta}$$

$$a \geq y\frac{\sin\gamma}{\sin\alpha} + z\frac{\sin\beta}{\sin\alpha}$$

togethe with

$$c \geq y\frac{\sin\alpha}{\sin\gamma} + x\frac{\sin\beta}{\sin\gamma}$$

we get

$$a + b + c \geq 2(x + y + z),$$

as desired.

In summary, we have used the following tools:

- The laws of sine and cosine;
- The trigonometric addition formula;
- The Lagrange's complete square;
- Most imporantly, the fact $a^2 + b^2 \geq 2ab$.

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To repeat, let a, b be real numbers. Then we have

$$a^2 + b^2 \geq 2ab.$$

However, for three real numbers, the inequality

$$a^2 + b^2 + c^2 \geq 2(ab + bc + ca)$$

is not correct. The correct inequality is

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

which can be proved by using

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0.$$

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We can prove that the above inequality is optimal in the sense that if

$$a^2 + b^2 + c^2 \geq s(ab + bc + ca)$$

is true for any s, then $s \leq 1$. For details, see

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An optimal inequality
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Lagrange's method revisited

We have the following algebraic result:

Theorem

Let $\alpha+\beta+\gamma=\pi$. *Then*

$$a+b+c \geq 2\sqrt{bc}\cos\alpha + 2\sqrt{ca}\cos\beta + 2\sqrt{ab}\cos\gamma$$

In particular, if $\alpha=\beta=\gamma=\pi/3$, we get the result we knew before

$$a+b+c \geq \sqrt{bc}+\sqrt{ca}+\sqrt{ab}.$$

The one line proof

As before, we have

$$\cos\gamma = -\cos(\alpha+\beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Using the Lagrange's complete square, we have

$$a+b+c-2\sqrt{bc}\cos\alpha-2\sqrt{ca}\cos\beta-2\sqrt{ab}\cos\gamma$$
$$=(\sqrt{c}-\sqrt{a}\cos\beta-\sqrt{b}\cos\alpha)^2+(\sqrt{a}\sin\beta-\sqrt{b}\sin\alpha)^2\geq 0.$$

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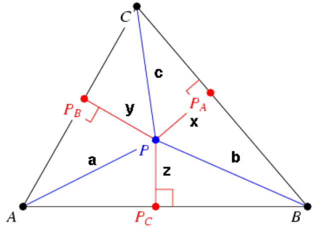

Let $\angle BPC = \alpha$, $\angle CPA = \beta$, and $\angle APB = \gamma$. Then we have

$$\alpha + \beta + \gamma = 2\pi$$

We can prove that

$$z \le \sqrt{ab}\cos(\gamma/2)$$
$$x \le \sqrt{bc}\cos(\alpha/2)$$
$$y \le \sqrt{ca}\cos(\beta/2)$$

Note that $\alpha/2 + \beta/2 + \gamma/2 = \pi$, the inequality $a + b + c \ge 2(x + y + z)$ follows.

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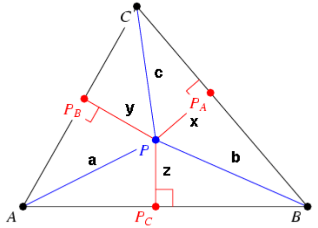

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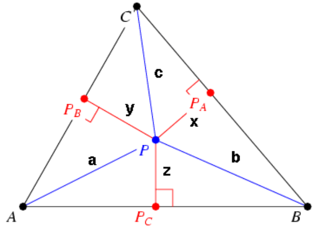

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We rewrite the above theorem

$$a+b+c \geq 2\sqrt{bc}\cos\alpha + 2\sqrt{ca}\cos\beta + 2\sqrt{ab}\cos\gamma$$

into the following form

$$a^2+b^2+c^2-2bc\cos\alpha-2ca\cos\beta-2ab\cos\gamma \geq 0$$

Or

$$(a+b+c)^2 \geq 2ab(1+\cos\gamma)+2bc(1+\cos\alpha)+2ca(1+\cos\beta).$$

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$$(a+b+c)^2 \geq 2ab(1+\cos\gamma) + 2bc(1+\cos\alpha) + 2ca(1+\cos\beta).$$

By the double angle formula, we have

$$(a+b+c)^2 \geq 4ab\cos^2\frac{\gamma}{2} + 4bc\cos^2\frac{\alpha}{2} + 4ca\cos^2\frac{\beta}{2}$$

which can be re-written as

$$(a^2+b^2+c^2)^2 \geq 4a^2b^2\cos^2\frac{\gamma}{2} + 4b^2c^2\cos^2\frac{\alpha}{2} + 4c^2a^2\cos^2\frac{\beta}{2}$$

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The key observation

We let $\vec{A}, \vec{B}, \vec{C}$ be vectors in R^3 with length a, b, c, respectively. Assume that the angle between $\vec{A}, \vec{B}, \vec{C}$ are $\pi/2+\alpha/2, \pi/2+\beta/2, \pi/2+\gamma/2$, respectivley. Then we have

$$(|\vec{A}|^2+|\vec{B}|^2+|\vec{C}|^2)^2 \geq 4(|\vec{A}\times\vec{B}|^2+|\vec{B}\times\vec{C}|^2+|\vec{C}\times\vec{A}|^2)$$

Review of the cross product

Let $\vec{A}, \vec{B}$ be two vectors. By definition $\vec{A} \times \vec{B}$ is a vector. Let the components of $\vec{A}, \vec{B}$ be (x_1, x_2, x_3) and (y_1, y_2, y_3), respectively. Then

$$\vec{A} \times \vec{B} = \vec{C},$$

where

$$\vec{C} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1).$$

We also have

$$|\vec{C}| = |\vec{A}| \cdot |\vec{B}| \sin\alpha,$$

where α is the angle between $\vec{A}$ and $\vec{B}$.

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The definition of the cross product is somewhat mysterious. We can make the following matrix interpretation. Recall that we can define the product of two $n \times n$ matrices. For example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$$

In general, for two matrices, we have $AB \neq BA$. Thus we define the commutator of the matrices as

$$[A, B] = AB - BA$$

which measures the non-commutativity of the matrices.

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An matrix definition of the cross product

We make the following identification:

$$(x_1, x_2, x_3) \longrightarrow \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

$$(y_1, y_2, y_3) \longrightarrow \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}$$

Then we have

$$\vec{A} \times \vec{B} \to \left[\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \right]$$

In mathematics, this phenomena is called isomorphism.

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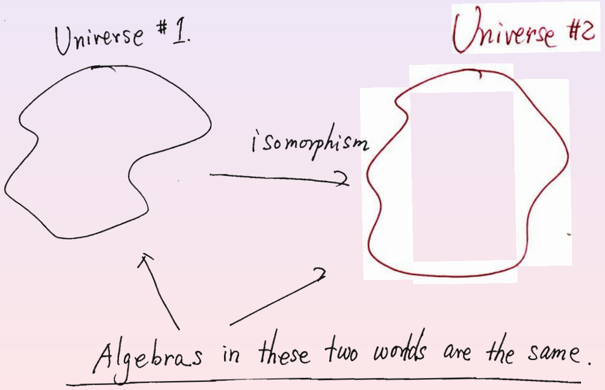
Interpretation of isomorphism



Using the notion of commutator, we have the following result

Theorem

Let A, B, C be 3×3 skew-symmetric matrices with zero diagonal parts. Then we have

$$(||A||^2 + ||B||^2 + ||C||^2)^2 \geq 8(||[A, B]||^2 + ||[B, C]||^2 + ||[C, A]||^2).$$

The above inequality implies the Erdös-Mordell inequality.

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Conjecture (Normal scalar curvature conjecture)

Let $A_1, \cdots, A_m$ be $n \times n$ symmetric matrices. Then we have

$$(\sum ||A_i||^2)^2 \geq 2 \sum_{i<j} ||[A_i, A_j]||^2.$$

Conjecture (Böttcher-Wenzel Conjecture)

Let A, B be two $n \times n$ matrices. Then

$$2||[A, B]||^2 \leq (||A||^2 + ||B||^2)^2.$$

In summer of 2007, I proved both conjectures.

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Conjecture (Zhiqin Lu)

Let A, B be the bounded trace class operators in a separable Hilbert space. Then we have

$$2||[A, B]||^2 \leq (||A||^2 + ||B||^2)^2,$$

where the normal is defined as

$$||A|| = \sqrt{Tr(A^* A)}.$$

Thank you!

Q.E.D.

(*quod erat demonstrandum*)