From trigonometry to elliptic functions

Zhiqin Lu

The Math Club
University of California, Irvine

March 31, 2010
Question

What is the area of a triangle with side lengths $a$, $b$, and $c$?
Heron’s formula [A.D. 60]

\[ A = \sqrt{p(p-a)(p-b)(p-c)}, \]

where \( p = (a + b + c)/2 \).
By the Pythagorean theorem, we have
\[ b^2 - d^2 = a^2 - (c - d)^2. \]

Therefore, we have
\[ d = \frac{1}{2} \left( c + \frac{b^2 - a^2}{c} \right). \]

By the Pythagorean theorem again, we have
\[ h^2 = b^2 - \frac{1}{4} \left( c + \frac{b^2 - a^2}{c} \right)^2. \]

The formula follows from the fact that
\[ A = \frac{1}{2} ch. \]
Using Trigonometry, the formula is a lot easier to prove:

\[ A = \frac{1}{2}bc \sin \alpha. \]

By the law of cosine, we have

\[ \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left( \frac{b^2 + c^2 - a^2}{2bc} \right)^2}. \]
What is \( \sin 20^\circ \)?
1. What is \( \sin 20^\circ \)?
2. We have to use the concept \textit{function}. 

1. What is $\sin 20^\circ$?

2. We have to use the concept *function*.

3. A function is an assignment: $f : A \rightarrow B$. 
1. What is $\sin 20^\circ$?
2. We have to use the concept *function*.
3. A function is an assignment: $f : A \rightarrow B$.
4. How to define trigonometric functions?
Definition:

- \( \sin A = \frac{a}{h} \)
- \( \cos A = \frac{b}{h} \)
- \( \tan A = \frac{a}{b} \)
- \ldots
Definition:

- $\sin A = \frac{a}{h}$
- $\cos A = \frac{b}{h}$
- $\tan A = \frac{a}{b}$
- ......

Such a definition can hardly be used in Calculus!
How to find the derivative of $\sin x$?

We have to use/assume the fact

$$
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,
$$

which is the derivative of $\sin x$ at 0.
In Stewart’s or Minton/Smith’s Calculus book, there are two ways to define the exp/log functions

1. Define $e^x$ first, and $\log x$ is the inverse function of $e^x$
   - Define $e^x$ for $x$ integers;
   - Define $e^x$ for $x$ rational numbers;
   - Define $e^x$ for any real number $x$ by continuity.

2. Define $\log x$ first and then define $e^x$ as the inverse of $\log x$

$$\log x = \int \frac{1}{x} dx + C.$$
Can we do that same thing for trigonometric functions?
Can we do that same thing for trigonometric functions?

\[
\arcsin x = \int \frac{1}{\sqrt{1 - x^2}} \, dx + C
\]
Can we do that same thing for trigonometric functions?

\[ \arcsin x = \int \frac{1}{\sqrt{1 - x^2}} \, dx + C \]

Compare with the integral

\[ \int \frac{1}{\sqrt{1 + x^2}} \, dx = \log(x + \sqrt{1 + x^2}) + C \]

We get the following mysterious formula

\[ \log(x + \sqrt{1 + x^2}) = \arcsin(\sqrt{-1}x) \]
A complex number $z$ is a pair of two real numbers. But we are more used to writing it as

$$z = x + iy = x + \sqrt{-1}y,$$

where $x, y$ are the two real numbers.
A complex number $z$ is a pair of two real numbers. But we are more used to writing it as

$$z = x + iy = x + \sqrt{-1}y,$$

where $x, y$ are the two real numbers.

The production of two complex numbers is somewhat interesting

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$
A complex number $z$ is a pair of two real numbers. But we are more used to writing it as

$$z = x + iy = x + \sqrt{-1}y,$$

where $x, y$ are the two real numbers.

