An Example

Let $X$ be a Kähler manifold, and let $\mathcal{U} = \{ U_\alpha \}$ be a cover of $X$. Assume that on each $U_\alpha$, there exists a holomorphic function $u = u_\alpha$ such that the Kähler metric $\omega$ can be represented by

$$
\omega = \sqrt{-1} \text{Im} \left( \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) dz_i \wedge d\bar{z}_j.
$$

Then $X$ is called a special Kähler manifold (in the sense of Strominger and Freed).
Theorem

The curvature of the metric is

$$R_{i\bar{j}k\bar{l}} = -\frac{1}{4} g^{m\bar{n}} u_{ikm} u_{jln}$$

In particular, the scalar curvature of the metric is

$$\rho = \frac{1}{4} g^{m\bar{n}} g^{i\bar{j}} g^{k\bar{l}} u_{ikm} u_{jln},$$

where

$$u_{ijk} = \frac{\partial^3 u}{\partial z_i \partial z_j \partial z_k}.$$
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In particular, the scalar curvature of the metric is 

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where 

\[ u_{i j k} = \frac{\partial^3 u}{\partial z_i \partial z_j \partial z_k}. \]

Thus \( X \) is Ricci nonnegative and the scalar curvature is nonnegative. Moreover, if the scalar curvature is zero, then so is the curvature tensor.
Theorem (L, 1999)

We have

\[ \Delta \rho \geq \frac{3}{n^3} \rho^2 \]
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**Proof.** By the Bochner formula, we have

\[ \Delta \rho \geq \frac{1}{4} |\nabla^4 u|^2 + \frac{3}{16} u_{nps} u_{nqs} u_{ikm} u_{jln} \]

\[ \geq \frac{3}{n^3 \rho^2}. \]
The generalized maximum principle

**Theorem**

Let $X$ be a complete non-compact Riemannian manifold. Assume that

$$\text{Ric}(X) \geq -k.$$

Then if $f$ is a nonnegative function such that

$$\Delta f \geq f^2$$

then $f = 0$. 

From the above argument, we proved that

**Theorem (L,1999)**

*If $X$ is a complete special Kähler manifold, then $X$ has to be a flat space.*

**Proof.** By the generalized maximum principle, we have $\rho \equiv 0$, which implies that the sectional curvatures are zero. Therefore the manifold has to be flat.
CY moduli has the similar Kähler metric structure.
We consider a polarized Calabi-Yau manifold \((X, L)\), where

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3. Why polarization?
The following theorem of Yau is classical in understanding the so-called Calabi-Yau manifold:
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**Theorem (Yau)**

There exists a unique Ricci flat Kähler metric $\omega$ in the cohomology class defined by $[\omega]$. 
Why polarization?

The Ricci flat metric is unique only when fixing the cohomology class of a Kähler metric.
Then why moduli space (CY moduli)?

Moduli space gives a way to compare CY manifolds that are close to each other. So it gives a platform of linearization.
We will study the differential geometry of CY moduli.
We will study the **differential geometry** of CY moduli.

1. local properties
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2. semi-local (semi-global) properties
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2. semi-local (semi-global) properties
3. global properties
A very quick review of Kodaira-Spencer theory

Let $Z$ be a compact complex manifold. A complex structure $J$ is a real operator

$$J : T_C Z \rightarrow T_C Z$$

such that $J^2 = -I$ (plus the integrability conditions).
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A variation of the complex structure is a real matrix $A$ such that

$$(J + \varepsilon A)^2 = -I$$
A very quick review of Kodaira-Spencer theory-2

Or we have

\[ AJ + JA = 0 \]

Thus we have

\[ A = \begin{pmatrix} 2A_1 \\ -2\bar{A}_1 \end{pmatrix} \]

for some \( A_1 : T^{1,0}Z \to T^{0,1}Z \), or equivalently \( A_1 \) can be represented by

\[ \varphi \in \Lambda^{0,1}(T^{1,0}Z) \]

(equivalent to Beltrami differential for RS)
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The Kodaira-Spencer map is defined as

\[ \frac{\partial}{\partial \varepsilon} \mapsto \varphi \]
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or using the notation of super-Lie bracket

\[\partial\varphi - \frac{1}{2}[\varphi, \varphi] = 0.\]
By Kodaira-Nirenberg-Spencer-Kuranishi, the following equation

\[ \varphi(t) = \varphi'(0)t + \frac{1}{2} \bar{\partial}^* G[\varphi(t), \varphi(t)] \]

is always solvable for small \( t \) for any given initial condition \( \varphi(0) = 0 \) and \( \varphi'(0) = \beta \).
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The above equation is equivalent to the integrability condition of the complex structure if and only if

\[ [\varphi(t), \varphi(t)] \]

is a co-boundary.
Sufficient conditions for the existence of complex structures

1. If $Z$ is a Riemann surface (Riemann)

2. If $H^2(Z, \mathbb{T}_Z) = 0$ (Kodaira-Nirenberg-Spencer)

3. If $Z$ is a Calabi-Yau manifold (Tian)

We remark that the proof of (3) depends heavily on the so-called Tian-Todorov Lemma, which states that when $\phi_1, \phi_2$ are harmonic, then $[\phi_2, \phi_2]$ is a $\bar{\partial} \ast$-coboundary.
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Weil-Petersson metric

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**Definition**

Let $Z$ be a polarized Calabi-Yau manifold with the Ricci flat Kähler metric $\mu$ whose Kähler form defines the polarization. Let $X, Y \in H^1(Z, T^{(1,0)}Z)$. Define the $L^2$ inner product by

$$ (X, Y) = \frac{1}{n!} \int_Z \langle X, Y \rangle \mu^n. $$

For a Calabi-Yau manifold, via the Kodaira-Spencer map:

$$ T_Z M \to H^1(Z, T^{(1,0)}Z), $$

which is an isomorphism, the above inner product defines a metric on the smooth part of $M$. The metric happens to be Kählerian, and is called the Weil-Petersson metric of $M$. 
The differential geometry of the Weil-Petersson metric on Calabi-Yau moduli is called the Weil-Petersson geometry.
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**The Aim:**
We want to be able to tell the properties of the moduli space through geometric analysis.
Extrinsic characterization of the WP metric

Theorem (Tian)

Let $\mathcal{M}$ be the CY moduli of $Z$. Let $\Omega$ be a holomorphic family of nonzero $(3, 0)$ forms. Then the WP metric can be expressed as

$$\omega = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \int \Omega \wedge \bar{\Omega}$$
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This is quite unique!

The Weil-Petersson metric can be expressed explicitly in terms of the variation of Hodge structures.
Hodge theory helps us!

Recall that for any compact complex manifold, we can define the cohomology groups $H^{p,q}$. By Hodge theorem, they are made from harmonic forms.
If we deform the CY manifolds, we deform the Hodge flags. This is called the variation of Hodge structure.
Let’s take the example of moduli space of a CY 3 fold. On the moduli space, we can define the following Hodge bundles:

\[
\begin{align*}
F^3 &= H^{3,0} \\
F^2 &= H^{3,0} \oplus H^{2,1} \\
F^1 &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \\
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\]

The last bundle is locally flat. By the Griffiths transversality, we have

\[\nabla F^k \subset F^{k-1}\]
Since $F^3$ is a subbundle of a locally flat bundle, we can write a holomorphic section $\Omega$ of $F^3$ as

$$\Omega = (1, \frac{1}{\sqrt{2}} z_1, \cdots, \frac{1}{\sqrt{2}} z_n, f_0, \cdots, f_n),$$

where $n = \dim H^{2,1} = \dim \mathcal{M}$, $(z_1, \cdots, z_n)$ can be used as holomorphic coordinates.
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Then $Q$ is skew-symmetric and can be written as

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
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$$Q(F^3, F^2) = 0$$
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Therefore we have

$$f_k = \frac{1}{\sqrt{2}} \frac{\partial u}{\partial z_k}$$

locally for some holomorphic function $u$. 
With the above theorem, we have

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The curvature of the metric is (for the CY moduli of CY 3 folds)

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where \( F_{ijk} \) is the Yukawa coupling.
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where $F_{ijk}$ is the Yukawa coupling. The curvature is similar to that of special Kähler manifolds.
Local Weil-Petersson geometry

Good properties
Local Weil-Petersson geometry

Good properties
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Bad property
- The curvature is neither positive nor negative.
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Rebuilding the local geometry

The following result was proved

**Theorem (L,1997)**

*Let $\mathcal{M}$ be the moduli space of a CY 3-fold with dimension $m$. Then we define the following metric

$$\omega_H = (m + 3)\omega_{WP} + \text{Ric}(\omega_{WP})$$

which I called it Hodge metric. The curvature has good properties (non-positive bisectional curvature, negative Ricci and holomorphic sectional curvature, etc).*
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An alternative way of defining the Hodge metric

Let $D$ be the classifying space of a given Hodge structure. That is, roughly speaking $D$ is the set of all subspaces $H^{p,q}$ of a given vector space $H$. 

Example.

Let $X$ be a compact Riemann surface. Then $H^{1,0} \subset H^1(X, \mathbb{C})$ determines the period matrix of $X$. In this case, the classifying space is the Siegel upper half space.
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**Theorem**

*With respect to the Hermitian connection, the holomorphic sectional curvature of the tangent bundle is negative away from zero.*
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Later Peters proved that the bisectional curvature is always non-positive.
Theorem (L, 1999)

1. The restriction/pull back of the invariant Hermitian metric on the moduli space is always Kählerian.
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1. The restriction/pull back of the invariant Hermitian metric on the moduli space is always Kählerian.
2. The metric is the Hodge metric defined before for the moduli space of CY 3 fold.
3. The curvature on the moduli space is smaller.
We are not able to prove that the sectional curvature is non-positive. However, we don’t expect that, otherwise the CY moduli may be too restrictive.
The BCOV Conjecture

BCOV = Bershadsky-Cecotti-Ooguri-Vafa
The BCOV Conjecture

BCOV=Bershadsky-Cecotti-Ooguri-Vafa

Definition

The mirror map is the holomorphic map from a neighborhood of $\infty \in \mathbb{P}^1$ to a neighborhood of $0 \in \Delta$ defined by the following formula

$$q := (5\psi)^{-5} \exp \left( \frac{5}{y_0(\psi)} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left\{ \sum_{j=n+1}^{5n} \frac{1}{j} \right\} \frac{1}{(5\psi)^{5n}} \right),$$

where $|\psi| \gg 1$, and

$$y_0(\psi) := \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}, \quad |\psi| > 1.$$

The inverse of the mirror map is denoted by $\psi(q)$. 
Define the multi-valued function \( F_{1,B}^{\text{top}}(\psi) \) as

\[
F_{1,B}^{\text{top}}(\psi) := \left( \frac{\psi}{y_0(\psi)} \right)^{\frac{62}{3}} (\psi^5 - 1)^{-\frac{1}{6}} q \frac{d\psi}{dq},
\]

and

\[
F_{1,A}^{\text{top}}(q) := F_{1,B}^{\text{top}}(\psi(q)).
\]
Conjecture (A)

Let $n_g(d)$ be the genus-$g$ degree-$d$ instanton number of a quintic in $\mathbb{CP}^4$ for $g = 0, 1$. Then the following identity holds:

$$-q \frac{d}{dq} \log F_{1,A}^{\text{top}}(q) =$$

$$\frac{50}{12} - \sum_{n,d=1}^{\infty} n_1(d) \frac{2nd q^{nd}}{1 - q^{nd}} - \sum_{d=1}^{\infty} n_0(d) \frac{2d q^d}{12(1 - q^d)}.$$
Conjecture (A) was proved by Aleksey Zinger.

Aleksey Zinger
The Reduced Genus-One Gromov-Witten Invariants of Calabi-Yau Hypersurfaces
ArXiv: 0705.2397v2, 2007. JAMS 08
Setup of Conjecture (B)

Let $X$ be a compact Kähler manifold.
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- Let $\Delta = \Delta_{p,q}$ be the Laplacian on $(p, q)$ forms;

Note: The definition $\det \Delta = \prod_{\lambda_i \neq 0} \lambda_i$ is not well-defined. Use regularization (e.g., Riemann $\zeta$-function).
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Bershadsky-Ceccotti-Ooguri-Vafa defined

$$ T \overset{\text{def}}{=} \prod_{p, q} (\det \Delta_{p, q}) (-1)^{p+q} pq. $$
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- Bershadsky-Ceccotti-Ooguri-Vafa defined

\[ T \overset{\text{def}}{=} \prod_{p, q} (\det \Delta_{p, q})(-1)^{p+q}pq. \]

- Why define such a strange quantity?
Conjecture

(B) Let \( \| \cdot \| \) be the Hermitian metric on the line bundle

\[
(\pi_* K_{W/CP^1}) \otimes 62 \otimes (T(CP^1)) \otimes 3 \big|_{CP^1 \setminus D}
\]

induced from the \( L^2 \)-metric on \( \pi_* K_{W/CP^1} \) and from the Weil-Petersson metric on \( T(CP^1) \). Then the following identity holds:

\[
\tau_{BCOV}(W_\psi) = \text{Const.} \left\| \frac{1}{F_{1, B}^\text{top}(\psi)^3} \left( \frac{\Omega_\psi}{y_0(\psi)} \right)^{62} \otimes \left( q \frac{d}{dq} \right)^3 \right\|_{\frac{2}{3}},
\]

where \( \Omega \) is the local holomorphic section of the \((3, 0)\) forms.
Conjecture B was proved by Fang-L-Yoshikawa.

Hao Fang (方浩)-L-Yoshikawa (吉川谦一)
Asymptotic behavior of the BCOV torsion of Calabi-Yau moduli
ArXiv: 0601411 JDG (80), 2008, 175-259,
From string theory point of view, there is only one conjecture: By counting the instanton numbers we get the function $F^{\text{top}}_{1,A}$, and by computing the BCOV torsion we get the function $F^{\text{top}}_{1,B}$. These two functions are the same.
From string theory point of view, there is only one conjecture: By counting the instanton numbers we get the function $\mathcal{F}^\text{top}_{1,A}$, and by computing the BCOV torsion we get the function $\mathcal{F}^\text{top}_{1,B}$. These two functions are the same.

Combining Conjecture A and B, we verified the Mirror Symmetry prediction of the case $g = 1$. For higher genus, the B-side of the conjectures have not been set up. (Yau-Yamaguchi, Huang-Klemm-Quakenbush, and many others)
Theorem

On the moduli space of a primitive CY $n$-fold, we have

$$(-1)^n \omega_H - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \tau_{BCOV} = \frac{\chi_X}{12} \omega_{WP},$$

where $\chi_{\text{top}}(X)$ is the Euler characteristic number of $X$. 

Reduction of the problem
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where $\chi_{top}(X)$ is the Euler characteristic number of $X$. 

Corollary

If $X$ is a primitive Calabi-Yau, $N \subset \mathcal{M}$ is a $k$-dimensional complete subvariety of $\mathcal{M}$ where $\mathcal{M}$ is the moduli space of $X$, then the following volume identity holds:

$$\text{Vol}_{H^n}(N) = \left[\frac{(-1)^n}{12} \chi(X)\right]^k \text{Vol}_{WP}(N).$$
Corollary

Assume that a polarized Calabi-Yau manifold $X$ is primitive, and that

$$(-1)^{n+1} \chi_X > -24.$$

Let $\mathcal{M}$ be the moduli space of $X$. Then there exists no complete curve in $\mathcal{M}$; hence, there exists no projective subvariety of $\mathcal{M}$ (of positive dimensions).
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Observation: $\omega_H \geq 2\omega_{WP}$!
Using the curvature formula, the BCOV torsion is determined up to a pluriharmonic function.
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\[
\log \tau_{BCOV}(X_t) = \frac{1}{6} \log |t|^2 + O(\log(-\log |t|^2)).
\]
Let $(X, L)$ be a polarized Calabi-Yau manifold. The Kuranishi space of $X$ is an open neighborhood of the vector space $H^1(X, TX)$.
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Let $c_1(L) = [\omega]$ be the polarization of $X$. Then the infinitesimal deformation space that keeps the polarization is

$$H^1_{\omega}(X, TX) = \{ \varphi \in H^1(X, TX) \mid L\varphi[\omega] = 0 \}$$
A remark on the WP metric on CY moduli

Let \((X, L)\) be a polarized Calabi-Yau manifold. The Kuranishi space of \(X\) is an open neighborhood of the vector space \(H^1(X, TX)\).

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H^1_{\omega}(X, TX) = \{ \varphi \in H^1(X, TX) \mid L\varphi[\omega] = 0 \}
\]

That is, if \((g_{i\bar{j}})\) is the Kähler metric matrix, and let

\[
\varphi = \varphi^i_j \frac{\partial}{\partial z_i} d\bar{z}_j.
\]

Then (in the cohomology class)

\[
\varphi^i_j g_{i\bar{k}} d\bar{z}_j \wedge d\bar{z}_k = 0.
\]
Let’s assume that the Ricci flat metric is given by $g_{i\bar{j}}$. The Kodaira-Spencer map is defined by

$$\frac{\partial}{\partial t_\alpha} \mapsto \left( \frac{\partial \Omega}{\partial t_\alpha} \right)^{2,1},$$

where $(\quad)^{2,1}$ is the $(2, 1)$-component. Here we identify

$$\wedge^{0,2}(TX) \longrightarrow H^{2,1}$$

by

$$\varphi \mapsto \iota(\varphi)\Omega.$$
A straightforward computation gives

\[-\partial_\alpha \bar{\partial}_\beta \log Q(\Omega, \bar{\Omega}) = -\frac{Q(\partial_\alpha \Omega - K_\alpha \Omega, \bar{\partial}_\beta \Omega - K_\beta \Omega)}{Q(\Omega, \bar{\Omega})},\]
A straightforward computation gives

$$-\partial_\alpha \bar{\partial}_\beta \log Q(\Omega, \bar{\Omega}) = -\frac{Q(\partial_\alpha \Omega - K_\alpha \Omega, \bar{\partial}_\beta \Omega - K_\beta \Omega)}{Q(\Omega, \bar{\Omega})},$$

where $K_\alpha$ is chosen such that

$$\partial_\alpha \Omega - K_\alpha \Omega \perp \Omega.$$
Note that $Q(\varphi, \psi) = \int \varphi \wedge \psi$, the key in the proof is that, assuming that the Calabi-Yau is simply connected, then the cohomology group $H^{2,1}$ is primitive. Therefore, up to a constant, we have

$$Q(\partial_\alpha \Omega - K_\alpha \Omega, \partial_\beta \Omega - K_\beta \Omega)$$

$$= \int_X < \partial_\alpha \Omega - K_\alpha \Omega, \partial_\beta \Omega - K_\beta \Omega > dV_X$$
Conclusion:

The metric \(-\partial\bar{\partial} \log Q(\Omega, \bar{\Omega})\) is well-defined on the Kuranishi space without polarization. Its restriction to any moduli space of polarized CY moduli space is the Weil-Petersson metric.
The metric $-\partial\bar{\partial} \log Q(\Omega, \bar{\Omega})$ is well-defined on the Kuranishi space without polarization. Its restriction to any moduli space of polarized CY moduli space is the Weil-Petersson metric.

If we assume that $H^2,0 = 0$, then $H^1_\omega(X, TX) = H^1(X, TX)$. 

Conclusion: