Nonlinear Analysis in Differential Geometry

UCI Graduate Seminar

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1 Introduction

Four years ago, I gave a talk in the graduate student colloquium here. The main point of that talk is that differential geometry is just Calculus on manifolds. It is a kind of Calculus that takes the underlying geometry or topology of the manifolds into accounts.

The main objects of differential geometry are manifolds. Unfortunately, other than two dimensional manifolds, we can't visualize manifolds. Thus, how to study manifolds.

Using Algebra, we have algebraic geometry. In geometric analysis, we use PDE to study manifolds, to detect their properties.

Successful stories:

1. Calabi-Yau Theorem was proved by Yau in 1976, by solving a complex Monge-Amperé equation.
2. Positive mass theorem solved by Yau and Scheon by using regularity theorem of Minimal Surfaces (1979).
3. Frankel Conjecture solved by Siu and Yau using harmonic maps, which is a system of elliptic equations (1982).
4. Poincaré Conjecture solved by Hamilton-Perelman using nonlinear parabolic equations (1982-2002).

2 Arzelá-Ascoli theorem

One of the most useful theorem in PDE is the Arezela-Ascoli theorem.

Theorem. *Let $\{f_n\}$ be a sequence of continuous functions on $\Omega \subset \mathbb{R}^n$ (Ω is a bounded domain with smooth boundary). If*

①. *$\{f_n\}$ is totally bounded.*

②. *$\{f_n\}$ is equi-continuous in the sense that $\forall\varepsilon, \exists\delta$ such that $\forall n, x, y$ with $d(x,y) < \delta$, we have*

$$|f_n(x) - f_n(y)| < \varepsilon$$

Then there is a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k} \to f$$

uniformly.

There are several important corollary of the above theorem.

Corollary 1. *If $|f_n| + |\nabla f_n| \leqslant C$. Then there is a subsequence such that $\{f_{n_k}\}$ convergence uniformly.*

Corollary 2. *If $\sum_{k \leqslant N} |\nabla^k f_n| \leqslant C(N)$. Then there is a subsequence such that $\{f_{n_k}\}$ convergence in the C^{k-1} norm.*

The proof of the Arzelá-Ascoli theorem is straightforward, and will be omitted.

3 A priori estimates

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. We consider the Laplacian operator Δ; where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Consider the following Dirichlet problem

$$\begin{cases} \Delta u &= f \\ u|_{\partial\Omega} &= \psi \end{cases}$$

where f and ψ are known functions.

It is well known that there is a constant C such that

$$|u|_{k+2,\alpha} \leqslant C(|f|_{k,\alpha} + |\psi|_{k+2,\alpha} + |u|_0)$$

where $\alpha > 0$.

Generalizing the above estimate, we have the Schauder estimate:
Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, let

$$\begin{cases} -a_{ij}D_{ij}u + b_iD_iu + Cu = f & \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

such that

$$\lambda|\zeta|^2 \leqslant a_{ij}\zeta_i\zeta_j \leqslant \wedge|\zeta|^2$$

$$\frac{1}{\lambda}\{\sum_{ij}|a_{ij}|_\alpha + \sum_i |b_i|_{\alpha,\Omega} + |C|_{\alpha,\Omega}\} \leqslant \wedge_\alpha$$

$$C \geqslant 0$$

Then if $f \in C^\alpha$ and $\varphi \in C^{2,\alpha}$, the Dirichlet problem has unique solution.

Proof. *We have the same estimates*

$$|u|_{2,\alpha} \leqslant C(|f|_\alpha + |\varphi|_{2,\alpha})$$

where C depending on λ, $\wedge$ and $\wedge_\alpha$. Let a_{ij}^N, b_i^N, C^N be smooth functions such that

$$a_{ij}^N \to a_{ij}, b_i^N \to b_i, C^N \to C$$

Then for each N, u_N exists.

By the Schauder estimates,

$$|u_N| \leqslant C(|f_N|_\alpha + |\varphi_N|_{2,\alpha})$$

where C is independent of N. By using Arzelá-Ascoli theorem, u_N converges to u in C^2 and using Schauder estimate again, $u \in C^{2,\alpha}$.

4 Nonlinear problem.

In the Schauder estimates, we need the coefficients to be at least continuous. Thus it is impossible to use them in nonlinear problems, where in geometry, almost all equations are nonlinear. In 1957 De Giorgi got the Hölder estimates where the coefficients are bounded measurable, in 1958, Nash independently got the estimate, in 1960, Moser simplified the proof and got the Harnark inequality. As an application in 1976, Yau used the Moser iteration and got the C^0 estimate of the complex Monge-Amperé equation.

The theorem of De Giorgi-Nash-Moser, in its simplest form, can be stated as follows:

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider the equation

$$\begin{cases} -D_j(a_{ij}(x)D_i u) = 0 & \\ u \geqslant 0 & \text{on} \quad \Omega \end{cases}$$

where

$$\lambda|\zeta|^2 \leqslant a_{ij}\zeta_i\zeta_j \leqslant \wedge|\zeta|^2$$

$$\sum_{ij} \|a_{ij}\|_{L^\infty} \leqslant \wedge$$

Then $u \in C^\alpha(\Omega)$.

Remark. *The above equation is called of divergence form, because of the following reason: let*

$$\mathscr{L} = -D_j(a_{ij}D_i)$$

Then

$$\langle \mathscr{L}f, g\rangle = \langle f, \mathscr{L}g\rangle.$$

for smooth functions f and g. If

$$\mathscr{L}' = -a_{ij}D_iD_j$$

Then $\mathscr{L}'$ is of non-divergence form. 1983, Krylov-Safonov got the Hölder estimates for elliptic equation of non-divergence form. 1984-1986, Caffarelli-Nirenberg-Kohn-Spruck got the same results. Let's sketch the Moser Iteration.

Consider

$$-u^pD_j(a_{ij}D_ju) \geqslant 0$$

Let ξ be a cut-off function

$$-\zeta^2u^pD_j(a_{ij}D_iu) \geqslant 0$$

Using Integration by parts, we have

$$\int_{B_R} \zeta^2|Dv^{\frac{p}{2}}|^2dx \leqslant C\int_{B_R} |D\zeta|^2v^pdx$$

Thus we have

$$\int_{B_R} |D(\zeta v^{\frac{p}{2}})|^2dx \leqslant C\int_{B_R} |D\zeta|^2v^pdx$$

Using Sobolev embedding theorem, we get

$$\left(\int_{B_R} (\zeta v^{\frac{p}{2}})^{2^*}dx\right)^{\frac{2}{2^*}} \leqslant C\int_{B_R} |D\zeta|^2v^pdx$$

where $2^* = \frac{2n}{n-2}$. Define

$$R_k = R(\theta + \frac{1-\theta}{2^k}), k = 0, 1, 2, \cdots$$

where $\theta \in (0,1)$. Let $\zeta_k \in C_0^\infty(B_{R_k})$, $0 \leqslant \zeta_k \leqslant 1$ and on B_{k+1}, $\zeta_k(x) \equiv 1$. Then

$$|D\zeta_k| \leqslant \frac{2}{R_k - R_{k+1}} = \frac{2^{k+1}}{(1-\theta)R}$$

Thus we have

$$\left(\int_{B_{R_{k+1}}} v^{\frac{np}{n-p}}dx\right)^{\frac{n-2}{n}} \leqslant \frac{C4^k}{(1-\theta)^2R^2}\int_{B_{R_k}} v^pdx$$

Let

$$P_k = p(\frac{n}{n-2})^k$$

Then

$$\|v\|_{L^{P_{k+1}}(B_{k+1})} \leqslant \left(\frac{C4^k}{(1-\theta)^2R^2}\right)^{\frac{1}{P_k}} \|v\|_{L^{P_k}(B_{R_k})}$$

Thus

$$\|v\|_{L^{P_{k+1}}(B_{k+1})} \leqslant \left(\frac{C}{(1-\theta)^2R^2}\right)^{\sum\frac{1}{P_k}} 4^{\sum\frac{k}{P_k}} \|v\|_{L^p(B_R)}$$

Let $k \to \infty$, we get

$$\text{ess} \sup_{B_{\theta R}} v \leqslant \frac{C}{((1-\theta)R)^{n/p}} \|v\|_{L^p(B_R)}$$

Using the same method, we can get

Weak Harnark Inequality: there is a $p_0 > 0$ such that

$$\inf_{B(\theta R)} u \geqslant \frac{1}{C}(\int_{B(R)} v^{p_0})^{\frac{1}{p_0}}$$

Combine the two inequality, we have

$$\sup_{B_R} u \leqslant C \inf_{B_R} u$$

if $u \geqslant 0$ on $B(2R)$. (Harnark Inequality)

We can prove that the Harnark Inequality implies Hölder continuity. To this purpose, let

$$M(R) = \sup_{B_R} u, m(R) = \inf_{B_R} u$$

Then

$$M(\frac{1}{2}R) - mR \leqslant C(m(\frac{1}{2}R) - m(R))$$

Thus

$$w(\frac{1}{2}R) = (1 - \frac{1}{C})w(R)$$

Iteration

$$w(\frac{R}{2^k}) \leqslant (1 - \frac{1}{C})^k w(R)$$

Let $x, y \in \Omega$. Let

$$\frac{R}{2^k} \leqslant |x-y| \leqslant \frac{R}{2^k - 1}$$

Thus

$$|u(x) - u(y)| \leqslant (1 - \frac{1}{C})^{k-1} w(R)$$

if

$$|x-y| \geqslant \frac{R}{2^k}.$$

The Hölder continuity is proved.

□