The production of two complex numbers is somewhat interesting

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$

The negative sign costs 99% of errors!
Euler’s formula

\[ e^{iz} = \cos z + i \sin z \]
Euler’s formula

\[ e^{iz} = \cos z + i \sin z \]

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}
\]

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}
\]
Theorem: \( \cos(x + y) = \cos x \cos y - \sin x \sin y. \)

Proof.

\[
\cos x \cos y - \sin x \sin y \\
= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} - \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} - e^{-iy}}{2i} \\
= \frac{1}{4} (e^{i(x+y)} + e^{-i(x+y)} + e^{i(x-y)} + e^{i(y-x)}) \\
+ \frac{1}{4} (e^{i(x+y)} + e^{-i(x+y)} - e^{i(x-y)} - e^{i(y-x)}) \\
= \cos(x + y).
\]

Q.E.D.
Theorem: \( \cos(x + y) = \cos x \cos y - \sin x \sin y. \)

Proof.

\[
\cos x \cos y - \sin x \sin y = \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} - \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} - e^{-iy}}{2i}
\]

\[
= \frac{1}{4} (e^{i(x+y)} + e^{-i(x+y)} + e^{i(x-y)} + e^{i(y-x)})
\]

\[
+ \frac{1}{4} (e^{i(x+y)} + e^{-i(x+y)} - e^{i(x-y)} - e^{i(y-x)})
\]

\[
= \cos(x + y).
\]

Q.E.D. (=quod erat demonstrandum)
**Theorem:** \( \cos(x + y) = \cos x \cos y - \sin x \sin y. \)

**Proof.**

\[
\begin{align*}
\cos x \cos y - \sin x \sin y &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} - \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} - e^{-iy}}{2i} \\
&= \frac{1}{4} \left( e^{i(x+y)} + e^{-i(x+y)} + e^{i(x-y)} + e^{i(y-x)} \right) \\
&\quad + \frac{1}{4} \left( e^{i(x+y)} + e^{-i(x+y)} - e^{i(x-y)} - e^{i(y-x)} \right) \\
&= \cos(x + y).
\end{align*}
\]

**Q.E.D.** (\( \text{=quod erat demonstrandum=} \text{=that which was to be demonstrated} \))
The above discussion hint us that we need to study

$$\int \frac{1}{\sqrt{1 - z^2}} dz,$$

where $z$ is a complex number.
The above discussion hint us that we need to study

\[ \int \frac{1}{\sqrt{1 - z^2}} \, dz, \]

where \( z \) is a complex number.

Or more precisely,

\[ \phi(z) = \int_0^z \frac{1}{\sqrt{1 - t^2}} \, dt. \]
The function $\frac{1}{\sqrt{1-z^2}}$ is multi-valued.
The function \( \frac{1}{\sqrt{1-z^2}} \) is multi-valued.

In order to make it single-valued, we need to construct a Riemann surface.
1. The function \( \frac{1}{\sqrt{1-z^2}} \) is multi-valued.

2. In order to make it single-valued, we need to construct a Riemann surface.

3. We cut two copies of \( \mathbb{C} \) along \([−1, 1]\) and glue them. The space \( \mathbb{C} \) is called a Riemann surface. \( \frac{1}{\sqrt{1-z^2}} \) is a single-valued function on the Riemann surface \( \mathbb{C} \).
The function $\frac{1}{\sqrt{1-z^2}}$ is multi-valued.

In order to make it single-valued, we need to construct a Riemann surface.

We cut two copies of $\mathbb{C}$ along $[-1, 1]$ and glue them. The space $C$ is called a Riemann surface. $\frac{1}{\sqrt{1-z^2}}$ is a single-valued function on the Riemann surface $C$.

$\phi(z)$ is multivalued even if on $C$, $\frac{1}{\sqrt{1-z^2}}$ is single-valued,
The function $\frac{1}{\sqrt{1-z^2}}$ is multi-valued.

In order to make it single-valued, we need to construct a Riemann surface.

We cut two copies of $\mathbb{C}$ along $[-1,1]$ and glue them. The space $C$ is called a Riemann surface. $\frac{1}{\sqrt{1-z^2}}$ is a single-valued function on the Riemann surface $C$.

$\phi(z)$ is multivalued even if on $C$, $\frac{1}{\sqrt{1-z^2}}$ is single-valued, because of the existence of Residue.
The function $\frac{1}{\sqrt{1-z^2}}$ is multi-valued.

In order to make it single-valued, we need to construct a Riemann surface.

We cut two copies of $\mathbb{C}$ along $[-1, 1]$ and glue them. The space $C$ is called a Riemann surface. $\frac{1}{\sqrt{1-z^2}}$ is a single-valued function on the Riemann surface $C$.

$\phi(z)$ is multivalued even if on $C$, $\frac{1}{\sqrt{1-z^2}}$ is single-valued, because of the existence of Residue.

We have

$$\oint_{|z|=R} \frac{dz}{\sqrt{1-z^2}} = 2\pi$$
Therefore, \( \phi : C \to \mathbb{C}/2\pi \mathbb{Z} \)
Therefore, $\phi : C \rightarrow \mathbb{C}/2\pi\mathbb{Z}$

$\phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \rightarrow C$ exists.
Therefore, \( \phi : C \to \mathbb{C}/2\pi \mathbb{Z} \)

\( \phi^{-1} : \mathbb{C}/2\pi \mathbb{Z} \to C \) exists.

C is the set of all points \( (t, \pm \sqrt{1 - t^2}) \). Therefore, C can be represented by the set

\[ x^2 + y^2 = 1 \]

in \( \mathbb{C}^2 \).
Therefore, $\phi : C \to \mathbb{C}/2\pi\mathbb{Z}$

$\phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \to C$ exists.

$C$ is the set of all points $(t, \pm\sqrt{1-t^2})$. Therefore, $C$ can be represented by the set

$$x^2 + y^2 = 1$$

in $\mathbb{C}^2$.

$C$ is a group

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1y_2 + x_2y_1, x_1x_2 - y_1y_2).$$
Therefore, \( \phi : C \rightarrow \mathbb{C}/2\pi\mathbb{Z} \)

\( \phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \rightarrow C \) exists.

\( C \) is the set of all points \((t, \pm \sqrt{1-t^2})\). Therefore, \( C \) can be represented by the set

\[ x^2 + y^2 = 1 \]

in \( \mathbb{C}^2 \).

\( C \) is a group

\[ (x_1, y_1) \oplus (x_2, y_2) = (x_1y_2 + x_2y_1, x_1x_2 - y_1y_2). \]

Check:

\[ (x_1y_2 + x_2y_1)^2 + (x_1x_2 - y_1y_2)^2 = 1. \]
The map $\phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \rightarrow \mathbb{C}$ is a group isomorphism:

$$(\alpha, \beta) \quad \longrightarrow \quad \alpha + \beta$$

$$(((x_1, y_1), (x_2, y_2)) \quad \longrightarrow \quad (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

where $x_1 = \cos \alpha, y_1 = \sin \alpha$, etc.
The map $\phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \to \mathbb{C}$ is a group isomorphism:

$$(\alpha, \beta) \quad \longrightarrow \quad \alpha + \beta$$

$$(x_1, y_1), (x_2, y_2) \quad \longrightarrow \quad (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

where $x_1 = \cos \alpha, y_1 = \sin \alpha$, etc.

Such an isomorphism is called Addition Theorem.
The map $\phi^{-1} : \mathbb{C}/2\pi\mathbb{Z} \to \mathbb{C}$ is a group isomorphism:

$$(\alpha, \beta) \quad \longrightarrow \quad \alpha + \beta$$

\[
((x_1, y_1), (x_2, y_2)) \quad \longrightarrow \quad (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)
\]

where $x_1 = \cos \alpha, y_1 = \sin \alpha$, etc.

