A complex geometric proof of Tian-Yau-Zelditch expansion

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October 21, 2010
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Let \((M, L)\) be a polarized Kähler manifold. Let \(h\) be a Hermitian metric on \(L\) such that \(c_1(L) = \omega\), the Kähler metric. Let \(E\) be a Hermitian vector bundle.
$H^0(M, L^m \otimes E)$ and $H^0(M, L^m)$ are Hermitian inner product spaces with respect to the $L^2$ metrics.
When $L$ is ample, we know that $L^m$ is very ample for $m$ big enough.
When $L$ is ample, we know that $L^m$ is very ample for $m$ big enough. Let $S_0, \cdots, S_d$ be an orthonormal basis of $H^0(M, L^m)$. Then the map

$$x \mapsto [S_0, \cdots, S_d]$$

gives an embedding $f_m$ of $M$ to $\mathbb{C}P^d$. The metric

$$\frac{1}{m} f_m^* (\omega_{FS})$$

is called the Bergman metric of $M$. 
Theorem (Tian)

\[ \frac{1}{m} f_m^*(\omega_{FS}) - \omega \to 0 \]

as \( m \to \infty \).
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\]

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(In fact, Tian proved the convergence rate to be \( m^{-\frac{1}{2}} \), and Ruan improved it to \( 1/m \).)
Let \( \{ S_1^m, \cdots, S_{d_m}^m \} \) be any orthonormal basis of \( H^0(M, L^m) \). Then

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\mathcal{B}(x) = \mathcal{B}_m(x) = \sum ||S_j||^2
\]

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is called the Bergman kernel of the polarized Kähler manifold. Important observation
\[
\frac{1}{m} f_m^*(\omega_{FS}) - \omega = \frac{1}{m} \partial \bar{\partial} \log \mathcal{B}(x).
\]
Theorem (Catlin, Zelditch)

There is an asymptotic expansion:

\[ \mathcal{B} \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \]

for certain smooth coefficients \( a_j(x) \in \text{Hom}(E, E) \) with \( a_0 = I \). More precisely, for any \( k \)

\[ \| \mathcal{B}(x) - \sum_{k=0}^{N} a_j(x)m^{n-k} \|_{C^\mu} \leq C_{N, \mu} m^{n-N-1}, \]

where \( C_{N, \mu} \) depends on \( N, \mu \) and the manifold \( M \) and the bundles \( L, E \).
Result of BBS

Robert Berman and Bo Berndtsson and Johannes Sjöstrand
A direct approach to Bergman Kernel Asymptotics for Positive Line Bundles
*Arkiv Math, 46(2): 197-217, 2008*
In the paper, they proved the existence of the Bergman kernel expansion directly, without using the deep result of Fefferman and Boutet de Monvel-Johannes Sjöstrand on Bergman kernel.
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Result of Ross-Thomas

Julius Ross and Richard Thomas
Weighted Bergman kernels on Orbifolds
ArXiv: 0907.515 v2
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$C^\infty$ convergence on orbifolds.
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3. The expansion when the metrics are real analytic;
4. Orbiford Cases?
5. (less important but more difficult) A recursive formula for all coefficients?
The $C^\infty$ convergence ($C^0$ convergence is proved in my paper “On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch expansion”, AJM 122, 2000.)
Technical Difficulties

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3. $\text{Hom}(E, E)$-valued objects.
Definition of Bergman kernel (vector bundle case)

Let $S_1, \ldots, S_d$ be any basis of $H^0(M, E)$. Let $F = (F_{ij}) = (S_i, S_j)$.

Let $P$ be the matrix such that $PFP^* = I$. If we write $S_i = \sum_{j=1}^r b_{ij} e_j$, then the Bergman kernel can be represented by $B = HB^* F^{-1} B$, where $B = (b_{ij})$ is a $d \times r$ matrix.

$B$ is an $r \times r$ matrix, where $r$ is the rank of $E$. $H$ is the Hermitian metric of $E$. Zhiqin Lu, Dept. Math, UCI A complex geometric proof of TYZ expansion
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$$ \mathcal{B} = HB^* F^{-1} B, $$

where $B = (b_{ij})$ is a $d \times r$ matrix. $\mathcal{B}$ is an $r \times r$ matrix, where $r$ is the rank of $E$. $H$ is the Hermitian metric of $E$. 
Important remark

For any linearly independent sections $S_1, \cdots, S_t$ (subspace of $H^0(M, E)$), Bergman kernels are defined.
Reduction of the problem

For Bergman kernel of \( H^0(M, L^m \otimes E) \)
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1. the $C^\infty$ asymptotic expansion of $B^* F^{-1} B$
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3. the $C^0$ asymptotic expansion at one point ($F^{-1}_{PQ} z^P \bar{z}^Q$).
4. the expansion of the inverse of the metric matrix
5. Effective version of Ruan’s lemma.
Definition

We say a sequence of functions $f_m(x)$ has a $C^\mu$ asymptotic expansion, if there exist matrix-valued functions $a_0(x), \cdots, a_s(x), \cdots$ such that for any $s, \mu,$

$$\left\| f_m(x) - m^n \left( a_0(x) + \frac{a_1(x)}{m} + \cdots + \frac{a_s(x)}{m^s} \right) \right\|_{C^\mu} \leq \frac{C}{m^{s+1}},$$

where $C$ is a constant independent to $m.$
Definition
We say a sequence of functions $f_m(x)$ has a $C^\mu$ asymptotic expansion at the point $x_0$, if there exist matrix-valued functions $a_0(x), \cdots, a_s(x), \cdots$ in a neighborhood of $x_0$ such that for any $s$, we have

$$\left| D^\mu \left( f_m(x) - m^n \left( a_0(x) + \frac{a_1(x)}{m} + \cdots + \frac{a_s(x)}{m^s} \right) \right) \right| (x_0) \leq \frac{C}{m^{s+1}},$$

where $C$ is a constant independent to $m$ and $x_0$. The derivative is taken with respect to a $K$-coordinate system.
Definition

In particular, we say $f_m(x)$ has a $C^0$ asymptotic expansion at the point $x_0$, if there exists matrices $a_0, \cdots, a_s, \cdots$ such that

$$
|f_m(x) - m^n \left(a_0 + \frac{a_1}{m} + \cdots + \frac{a_s}{m^s}\right)| (x_0) \leq \frac{C}{m^{s+1}},
$$

where $C$ is independent to $m$.

If a sequence of functions has a $C^\mu$ asymptotic expansion, then it has the $C^\mu$ asymptotic expansion at any point $x_0$. 

Lemma

A sequence of functions $f_m$ has a $C^\mu$ asymptotic expansion if and only if for every $\mu \geq 0$ and at each point $x_0 \in M$, $f_m$ has a $C^\mu$ asymptotic expansion at $x_0$. 
Definition of $K$-coordinates and $K$-frames

Definition
Let $p > 0$ be any positive integer. Let $x_0 \in M$ be a point. Let $(z_1, \cdots, z_n)$ be a holomorphic coordinate system centered at $x_0$. Let $(g_{\alpha \overline{\beta}})$ be the Kähler metric matrix. If

$$g_{\alpha \overline{\beta}}(x_0) = \delta_{\alpha \beta},$$

$$\frac{\partial^{p_1+\cdots+p_n} g_{\alpha \overline{\beta}}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0$$

for $\alpha, \beta = 1, \cdots, n$ and any nonnegative integers $(p_1, \cdots, p_n)$ with $p > p_1 + \cdots + p_n \neq 0$. Then we call the coordinate system a $K$-coordinate system of order $p$. 
Definition
Let $e_L$ be a local holomorphic frame of $L$ at $x_0$. If for $p > 0$, the local representation function $a$ of the Hermitian metric $h_L$ satisfies

$$a(x_0) = 1, \quad \frac{\partial^{p_1 + \cdots + p_n} a}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0$$ (1)

for any nonnegative integers $(p_1, \cdots, p_n)$ with $p > p_1 + \cdots + p_n \neq 0$. Then we call $e_L$ is a $K$-frame of order $p$. If $a$ is analytic, then again we can take $p = +\infty$. 
Under $K$-coordinates, the coefficients of the Taylor expansions of the metrics only contain the curvatures and their derivatives.
Let $P = (p_1, \cdots, p_n)$ be a multiple index and let $1 \leq j \leq r$. Define the lexicographical order on the set of $(P, j)$’s. That is, $(P, j) < (Q, k)$ if

1. $\sum p_i < \sum q_i$, or
2. $p_1 = q_1, \cdots, p_\ell = q_\ell$ but $p_{\ell+1} < q_{\ell+1}$ for some $0 \leq \ell \leq n - 1$, or
3. $j < k$.

Such an order gives rise to the function $P = P(j)$. For example, $P(1) = ((0, \cdots, 0), 1)$, $P(2r + 2) = ((0, 1, \cdots, 0), 2)$, etc.
Definition

Let $S_1, \cdots, S_k, S_{k+1}, \cdots, S_d$ be a basis of $H^0(M, L^m \otimes E)$. We say that it is a regular basis at $x_0$ of order $\mu$, if under the local $K$-coordinates at $x_0$

1. for $1 \leq j \leq k$, $S_j(z) = z^{P(j)} + o(|z|^\mu)$;
2. for $j > k$, $S_j(z) = o(|z|^\mu)$.

Moreover, the $(i, j)$-th entry of $F^{-1}$ has a $C^0$ asymptotic expansion at $x_0$. 
Lemma

If a regular basis exists, then the Catlin-Zelditch’s result is valid.

Proof. The Taylor expansion for the smooth vector-valued function $H$ gives the asymptotic expansion. Thus in order to prove the result, we only need to prove the existence of the $C^\mu$ expansion of $B^* F^{-1} B$. It is not hard to see that if

$$\frac{\partial |P| + |Q| \mathfrak{B}}{\partial z^P \partial \bar{z}^Q}$$

has the $C^0$ asymptotic expansion at $x_0$ for all $|P| + |Q| \leq \mu$. 

\[\square\]
Peak Sections

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Graphs of \( f_i(x) = x^i e^{-x} \) for \( i = 1, \cdots, 4 \).
Peak Sections

1, $z, z^2, \cdots, z^p$ are the “peak” functions on $\mathbb{C}$ with respect to the norm $e^{-|z|^2}$.

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1, z, z^2, \cdots, z^p are the “peak” functions on \( \mathbb{C} \) with respect to the norm \( e^{-|z|^2} \).

Graphs of \( f_i(x) = x^i e^{-x} \) for \( i = 1, \cdots, 4 \).

\[ x^i e^{-x} \text{ goes to zero but not uniformly.} \]
In $\mathbb{C}^n$, $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ are peak sections for the norm $e^{-|z|^2}$. 
In $\mathbb{C}^n$, $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ are peak sections for the norm $e^{-|z|^2}$.

On the polarized manifold, the fact that $c_1(L) = \omega$ implies that in a neighborhood of $x_0$, the metric is close to $e^{-|z|^2}$.

$$(\partial \bar{\partial} \log e^{-|z|^2} = \sum_j dz_j \wedge d\bar{z}_j)$$
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The metric on $L^m$ is close to $e^{-m|z|^2}$. After rescaling

$$\left\{ |z| \leq \frac{\log m}{\sqrt{m}} \right\}, e^{-m|z|^2} \leftrightarrow \left\{ |z| \leq \log m \right\}, e^{-|z|^2}.$$
Theorem (Peak Section Theorem)

We can find holomorphic sections $S \in H^0(M, L^m)$ such that

1. in a neighborhood of $x_0$, $S$ is close to $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$
2. Outside the neighborhood, $S$ is very small.

Here $\sum \alpha_j < \varepsilon \log m$. 
Theorem

Let $S_1, \ldots, S_k$ be peak sections $k = [\varepsilon \log m]$. Let $\mathcal{B}_\text{peak}^k$ be the Bergman kernel with respect to the peak sections $S_1, \ldots, S_k$. Then

1. $\mathcal{B}_\text{peak}^k$ has an $C^\infty$ asymptotic expansion
2. The expansion stables to the TYZ expansion:

$$\left\| \mathcal{B}(x) - \mathcal{B}_\text{peak}^k \right\|_{C^\mu} \leq \frac{C}{m^{\varepsilon_0} \log k}.$$
W.D. Ruan’s Lemma

Lemma

Let $S_P$ be peak sections. Let $T$ be another section of $L^m$.

Near $x_0$, $T = f e^m_L$ for a holomorphic function $f$. When we say $T$’s Taylor expansion at $x_0$, we mean the Taylor expansion of $f$ at $x_0$ under the coordinate system $(z_1, \ldots, z_n)$.

1. If $z^P$ is not in $T$’s Taylor expansion at $0$, then

$$(S_P, T) = O \left( \frac{1}{m} \right) \|S\| \cdot \|T\|.$$ 

2. If $T$ contains terms $z^Q$ for $|Q| \geq |P| + \sigma$ in the Taylor expansion, then

$$(S_P, T) = O \left( \frac{1}{m^{1+\frac{\sigma}{2}}} \right) \|S\| \cdot \|T\|.$$
We worked out the effective version of Ruan’s Lemma.
Let $S_{k+1}, \cdots, S_d$ be an orthonormal basis of the space $V_{s+1} = \{ T \in H^0(M, L^m \otimes E) \mid T \text{ vanishes at } x_0 \text{ of order at least } s + 1 \}$.

Let $S_1, \cdots, S_k$ be peak sections.
Define the matrix $A_{ij}$ to be

$$
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Then we have

$$F_1 = I - A.$$ 

By our effective Ruan's Lemma, we have

$$||A_{\alpha,\beta}|| \leq \frac{C}{m^{1+|\alpha-\beta|/2}}.$$ 

(This is a little over-simplified version)
We represent $F_1^{-1} = (C_{\alpha\beta})$ as a block matrix using the same partition as in the matrix $A$. Using the expansion

$$F^{-1} = I + A + A^2 + \cdots,$$

for any fixed $(\alpha_0, \beta_0)$, we have

$$C_{\alpha\beta} - I = \sum_{k=1}^{\infty} \sum_{i_1, \cdots, i_{k-1}} A_{\alpha_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \beta_0}.$$

We have

$$\left\| \sum_{k=s+1}^{\infty} \sum_{i_1, \cdots, i_{k-1}} A_{\alpha_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \beta_0} \right\| \leq \frac{C}{m^{s+1}}.$$
Similarly, we consider the terms

$$\sum_{k=1}^{s} \sum_{\text{some } i_j=s+1} A_{\alpha_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \beta_0}.$$ 

If some $i_j = s + 1$, we must have

$$|\alpha_0 - i_1| + |i_1 - i_2| + \cdots + |i_{k-1} - \beta_0| \geq 2s + 2 - \alpha_0 - \beta_0.$$ 

Thus we have

$$\left\| \sum_{k=1}^{s} \sum_{\text{some } i_j=s+1} A_{\alpha_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \beta_0} \right\| \leq \frac{C}{m^{s+1 - \frac{1}{2}(\alpha_0 + \beta_0)}}.$$
Real analytic case

Theorem (Liu-L)

Assume that the metrics are real analytic. Then

1. The TYZ expansion

\[ \sum_{j=0}^{\infty} \frac{a_j}{m^j} \]

is convergent for \( m \) large.

2. \[ \| \mathfrak{B}(x) - m^n \sum_{j=0}^{\infty} \frac{a_j}{m^j} \|_{C^\mu} \leq C m^{-\varepsilon (\log m)^{1/n}}. \]

(Fefferman-Boutet de Monvel-Sjöstrand’s method possible)
The technical heart is that, if we work harder, we can have

\[ \| \mathfrak{B}(x) - m^n \sum_{j=0}^{N} \frac{a_j}{m^j} \|_{C^\mu} \leq \frac{C'^N}{m^{N+1}}, \]

where \( N \) is up to \( \epsilon \log m \) and \( C' \) is independent to \( N \).
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$$\| \mathfrak{B}(x) - m^n \sum_{j=0}^{N} \frac{a_j}{m_j} \|_{C^\mu} \leq \frac{C'^N}{m^{N+1}},$$

where $N$ is up to $\varepsilon \log m$ and $C'$ is independent to $N$. By setting $N = [\varepsilon \log m]$, we can get the result.
The orbifold case

Assume $M$ is an orbifold with only one orbifold singularity point. Assume that the local group is $\mathbb{Z}_2$. Such an orbifold may not exist. Just use it as an example.
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It is not possible to have the $C^\infty$ convergence of the Bergman kernel at the orbifold point. What Ross-Thomas did was to “average” several Bergman kernels.
In terms of peak sections, this is clear. The set of peak sections can be decomposed into two parts: \( \{ z^{2k+1} \} \) and \( \{ z^{2k} \} \) and these two sets are perpendicular to each other. We can make the Begman kernel for each of them. The sum of the Bergman kernels is equal to the Bergman kernel of the local uniformization. Since the Bergman kernel expansion can be localized, using the ordinary method on manifold, we can recover the result of Ross-Thomas.
Thank you!