Definition and basic properties of heat kernels III, Constructions

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April 27, 2010
Theorem

Let $M$ be a complete Riemannian manifold. Then there is a heat kernel

$$H(x, y, t) \in C^\infty(M \times M \times \mathbb{R}^+)$$

such that

$$(e^{\Delta t} f)(x) = \int_M H(x, y, t)f(y) \, dy$$

satisfying

1. $H(x, y, t) = H(y, x, t)$,
2. $\lim_{t \to 0^+} H(x, y, t) = \delta_x(y)$;
3. $(\Delta - \frac{\partial}{\partial t})H = 0$;
4. $H(x, y, t) = \int_M H(x, z, t - s)H(z, y, s) \, dz \text{ for any } 0 < s \leq t$. 
The proof composed of two steps

1. Construct the *paramatrix*;
2. Using the paramatrix to construct the heat kernel.
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The $k$-simplex $\Delta_k$ is the following subset of $\mathbb{R}^k$

\[ \{(t_1, \cdots, t_k) \mid 0 \leq t_1 \leq \cdots \leq t_k \leq 1\} \]

For $t > 0$, we write $t\Delta_k$ for the rescaled simplex

\[ \{(t_1, \cdots, t_k) \mid 0 \leq t_1 \leq \cdots \leq t_k \leq t\} \]
We assume that $U(x, y, t)$ be a function on $M \times M \times \mathbb{R}^+$ such that $\lim_{t \to 0^+} U(x, y, t) = \delta_x(y)$ be the Dirac function. Let

$$R(x, y, t) = \frac{dU(x, y, t)}{dt} - \Delta_x U(x, y, t),$$

where $\Delta_x$ means the Laplace operator for the variable $x$. Note that formally, any function $g(x, y, t)$ defines one-parameter family of operators $G_t$ by the formula

$$G_t f(x) = \int_M g(x, y, t) f(y) dy.$$
We use the $R_t, U_t$ to denote the corresponding families of operators with respect to the functions $R(x, y, t)$ and $U(x, y, t)$, respectively. For any $k \geq 1$, define the operator

$$Q^k_t = \int_{t\Delta_k} U_{t-t_k} R_{t_k-t_{k-1}} \cdots R_{t_2-t_1} R_{t_1} dt_1 \cdots dt_k,$$

and $Q^0_t = U_t$. Let

$$R^{(k)}(s) = \int_{s\Delta_{k-1}} R_{s-t_{k-1}} \cdots R_{t_2-t_1} R_{t_1} dt_1 \cdots dt_{k-1},$$

and $R^{(0)}(s) = 0$. 
Since the derivative of the integral of the form
\[ \int_0^t a(t - s)b(s)ds \] is equal to

\[ \int_0^t \frac{da}{dt}(t - s)b(s)ds + a(0)b(t), \]

we have

\[ \left( \frac{\partial}{\partial t} - \Delta \right) Q_k^t = R^{(k+1)}(t) + R^{(k)}(t). \]

As a result, we have

\[ \left( \frac{\partial}{\partial t} - \Delta \right) \sum_{k=0}^{\infty} (-1)^k Q_k^t = 0. \]
We wish to find the following form of the fundamental solution of the heat equation:

\[
U(x, y, t) \sim (4\pi t)^{-\frac{n}{2}} e^{-d^2(x,y)/4t} \left\{ \sum_{i \geq 0} \phi_i(x, y)t^i \right\}
\]

where \(d(x, y)\) is the distance function on the Riemannian manifold. \(U(x, y, t)\) should satisfy

1. \(\lim_{t \to 0^+} U(x, y, t) = \delta_x(y)\), where \(\delta_x(y)\) is the Dirac function at \(x\);

2. For any \(N\), \(\lim_{t \to 0^+} (\Delta - \frac{\partial}{\partial t})U(x, y, t) = O(t^N)\).

The function \(U(x, y, t)\) is called the \textit{paramatrix} of the heat kernel.
We pick a normal coordinate system \((y_1, \cdots, y_n)\), and let \(r = d(x, y)\) be the Riemannian distance. We identify a neighborhood of \(y\) to a small ball of \(T_y(M)\) by the exponential map. Under this map, the coordinates of \(x\) can be written as \((x_1, \cdots, x_n)\). On the other hand, let \((\theta_1, \cdots, \theta_{n-1})\) be a coordinate system on \(S^{n-1}\), then \((r, \theta_1, \cdots, \theta_{n-1})\) gives a coordinate system at \(y\) also, and this coordinate system is called the polar coordinates.
Let $\psi(r)$ be a function of $r$, and let $g = \det(s_{ij})$. Then we have

$$
\Delta \psi = \frac{d^2 \psi}{dr^2} + \left( \frac{d \log \sqrt{g}}{dr} \right) \frac{d\psi}{dr},
$$

$$
\Delta(\phi \psi) = \phi \Delta \psi + \psi \Delta \phi + 2 \frac{d\phi}{dr} \frac{d\psi}{dr}.
$$

We let

$$
\psi = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}},
$$

$$
\phi = \phi_0 + \phi_1 t + \cdots + \phi_N t^N,
$$

where $\phi_j = \phi_j(x, y)$ are smooth functions on $M \times M$, and

$$
u_N = \psi \phi = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \sum_{i=0}^{N} \phi_i t^i.$$
Then
\[
\left( \Delta - \frac{\partial}{\partial t} \right) u_N = \phi \left( \Delta \psi - \frac{\partial \psi}{\partial t} \right) + \psi \left( \Delta \phi - \frac{\partial \phi}{\partial t} \right) + 2 \frac{\partial \phi}{\partial r} \frac{d\psi}{dr}
\]

Since
\[
\Delta \psi - \frac{\partial \psi}{\partial t} = \frac{d \log \sqrt{g}}{dr} \frac{d\psi}{dr},
\]
\[
\frac{d\psi}{dr} = -\frac{r}{2t} \psi.
\]
we have
\[
\left( \Delta - \frac{\partial}{\partial t} \right) u_N = \frac{\psi}{t} \sum_{k=0}^{N} \left[ \Delta \phi_{k-1} - \left( k + \frac{r}{2} \frac{d \log \sqrt{g}}{dr} \right) \phi_k - r \frac{d\phi_k}{dr} \right] t^k
\]
Thus in order to find the paramatrix, we set
\[
rd\phi_k dr + \left( k + \frac{r d\log \sqrt{g}}{2} dr \right) \phi_k = \Delta \phi_{k-1}
\]
for \( k = 0, \cdots, N \), where we let \( \phi_{-1} = 0 \). The solutions of the above ODEs are
\[
\phi_0(x, y) = g^{-\frac{1}{4}}(x);
\]
\[
\phi_k(x, y) = g^{-\frac{1}{4}}(x)r(x, y)^{-k}
\]
\[
\times \int_0^{r(x,y)} r^{k-1}(\Delta \phi_{k-1}) \left( \frac{rx}{r(x, y)} \right) g \left( \frac{rx}{r(x, y)} \right)^{\frac{1}{4}} dr.
\]
Thus we have
\[
\left( \Delta - \frac{\partial}{\partial t} \right) u_N = \frac{\psi}{t}(\Delta \phi_N)t^N.
\]
From the above, we proved that

**Lemma**

*There is a unique formal solution* \( U(x, y, t) \) *of the heat equation*

\[
\left( \Delta - \frac{\partial}{\partial t} \right) U(x, y, t) = 0
\]

*of the form*

\[
u_N = \psi \phi = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \sum_{i=0}^{N} \phi_i t^i.
\]
Let \( \eta \) be a smooth function such that \( \eta = 1 \) for \( t < 1 \) and \( \eta = 0 \) for \( t > 2 \). Let

\[
p(x, y) = \eta \left( \frac{2r(x, y)}{\delta} \right)
\]

be the cut-off function, where \( \delta \) is the injectivity radius. Then for any \( N \), we consider the function

\[
u_N(x, y, t) = p(x, y)u_N(x, y, t).
\]

We shall prove that

**Lemma**

For any \( N \) sufficiently large, we have

1. \( \lim_{t \to 0^+} u_N(x, y, t) = \delta_x(y) \);
2. The kernel \( R_N(x, y, t) = (\Delta_y - \frac{\partial}{\partial t}) u_N(x, y, t) \) satisfies the estimate

\[
||R_N(x, y, t)||_{C^l} \leq C(l) t^{N-l/2-1}.
\]
Using the above estimate, we know that for $N \gg 0$, we have

$$\| R_N(x, y, t) \|_{C^l} \leq C t^\alpha.$$ 

It follows that

$$\| R^{(k)}(s) \|_{C^l} \leq \frac{C^k \alpha!}{(\alpha + k)!}.$$ 

Since

$$\sum_k \frac{C^k \alpha!}{(\alpha + k)!} < +\infty,$$

our formal construction is convergent to the heat kernel.

**Theorem**

The $\Delta_p$ on $L^p(M)$ is well defined as the infinitesimal generator of the heat semi-group.
The following equation is true

\[ H(x, y, t) = U(x, y, t) - \int_0^t e^{\Delta_x(t-s)} \left( \frac{\partial}{\partial s} - \Delta_x \right) U(x, y, s) \, ds \]
The following equation is true

\[ H(x, y, t) = U(x, y, t) - \int_0^t e^{\Delta x(t-s)} \left( \frac{\partial}{\partial s} - \Delta_x \right) U(x, y, s) ds \]

Two problems

1. The self-adjoint extension of \( \Delta \) has to be assumed first;
2. The convergence of \( U(x, y, t) \).
Let $u$ be a positive harmonic function on $M$. 

\[
\begin{align*}
\phi & = |\nabla \log u|, \\
\phi_i & = \frac{1}{u} |\nabla u| u^j u_{ji} - |\nabla u|^2 u^i u_j u_{ij} - \frac{1}{u} |\nabla u|^3 \sum_i (\sum_j u_j u_{ji})^2 + 2 \phi^3
\end{align*}
\]
Let $u$ be a positive harmonic function on $M$. Define $\phi = |\nabla \log u|$.
Let $u$ be a positive harmonic function on $M$. Define $\phi = |\nabla \log u|$. Then we have

$$
\phi_i = \frac{1}{u|\nabla u|} u_j u_{ji} - \frac{|\nabla u|}{u^2} u_i
$$

$$
\Delta \phi = \phi_{ii} = \frac{1}{u|\nabla u|} \sum u_j^2 + \frac{1}{u|\nabla u|} u_j u_{jii}
$$

$$
- \frac{2}{u^2|\nabla u|} u_i u_j u_{ij} - \frac{1}{u|\nabla u|^3} \sum_i \left( \sum_j u_j u_{ji} \right)^2 + 2\phi^3
$$
We assume at a point $u_j = 0$ for $j > 1$. Then we have

\[
\phi_1 = \frac{1}{u|\nabla u|} u_1 u_{11} - \frac{|\nabla u|}{u^2} u_1
\]

\[
\Delta \phi = \frac{1}{u|\nabla u|} \sum u_{ji}^2 + \frac{1}{u|\nabla u|} u_j u_{jii}
\]

\[
- \frac{2u_1}{u^2} u_{11} - \frac{1}{u|\nabla u|} \sum_i u_{1i}^2 + 2\phi^3
\]
We have

\[ \frac{1}{u |\nabla u|} u_j u_{jii} \geq \frac{1}{u |\nabla u|} u_j (\Delta u)_j - K \phi \]
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\]
and
\[
\frac{1}{u|\nabla u|} \sum u^2_{ji} - \frac{1}{u|\nabla u|} \sum_i u^2_{1i} \\
\geq \frac{1}{u|\nabla u|} \sum_{j \geq 2} u^2_{jj} \geq \frac{1}{(n - 1)u|\nabla u|} u^2_{11}
\]
Thus we have

$$\Delta \phi \geq 2\phi^3 - \frac{2u_1}{u^2} u_{11} + \frac{1}{(n - 1)u|\nabla u|} u_{11}^2 - K\phi$$
Thus we have

\[ \Delta \phi \geq 2\phi^3 - \frac{2u_1}{u^2} u_{11} + \frac{1}{(n - 1)u|\nabla u|} u_{11}^2 - K\phi \]

We have

\[ \phi^3 - \frac{|\nabla u|}{u^2} u_{11} = -\frac{\nabla \phi \nabla u}{u} \]

Thus we have

\[ \Delta \phi \geq A \frac{\nabla \phi \nabla u}{u} \]

\[ + (2 + A)\phi^3 - \frac{2 + A}{u^2} u_{11} + \frac{1}{(n - 1)u|\nabla u|} u_{11}^2 - K\phi \]
Thus we have
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\Delta \phi \geq 2\phi^3 - \frac{2u_1}{u^2} u_{11} + \frac{1}{(n-1)u|\nabla u|} u_{11}^2 - K\phi
\]

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\]

Thus we have
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\Delta \phi \geq A \frac{\nabla \phi \nabla u}{u}
\]
\[
+ (2 + A)\phi^3 - \frac{(2 + A)u_1}{u^2} u_{11} + \frac{1}{(n-1)u|\nabla u|} u_{11}^2 - K\phi
\]

If we choose $A$ close to 2 enough, we can find an $\epsilon > 0$ such that
\[
\Delta \phi \geq A \frac{\nabla \phi \nabla u}{u} + \frac{1}{2} \epsilon \phi^3 - K\phi.
\]
The Bakry-Émery geometry was introduced to study the diffusion processes. For a Riemannian manifold \((M, g)\) and a smooth function \(\phi\) on \(M\), the Bakry-Émery manifold is a triple \((M, g, \phi)\), where the measure on \(M\) is the weighted measure \(e^{-\phi}dV_g\). The Bakry-Émery Ricci curvature is defined to be

\[
\text{Ric}_\infty = \text{Ric} + \text{Hess}(\phi),
\]

and the Bakry-Émery Laplacian is

\[
\Delta_\phi = \Delta - \nabla \phi \cdot \nabla.
\]

The operator can be extended as a self-adjoint operator with respect to the weighted measure \(e^{-\phi}dV_g\).
The B-E Laplacian

We have

$$\int \Delta \phi f g e^{-\phi} = - \int \nabla f \nabla g \cdot e^{-\phi}$$
The B-E Laplacian

We have
\[ \int \Delta_\phi f e^{-\phi} = - \int \nabla f \nabla g \cdot e^{-\phi} \]

Therefore, it can be extended as a self-adjoint elliptic operator.
The B-E Laplacian

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\[ \int \Delta_\phi f g e^{-\phi} = - \int \nabla f \nabla g \cdot e^{-\phi} \]

Therefore, it can be extended as a self-adjoint elliptic operator. Hodge theorem is also valid for this new operator.
Theorem (Brighton)

Let $f$ be a bounded function and let $u$ be a $f$-harmonic function.

$\Delta u - \nabla f \cdot \nabla u = 0$

Then if $M$ is a Ricci nonnegative manifold and $u$ is positive, $u$ must be a constant.
Theorem (Brighton)

Let $f$ be a bounded function and let $u$ be a $f$-harmonic function. That is

$$\Delta u - \nabla f \nabla u = 0$$

Then if $M$ is a Ricci nonnegative manifold and $u$ is positive, $u$ must be a constant.
Define a function $U(x, y)$ on $M \times \mathbb{R}^+$. 
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$$U(x, y) = u(x) - \frac{1}{2} y^2 \nabla f \nabla u$$
Define a function $U(x, y)$ on $M \times \mathbb{R}^+$. 

$$U(x, y) = u(x) - \frac{1}{2} y^2 \nabla f \nabla u$$

Then

$$\tilde{\Delta} U = 0$$

at $(x, 0)$. 
Define a function $U(x, y)$ on $M \times \mathbb{R}^+$. 

$$U(x, y) = u(x) - \frac{1}{2} y^2 \nabla f \nabla u$$

Then

$$\tilde{\Delta} U = 0$$

at $(x, 0)$. Let $\Phi = |\nabla \log U|$. Then we have

$$\tilde{\Delta} \Phi \geq A \Phi^3 + C \frac{\nabla \Phi \nabla U}{U}$$

for $A > 0$ and $\tilde{\Delta}$ the Laplacian on the $n + 1$ dimensional manifold $M \times \mathbb{R}^+$. 