Such an isomorphism is called Addition Theorem.
The following is another version of the addition theorem for the sine function:

\[
\int_0^{\sin u} \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^{\sin v} \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{\sin(u+v)} \frac{1}{\sqrt{1 - x^2}} \, dx
\]
The following is another version of the addition theorem for the sine function:

\[
\int_0^{\sin u} \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^{\sin v} \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{\sin(u+v)} \frac{1}{\sqrt{1 - x^2}} \, dx
\]

If we set \( \sin u, z = \sin v \), then we have

\[
\int_0^y \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^z \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{T(y,z)} \frac{1}{\sqrt{1 - x^2}} \, dx,
\]

where

\[
T(y, z) = y \sqrt{1 - z^2} + z \sqrt{1 - y^2}.
\]
The following is another version of the addition theorem for the sine function:

\[
\int_0^{\sin u} \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^{\sin v} \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{\sin(u+v)} \frac{1}{\sqrt{1 - x^2}} \, dx
\]

If we set \( \sin u, z = \sin v \), then we have

\[
\int_0^y \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^z \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{T(y,z)} \frac{1}{\sqrt{1 - x^2}} \, dx,
\]

where

\[
T(y, z) = y \sqrt{1 - z^2} + z \sqrt{1 - y^2}.
\]

Any other functions satisfy the above addition theorem?
The following is another version of the addition theorem for the sine function:

\[ \int_0^{\sin u} \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^{\sin v} \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{\sin(u+v)} \frac{1}{\sqrt{1 - x^2}} \, dx \]

If we set \( \sin u, z = \sin v \), then we have

\[ \int_0^y \frac{1}{\sqrt{1 - x^2}} \, dx + \int_0^z \frac{1}{\sqrt{1 - x^2}} \, dx = \int_0^{T(y,z)} \frac{1}{\sqrt{1 - x^2}} \, dx, \]

where

\[ T(y, z) = y \sqrt{1 - z^2} + z \sqrt{1 - y^2}. \]

Any other functions satisfy the above addition theorem? **Elliptic Functions!**
Consider the integral
\[ \int \frac{dt}{\sqrt{t(t-1)(t-\lambda)}}, \]
Consider the integral
\[
\int \frac{dt}{\sqrt{t(t-1)(t-\lambda)}},
\]
where \(0 < \lambda < 1\).

**Historic Remarks**
Legendre studied these kind of integral extensively and wrote 3 volumes (4 volumes?) of *Traite des fonctions elliptiques*. But he failed in finding the most important properties of the integrals: they are the inverse functions of some doubly periodic functions! This fact was found by Abel and Jocobi independently.
The Riemann surface of the function
The Riemann surface of the function

\[
\frac{1}{\sqrt{z(z - 1)(z - \lambda)}}
\]
The Riemann surface of the function

\[
\frac{1}{\sqrt{z(z - 1)(z - \lambda)}}
\]

\[y^2 = z(z - 1)(z - \lambda)\]
Let

\[ \omega_1 = \oint_A \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \]
\[ \omega_2 = \oint_B \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \]

Then the inverse function of the elliptic integral

\[ \int \frac{dt}{\sqrt{t(t-1)(t-\lambda)}} \]

is a doubly period function of periods \( \omega_1, \omega_2 \).
Let
\[ \omega_1 = \oint_A \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \]
\[ \omega_2 = \oint_B \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \]

Then the inverse function of the elliptic integral
\[ \int^{z} \frac{dt}{\sqrt{t(t-1)(t-\lambda)}} \]
is a doubly period function of periods \( \omega_1, \omega_2 \).
Theorem: Entire (holomorphic) doubly periodic functions are constants.

Proof. Bounded holomorphic functions are constant by Liouville Theorem.
Theorem: Entire (holomorphic) doubly periodic functions are constants.

Proof. Bounded holomorphic functions are constant by Liouville Theorem.

Therefore, elliptic functions are meromorphic.
Theorem: Entire (holomorphic) doubly periodic functions are constants.

**Proof.** Bounded holomorphic functions are constant by Liouville Theorem.

Therefore, elliptic functions are *meromorphic* (singularities are poles).
**Theorem**

Any two tori $T(\omega_1, \omega_2)$ and $T(\omega_3, \omega_4)$ are biholomorphic if and only if there exists integers $a, b, c, d$ such that $ad - bc = 1$, and

$$\frac{\omega_4}{\omega_3} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}.$$
References: