§ 4 Eigenvalue Problems (II)

By the variational characterizing of the eigenvalues, we know that it is usually more difficult to get the lower bound estimate of eigenvalues. Among all the eigenvalues, the lower bound of the first eigenvalue is particularly important.

The Cheeger's result did give a lower bound estimate of the first eigenvalues. But the bounds are not "computable". In geometry, "computable" bounds provide effective versions of Poincaré and Sobolev inequalities.

The following Lichnerowicz theorem gives a good lower bound of the first eigenvalue of the first for closed manifold.

Theorem (Lichnerowicz) Let $M$ be a closed $n$-dimensional Riemannian manifold. Assume that

$$\text{Ric}(M) \geq (n-1)k > 0$$

Then $\lambda_1 \geq nk$

Proof: One line proof: let $u$ be the first eigenfunction. Then use the Ricci identity we have
\[ \frac{1}{2} \Delta |\nabla u|^2 \geq \sum u_i^2 + \nabla u \nabla u + \text{Ric}(\nabla u, \nabla u) \]

We have
\[ \sum u_i^2 = \sum u_{ii}^2 = \frac{1}{n} (\sum u_{ii})^2 = \frac{A_1}{n} u^2 \]
\[ \text{Ric}(\nabla u, \nabla u) \geq (n-1) k |\nabla u|^2 \]

Thus we have
\[ \frac{1}{2} \Delta |\nabla u|^2 \geq \frac{A_1}{n} u^2 - \lambda_1 |\nabla u|^2 + (n-1) k |\nabla u|^2 \]

Taking integration on both sides, we get
\[ \frac{A_1}{n} - \lambda_1 + (n-1) k \lambda_1 \leq 0 \]

The theorem follows.

In 1962, Otsu proved, if \( \lambda = nk \), then \( M \) has to be the standard sphere.

For the rest of this section, we use the gradient estimates to find "computable" lower bounds of the first eigenvalue.

We prove the following theorem.

Theorem (Li-Yau) Let \( M \) be a closed manifold and \( \text{Ric}(M) \geq 0 \).

Then \( \lambda_1 \geq \kappa^2 / d^2 \), \( d \) is the diameter.
Proof: Let $u$ be the first eigenfunction. After normalization we may assume that
\[
1 = \sup u > \inf u = -k > 1
\]
for some $1 > k > 0$. Let
\[
\tilde{u} = \frac{u - \frac{1-k}{1+k}}{1+k}
\]
The after this linear change of $u$, we have
\[
\begin{cases}
\Delta \tilde{u} = -\lambda_1 (\tilde{u} + a) \\
\sup \tilde{u} = 1 \\
\inf \tilde{u} = -1
\end{cases}
\]
for $a = \frac{1-k}{1+k}$. $1 > a > 0$.

Let $g = \frac{1}{2} \left( |\nabla \tilde{u}|^2 + (\lambda_1 + \epsilon) \tilde{u}^2 \right)$ for some $\epsilon > 0$ to be determined later. Assume that at $x_0$
\[
g(x_0) = \max g
\]
Using the maximum principle, at $x_0$, we have
\[
\tilde{u}_i \tilde{u}_i + (\lambda_1 + \epsilon) \tilde{u} \tilde{u}_i = 0
\]
and
\[
0 \geq \Delta g = \tilde{u}_{ij}^2 + \text{Ric} (\nabla \tilde{u}, \nabla \tilde{u}) + \nabla \tilde{u} \nabla \Delta \tilde{u}
+ (\lambda_1 + \epsilon) \left( |\nabla \tilde{u}|^2 + (\lambda_1 + \epsilon) \tilde{u} \Delta \tilde{u} \right)
\geq \tilde{u}_{ij}^2 - \lambda_1 (|\nabla \tilde{u}|^2 + (\lambda_1 + \epsilon) |\nabla \tilde{u}|^2)
\geq -\lambda_1 (\lambda_1 + \epsilon) \tilde{u} (\tilde{u} + a)
\]
If at $x_0$, $\nabla \tilde{u} = 0$. Then we have
\[
(|\nabla \tilde{u}|^2 + (\lambda_1 + \epsilon) \tilde{u}^2) \leq (\lambda_1 + \epsilon)
\]
In particular, we have
\[(\nabla \tilde{w})^2 + \lambda_i (1 + \varepsilon) \tilde{w}^2 \leq \lambda_i (1 + \varepsilon)\]

if \( \nabla \tilde{w} (x_0) \neq 0 \). Then using the Cauchy inequality
\[
\nabla \tilde{w}^2 \geq \frac{(\nabla \tilde{w} \cdot \nabla \tilde{w})^2}{\| \nabla \tilde{w} \|^2} = (\lambda_i + \varepsilon)^2 \tilde{w}^2
\]

Thus we have
\[
0 \geq \Delta g (x_0) > (\lambda_i + \varepsilon)^2 \tilde{w}^2 + \varepsilon \| \nabla \tilde{w} \|^2 - \lambda_i (\lambda_i + \varepsilon) \tilde{w}^2 - \lambda_i (\lambda_i + \varepsilon) \quad \text{if we let} \quad \varepsilon \rightarrow \lambda_i a \quad \text{then we have} \quad \Phi \geq 2 \varepsilon g - \lambda_i (\lambda_i + \varepsilon) a
\]

For any \( \varepsilon > \lambda_i a \), the above gives
\[
\| \nabla \tilde{w} \|^2 + \lambda_i (1 + \varepsilon) \tilde{w}^2 \leq \lambda_i (1 + \varepsilon)
\]

Let
\[
f (t) = \arcsin \tilde{w} (o(t))
\]

where \( o(t) \) is the arc-length curve connecting the minimal point and the maximum point of \( \tilde{w} \). Then by the above argument, we have
\[
|f' (t)| \leq \sqrt{\lambda_i (1 + a)}
\]

Let \( d \) be the diameter of the manifold, then we have
\[
d \sqrt{\lambda_i (1 + a)} \geq \int_0^d |f' (t)| \, dt \geq \arcsin 1 - \arcsin (-1) = \pi
\]
Thus
\[ \lambda_1 \geq \frac{1}{1+a} \frac{\pi^2}{d^2} \]

Since \( a < 1 \), this gives
\[ \lambda_1 \geq \frac{\pi^2}{2d^2} \]

(\( H \))

We let \( \theta = \text{arc} \sin \tilde{u} \). Then Zhong-Yau proved the following surprising theorem:

Theorem (Zhong-Yau) Let
\[ \Psi(\theta) = \frac{4}{\pi} \left( \theta + \cos \theta \sin \theta \right) - 2 \sin \theta \]

Then
\[ \frac{\sqrt{\tilde{u}^2}}{1-\tilde{u}^2} \leq \lambda_1 \left( 1 + a \Psi(\theta) \right) \]

The method is maximal principle, very surprising and mysterious.

Using the above sharpened inequality, observed that \( \Psi(\theta) \) is an odd function, we can prove that
\[ \lambda_1 \geq \frac{\pi^2}{d^2} \]

Assume that \( \text{Ric}(M) \geq -(n-1)k \) for \( k > 0 \)

Then by estimating \( (\partial u)^2 + \lambda_1 (\theta - u)^2 \), Li-Yau
was above to prove that
\[ \lambda_1 \geq \frac{c}{d^2} \exp \left( -C_1 \sqrt{kd^2} \right) \]

Yang was able to modified the above and proved that
\[ \lambda_1 \geq \frac{\pi^2}{d^2} \exp \left( -C \sqrt{kd^2} \right) \]

When \( \text{Ric}(M) > 0 \), or \( \text{Ric}(M) \geq (n-1)k > 0 \), the above inequality is not optimal. In fact, it is far from being optimal. Let \( M = S^n \). Then by Lichnerowicz theorem, \( \lambda_1 \geq n \) (in fact, \( \lambda_1 = n \)). \( d(S^n) = \pi \). Thus, Yang gives
\[ \lambda_1 \geq 1 \]

In this direction, we have the following Peter Li Conjecture

Conjecture (P. Li) If \( \text{Ric}(M) \geq (n-1)k \). Then
\[ \lambda_1 \geq \frac{\pi^2}{d^2} + (n-1)k \]

Such a conjecture, if true, will sharpen both the result of Zhong-Yang and Lichnerowicz because by Myer's theorem
\[ \frac{\pi^2}{d^2} \geq k \]
Not much was known to the proof of the conjecture. D. Yang proved that
\[ \lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{4} (n-1)k \]
LingJun proved a bigger number.
\[ \lambda_1 \geq \frac{\pi^2}{d^2} + \alpha (n-1)k \quad \alpha > \frac{1}{4} \]

On the other end, if \( \text{Ric}(M) \geq -(n-1)k \), \( k > 0 \). Then
\[ \lambda_1 \geq \frac{\pi^2}{d^2} \text{ } \Box \text{ } (n-1)k \]

Recently, 王晓东 - 林凤波 was able to prove that
\[ \lambda_1 > \frac{\pi^2}{d^2} \]

Very interesting result.

We end this section by citing a result of Li and Croke.

**Theorem** Let \( M \) be a \( \mathbb{R} \) manifold with boundary. Then there is a constant \( c = (n, d, V, k) > 0 \) such that the Sobolev constant is \( > c > 0 \).