From trigonometry to elliptic functions

Zhiqin Lu

The Math Club
University of California, Irvine

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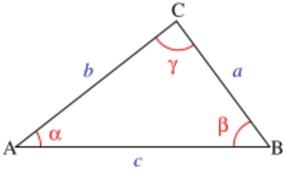
Question

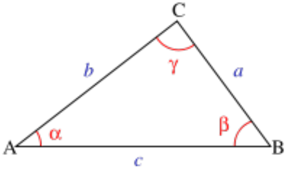
What is the area of a triangle with side lengths a, b, and c?



Heron's formula [A.D. 60]

$$A = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = (a+b+c)/2$.



By the Pythagorean theorem, we have

$$b^2 - d^2 = a^2 - (c - d)^2.$$

Therefore, we have

$$d = \frac{1}{2}\left(c + \frac{b^2 - a^2}{c}\right).$$

By the Pythagorean theorem again, we have

$$h^2 = b^2 - \frac{1}{4}\left(c + \frac{b^2 - a^2}{c}\right)^2.$$

The formula follows from the fact that

$$A = \frac{1}{2}ch.$$

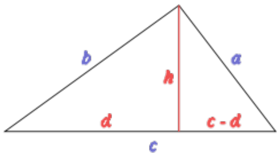


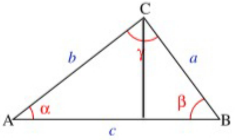
Figure: Triangle with altitude h cutting base c into d and (c - d).

Using Trigonometry, the formula is a lot easier to prove:

$$A = \frac{1}{2} bc \sin \alpha.$$

By the law of cosine, we have

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{b^2 + c^2 - a^2}{2bc} \right)^2}.$$



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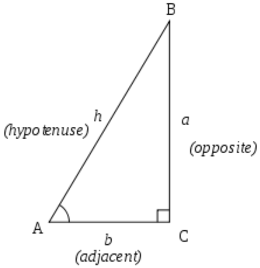
4. How to define trigonometric functions?

Definition:

- $\sin A = a/h$
- $\cos A = b/h$
- $\tan A = a/b$
- $\cdots\cdots$



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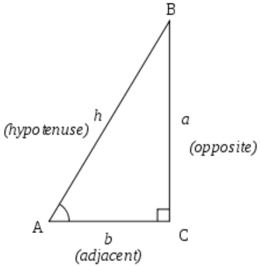

Such a definition can hardly be used in Calculus!

How to find the derivative of $\sin x$*?*

We have to use/assume the fact

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,$$

which is the derivative of $\sin x$ at 0.

In Stewart's or Minton/Smith's Calculus book, there are two ways to define the exp/log functions

1. Define e^x first, and $\log x$ is the inverse function of e^x
   1. Define e^x for x integers;
   2. Define e^x for x rational numbers;
   3. Define e^x for any real number x by continuity.
2. Define $\log x$ first and then define e^x as the inverse of $\log x$

$$\log x = \int \frac{1}{x} dx + C.$$

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Compare with the integral

$$\int \frac{1}{\sqrt{1+x^2}} dx = \log(x + \sqrt{1+x^2}) + C$$

We get the following mysterious formula

$$\log(x + \sqrt{1+x^2}) = \arcsin(\sqrt{-1}x)$$

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$$z = x + iy = x + \sqrt{-1}y,$$

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$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$

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The negative sign costs 99% of errors!

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$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Theorem: $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Proof.

$$\begin{aligned}
&\cos x \cos y - \sin x \sin y \\
&= \frac{e^{ix}+e^{-ix}}{2}\cdot\frac{e^{iy}+e^{-iy}}{2} - \frac{e^{ix}-e^{-ix}}{2i}\cdot\frac{e^{iy}-e^{-iy}}{2i} \\
&= \frac{1}{4}(e^{i(x+y)}+e^{-i(x+y)}+e^{i(x-y)}+e^{i(y-x)}) \\
&\qquad + \frac{1}{4}(e^{i(x+y)}+e^{-i(x+y)}-e^{i(x-y)}-e^{i(y-x)}) \\
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Or more precisely,

$$\phi(z) = \int_0^z \frac{1}{\sqrt{1-t^2}}\,dt.$$

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6. We have

$$\oint_{|z|=R} \frac{dz}{\sqrt{1-z^2}} = 2\pi$$

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Check:
$$(x_1 y_2 + x_2 y_1)^2 + (x_1 x_2 - y_1 y_2)^2 = 1.$$

The map $\phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \to C$ is a group isomorphism:

$$\begin{array}{ccc} (\alpha,\beta) & \longrightarrow & \alpha+\beta \\ \big\downarrow & & \big\downarrow \\ ((x_1,y_1),(x_2,y_2)) & \longrightarrow & (x_1x_2-y_1y_2, x_1y_2+x_2y_1) \end{array},$$

where $x_1 = \cos\alpha, y_1 = \sin\alpha$, etc.

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The following is another version of the addition theorem for the sine function:

$$\int_0^{\sin u} \frac{1}{\sqrt{1-x^2}}dx + \int_0^{\sin v} \frac{1}{\sqrt{1-x^2}}dx = \int_0^{\sin(u+v)} \frac{1}{\sqrt{1-x^2}}dx$$

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If we set $\sin u, z = \sin v$, then we have

$$\int_0^{y} \frac{1}{\sqrt{1-x^2}}dx + \int_0^{z} \frac{1}{\sqrt{1-x^2}}dx = \int_0^{T(y,z)} \frac{1}{\sqrt{1-x^2}}dx,$$

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Any other functions satisfy the above addition theorem? Elliptic Functions!

Consider the integral

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where $0 < \lambda < 1$.

Historic Remarks

Legendre studied these kind of integral extensively and wrote 3 volumes (4 volumes?) of *Traite des fonctions elliptiques*. But he failed in finding the most important properties of the integrals: they are the inverse functions of some doubly periodic functions! This fact was found by Abel and Jocobi independently.

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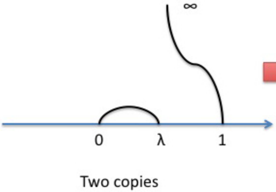
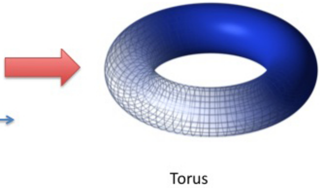
$$\frac{1}{\sqrt{z(z-1)(z-\lambda)}}$$





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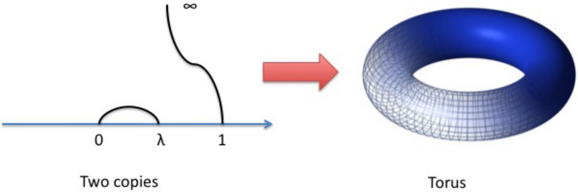
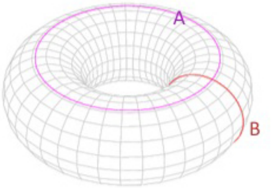


$$y^2 = z(z-1)(z-\lambda)$$

Let

$$\omega_1 = \oint_A \frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$$

$$\omega_2 = \oint_B \frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$$



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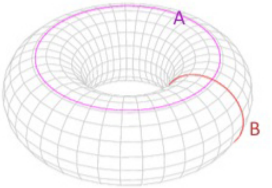

Then the inverse function of the elliptic integral

$$\int^z \frac{dt}{\sqrt{t(t-1)(t-\lambda)}}$$

is a doubly period function of periods ω_1, ω_2.

Basic property of elliptic functions

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Proof. Bounded holomorphic functions are constant by Liouivile Theorem. □

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Therefore, elliptic functions are meromorphic (singularities are poles).

Theorem

Any two tori $T(\omega_1, \omega_2)$ and $T(\omega_3, \omega_4)$ are biholomorphic if and only if there exists integers a, b, c, d such that $ad - bc = 1$, and

$$\frac{\omega_4}{\omega_3} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}.$$

References:

1. http://www.rose-hulman.edu/mathjournal/archives/2009/vol10-n2/paper2/v10n2-2pd.pdf
2. http://websites.math.leidenuniv.nl/algebra/ellipticfunctions.pdf