Hearing the size of a triangle

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Fourier expansion

We assume that $f(x)$ is a smooth period function on $[0, 2\pi]$. Then we have the well-known Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos\ kx + b_k \sin kx)$$

Good thing of the Fourier expansion

1. Split a function into simplier pieces;
2. Geometry interpretation;
3. Connecting differential operators to linear algebra=functional analysis.

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Geometric intepretation

Pythagorean theorem or Parseval equality (Bessel inequality)

$$a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{\pi}\int_0^{2\pi} f(x)^2 dx$$

Euclidean geometry on the space spanned by $\{1, \cos kx, \sin kx\}$.

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We consider the differential operator

$$\Delta = \frac{\partial^2}{\partial t^2}$$

We have

$$\Delta\, 1 = 0$$
$$\Delta \cos kx = -k^2 \cos kx$$
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Under the basis $\{1, \cos kx, \sin kx\}$, Δ is an infinite dimensional matrix

$$\Delta = \begin{pmatrix} 0 &&&&&&\\ & -1 &&&&&\\ && -1 &&&&\\ &&& -4 &&&\\ &&&& -4 &&\\ &&&&& \ddots &\\ &&&&&& \ddots \end{pmatrix}$$

Consider Δ as a differential operator AND an linear operator. Warning: Infinite dimensional space is quiet different from finite dimensional space. If we don't have *estimate*, strange things could happen.

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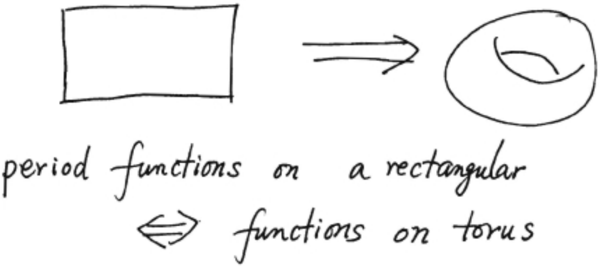
How to work on high dimensions?

Let Ω be a bounded domain in R^n with smooth boundary. What do we mean by period functions on Ω?



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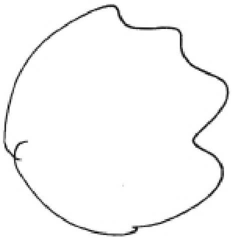

general domain

Boundary conditions!

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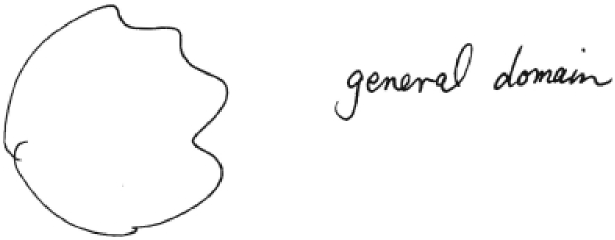

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Tow kinds of boundary conditions:

1. Dirichlet boundary condition

$$f|_{\partial\Omega} = 0$$

2. Neumann boundary condition

$$\frac{\partial f}{\partial n}|_{\partial\Omega} = 0$$

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Let the Laplace operator be defined

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We need to make the operator "symmetric", or self-adjoint. Let f, g be functions vanished on $\partial\Omega$. Then by the second Green's theorem, we have

$$\int_\Omega f\,\Delta g = \int_\Omega \Delta f\, g.$$

That is, in the abstract notation, we have

$$(\Delta f, g) = (f, \Delta g).$$

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If a function f is not smooth, then Δf may not make sense.

We have to be happy (or familiar with?) a linear operator that only defined on a subspace (densely defined).

Question: Δ is an operator defined on $C^\infty(\Omega)$, which is dense in $L^2(\Omega)$. Is it possible to extend the operator to the whole space?

Answer: not possible. Because Δ is a closed graph operator. If possible to extend, then the operator is in fact bounded. (closed graph theorem)

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Definition

A non-zero function f is called an eigenfunction, if there is a real number λ, such that

$$\Delta f = -\lambda f.$$

Fact: There is a basis of $L^2(\Omega)$ made from eigenfunctions!

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Under the above basis, we get

$$\Delta = \begin{pmatrix} -\lambda_1 & & & \\ & -\lambda_2 & & \\ & & -\lambda_3 & \\ & & & \ddots \end{pmatrix}$$

The inverse of the operator is

$$\Delta^{-1} = \begin{pmatrix} -\lambda_1^{-1} & & & \\ & -\lambda_2^{-1} & & \\ & & -\lambda_3^{-1} & \\ & & & \ddots \end{pmatrix}$$

Thus estimating the eigenvalues from above or from below are very interesting, because they tell the size of the Laplacian operator.

There are a lot of questions in the field that can be solved using the knowledge at the first-year graduate level.

For example, there is a very old theorem of Karen Uhlenbeck, stating that a generic domain in R^2 is *simple*. That is, the Laplacian, as a infinite matrix, whose eigenvalues are all of multiplicity one.

More recently, Hillairet and Judge proved that generic polygons of $n > 3$ are simple:

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They left the following conjecture: Generic triangle is simple.

Using Math 210, we can solve the problem.
(joint with Rowlett)

Theorem

Generic triangle is simple!

Proof. The parameter space (In the terminology of geometry, the moduli space is a 2-d orbifold.) of a triangle of fixed area is a 2-dimensional object.
The set F_m such that the first m eigenvalues are distinct is a dense open set on the parameter space.
Thus by the Baire category theorem,

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is dense and open, thus generic.

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Very famous paper

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ζ function regularization (Math 220 required).

Let

$$\zeta(s) = \sum_{\lambda_i \neq 0} \frac{1}{\lambda^s}$$

Then we can prove that the function is holomorphic, if $Re(s)$ is very negative.

By Math 220, we know that we can extend the function as a meromorphic function on C. In particular, at the original point the function $\zeta(s)$ is holomorphic.

Formally, we have

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Riemann ζ function

Define

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For $Re(z) > 1$, the above series converges absolutely. Thus $\zeta(z)$ defines a holomorphic function on $Re(z) > 1$.

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The same method is true for the ζ function of eigenvalues. We can define

$$\zeta(z) = \sum_{\lambda_i \neq 0} \frac{1}{\lambda_i^z}.$$

By Mirror Symmetry, we know that for a compact CY 3 fold X, there is the mirror pair X' such that the quantum field theories on both manifolds are identical.

We introduce the so-called BCOV conjecture.

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We consider the quintics in CP^4

$$Z_0^5 + \cdots + Z_4^5 - 5\lambda Z_0 \cdots Z_4 = 0.$$

For general complex number λ, it is a smooth hypersurface. In fact, it is a CY 3-fold.
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The BCOV Conjecture is related to the $g = 1$ Mirror symmetry. Based on physics observation, they conjecture a relation between two (presumably unrelated) Calabi-Yau threefolds.

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Definition

The mirror map is the holomorphic map from a neighborhood of $\infty \in \mathbb{P}^1$ to a neighborhood of $0 \in \Delta$ defined by the following formula

$$q := (5\psi)^{-5} \exp\left(\frac{5}{y_0(\psi)} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left\{ \sum_{j=n+1}^{5n} \frac{1}{j} \right\} \frac{1}{(5\psi)^{5n}} \right),$$

where $\quad |\psi| \gg 1$, and

$$y_0(\psi) := \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}, \quad |\psi| > 1.$$

The inverse of the mirror map is denoted by $\psi(q)$.

Define the multi-valued function $F_{1,B}^{\mathrm{top}}(\psi)$ as

$$F_{1,B}^{\mathrm{top}}(\psi) := \left(\frac{\psi}{y_0(\psi)}\right)^{\frac{62}{3}} (\psi^5 - 1)^{-\frac{1}{6}} q\frac{d\psi}{dq},$$

and

$$F_{1,A}^{\mathrm{top}}(q) := F_{1,B}^{\mathrm{top}}(\psi(q)).$$

Conjecture

(A) *Let $n_g(d)$ be the genus-g degree-d instanton number of a quintic in CP^4 for $g = 0, 1$. Then the following identity holds:*

$$-q\frac{d}{dq}\log F_{1,A}^{\mathrm{top}}(q) =$$

$$\frac{50}{12} - \sum_{n,d=1}^{\infty} n_1(d)\,\frac{2nd\,q^{nd}}{1-q^{nd}} - \sum_{d=1}^{\infty} n_0(d)\,\frac{2d\,q^d}{12(1-q^d)}.$$

Conjecture (A) was proved by Aleksey Zinger.

Aleksey Zinger
The Reduced Genus-One Gromov-Witten Invariants of Calabi-Yau Hypersurfaces
ArXiv: 0705.2397v2, 2007.

Setup of Conjecture (B)

Let X be a compact Kähler manifold.

- Let $\Delta = \Delta_{p,q}$ be the Laplacian on (p, q) forms;
- By compactness, the spectrum of Δ are eigenvalues:

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \to +\infty.$$

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- Let $\Delta = \Delta_{p,q}$ be the Laplacian on (p, q) forms;
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$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \to +\infty.$$

Setup of Conjecture B

- Bershadsky-Ceccotti-Ooguri-Vafa defined

$$T \stackrel{\text{def}}{=} \prod_{p,q} (\det \Delta_{p,q})^{(-1)^{p+q} pq}.$$

- Why define such a strange quantity?
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Conjecture

(B) *Let $\|\cdot\|$ be the Hermitian metric on the line bundle*

$$(\pi_* K_{\mathcal{W}/CP^1})^{\otimes 62} \otimes (T(CP^1))^{\otimes 3}|_{CP^1 \setminus \mathcal{D}}$$

induced from the L^2-metric on $\pi_* K_{\mathcal{W}/CP^1}$ and from the Weil-Petersson metric on $T(CP^1)$. Then the following identity holds:

$$\tau_{\mathrm{BCOV}}(W_\psi) = \mathrm{Const.} \left\| \frac{1}{F_{1,B}^{\mathrm{top}}(\psi)^3} \left(\frac{\Omega_\psi}{y_0(\psi)} \right)^{62} \otimes \left(q \frac{d}{dq} \right)^3 \right\|^{\frac{2}{3}},$$

where Ω is the local holomorphic section of the $(3,0)$ forms.

Conjecture B was proved by Fang-L-Yoshikawa.

H. Fang, Z. Lu, and K-I, Yoshikawa
Asymptotic behavior of the BCOV torsion of Calabi-Yau moduli
ArXiv: math/0601411, 2006, Journal of Diff. Geom. 2008.

Combining Conjecture A and B, we verified the Mirror Symmetry prediction to the case $g = 1$. For higher genus, the B-side of the conjectures have not been set up.

On the other hand, the $g = 0$ Mirror Symmetry Conjecture was proved by Lian-Liu-Yau and Givental.

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A life time question: Can we hear the shape of our Universe?

Not known, but joint with physicists Mike Douglas, we proved the number of Universes is finite, if the string theory is true.

Thank you!