Pinching Theorems on minimal submanifolds of sphere

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Böttcher-Wenzel Conjecture

Conjecture

Let $X, Y$ be $n \times n$ real matrices. Then

$$||[X, Y]||^2 \leq 2||X||^2||Y||^2.$$ 

Here the norm is the Hilbert-Schmidt norm:

$$||X||^2 = \text{tr}(A^*A)$$
1. The conjecture was proved by L in 2007 (posted on arXiv on 11/22/2007); after that there are three different proofs by Vong-Jin (黄锡荣—金小庆), Böttcher-Wenzel, and Audenaert.

2. Obviously, the result can be generalized to operators on separable Hilbert spaces.

3. We sharpened the Schrödinger-Robertson relation, which was

$$ \| [A, B] \| \leq 2 \| A \| \| B \| . $$

The optimal constant is $\sqrt{2}$. 

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Equality case

Let

\[ X = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \]

Then

\[ \|[X, Y]|| = \sqrt{2}||X|| \cdot ||Y||. \]
A result of Chern-do Carmo-Kobayashi (in 70’s)

Theorem

If $A, B$ are symmetric matrices, then

$$||[A, B]||^2 \leq 2||A||^2 ||B||^2$$

Since the inequality is invariant under orthogonal transformation, we can assume that $A$ is diagonalized. Then the inequality becomes

$$\sum (\lambda_i - \lambda_j)^2 b_{ij}^2 \leq 2||A||^2 ||B||^2.$$

$$(\lambda_i - \lambda_j)^2 \leq 2(\sum \lambda_k^2)$$
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Proof of the BW conjecture.

Let

\[ X = Q_1 \Lambda Q_2 \]

be the singular decomposition of \( X \), where \( Q_1, Q_2 \) are orthogonal matrices and \( \Lambda \) is a diagonal matrix. Let

\[ B = Q_2 Y Q_2^{-1}, \quad C = Q_1^{-1} Y Q_1. \]

Then we have

\[ ||[X, Y]||^2 = ||\Lambda B - CA||^2, \quad \text{expect to be} \leq 2 ||\Lambda||^2 ||B||^2 \]
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1. If $b_{11} = 0$, then the conjecture can be proved
2. We can make $b_{11} = 0$
Since $b_{11} = 0$, we have

($\Lambda = \text{diag}(s_1, \cdots, s_n), \sum s_j^2 = 1, s_1$ is the biggest)

$$
\|\Lambda B - C\Lambda\|^2 = c_{11}^2 s_1^2 + \sum_{i=2}^{n} (s_i b_{i1} - s_1 c_{i1})^2 + \sum_{j=2}^{n} (s_1 b_{1j} - s_j c_{1j})^2 + \Delta_1,
$$

where

$$
\Delta_1 = \sum_{i,j=2}^{n} (s_i b_{ij} - s_j c_{ij})^2.
$$

Since $s_2^2 \leq 1/2$, we have

$$
\Delta_1 \leq \sum_{i,j=2}^{n} (b_{ij}^2 + c_{ij}^2).
$$
The conjecture is implied by the following inequality

\[
c_{11}^2 s_1^2 + \sum_{i=2}^{n} (s_i b_{i1} - s_1 c_{i1})^2 + \sum_{j=2}^{n} (s_1 b_{1j} - s_j c_{1j})^2 \leq \Delta + \sum_{i=2}^{n} b_{i1}^2 + \sum_{j=2}^{n} c_{1j}^2,
\]

where

\[
\Delta = \sum_{i=2}^{n} b_{i1}^2 + \sum_{j=2}^{n} c_{1j}^2 + c_{11}^2.
\]
We consider the matrix

$$
\begin{pmatrix}
\Delta & -b_{12}c_{12} - b_{21}c_{21} & \cdots & -b_{1n}c_{1n} - b_{n1}c_{n1} \\
-b_{12}c_{12} - b_{21}c_{21} & b_{21}^2 + c_{12}^2 & & \\
& \ddots & & \\
-b_{1n}c_{1n} - b_{n1}c_{n1} & & b_{n1}^2 + c_{1n}^2 & \\
\end{pmatrix}
$$

The conjecture is equivalent to that the maximum eigenvalue of the above matrix is no more than $\Delta + \sum_{i=2}^{n} b_{i1}^2 + \sum_{j=2}^{n} c_{1j}^2$. 
We fix $X$ and assume that $||X|| = 1$. Let $V = \mathfrak{gl}(n, \mathbb{R})$. Define a linear map

$$T : V \rightarrow V, \quad Y \mapsto [X^T, [X, Y]],$$

where $X^T$ is the transpose of $X$. Then we have

**Lemma**

$T$ is a semi-positive definite symmetric linear transformation of $V$.
The conjecture is equivalent to the statement that the maximum eigenvalue of $T$ is not more than 2.

Let $\alpha$ be the maximum eigenvalue of $T$. Then $\alpha > 0$. Let $Y$ be an eigenvector of $T$ with respect to $\alpha$. Then we have

$$T(Y) = \alpha Y.$$

A straightforward computation gives

$$T([X^T, Y^T]) = \alpha [X^T, Y^T].$$

$Y$ and $Y_1 = [X^T, Y^T]$ are linearly independent. Linear combine the two eigenfunctions, we get $b_{11} = 0$. 

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Consider minimal submanifolds of sphere

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For the second gap...

1. Chern (陈省身) Conjecture: if the scalar curvature of a minimal hypersurface is constant, then the length of the second fundamental form is discrete.

2. Peng and Terng (彭家贵－滕楚莲) proved that, if the length of the second fundamental form is bigger than \( n \), then it must be bigger than \( n + \frac{1}{9n} \).

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Basic Facts

Let $M$ be a minimal submanifold of $S^{n+p}$.

Let $h^\alpha_{ij}$ be the second fundamental form (wrt orthonormal frames)

Let $A^\alpha = (h^\alpha_{ij})$

We have

$$\Delta A^\alpha = nA^\alpha - \langle A^\alpha, A^\beta \rangle A^\beta - [A^\beta, [A^\beta, A^\alpha]].$$
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Simons-Bochner formula

\[ \frac{1}{2} \Delta \| \sigma \|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n \| \sigma \|^2 - \sum_{\alpha,\beta} \| [A_\alpha, A_\beta] \|^2 - \sum_{\alpha,\beta} \| \langle A_\alpha, A_\beta \rangle \|^2. \]

The Simons (1969) pinching theorem

\[ \frac{1}{2} \Delta \| \sigma \|^2 \geq n \| \sigma \|^2 - 2 \sum_{\alpha \neq \beta} \| A_\alpha \|^2 \| A_\beta \|^2 - \sum_\alpha \| A_\alpha \|^4 \]

\[ \geq n \| \sigma \|^2 - (2 - \frac{1}{p}) \| \sigma \|^4 \]
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Some examples of minimal submanifolds:

1. Totally geodesic minimal submanifolds; \( ||\sigma||^2 = 0 \)

2. \( M_{r,n-r} = S^r \left( \sqrt{\frac{r}{n}} \right) \times S^{n-r} \left( \sqrt{\frac{n-r}{n}} \right); \ ||\sigma||^2 = n \)

3. The Veronese surface is defined as the immersion; \( ||\sigma||^2 = 4/3 = \frac{2}{3}n \)

\[
\begin{align*}
u^1 &= \frac{1}{\sqrt{3}}yz, \quad u^2 = \frac{1}{\sqrt{3}}zx, \quad u^3 = \frac{1}{\sqrt{3}}xy, \quad u^4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \\
u^5 &= \frac{1}{6}(x^2 + y^2 - 2z^2).
\end{align*}
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\((x, y, z) \in S^2\).
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$$(x, y, z) \in S^2.$$
It was found by J. Simons that there are rigidity theorems (pinching theorems) for minimal submanifolds of $S^N$. More precisely, the optimal results were

1. For minimal hypersurface of dim $n$, if $0 \leq ||\sigma||^2 \leq n$, then either $||\sigma|| = 0$ or $||\sigma||^2 = n$ (well known, Chern (陈省身)—do Carmo-Kobayashi);

2. For high co-dimensional minimal submanifolds, if $0 \leq ||\sigma||^2 \leq \frac{2}{3}n$, then they are totally geodesic or Veronese surface. (Li-Li (李基民—李安民), Chen-Xu (陈卿—徐森林))
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The new method

Taking $\Delta$ on $\max \| A_r \|^2$ instead of $\| \sigma \|^2$.

New J. Simons formula:

$$\frac{1}{2p} \Delta f_p = \frac{1}{2} \sum_{s+t=p-2} \sum_{k, \alpha, \beta} \lambda_s^\alpha \lambda_t^\beta \left( D \frac{\partial}{\partial x_k} a_{\alpha \beta} \right)^2 + \sum_{\alpha} \left( \lambda_p^{p-1} \sum_{i,j,k} (h_{ijk}^\alpha)^2 \right)$$

$$+ n f_p - f_{p+1} - \sum_{\beta \neq \alpha} \|[A^\beta, A^\alpha]\|^2 \|A^\alpha\|^{p-1},$$

where $f_p = \text{tr}(F^p)$, $F = (\langle A_i, A_j \rangle)$.

$$\lim \frac{1}{f_p} = \max \| A_r \|.$$
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where $f_p = \text{tr}(F^p), F = (\langle A_i, A_j \rangle)$.

$$\lim_{p \to \infty} \frac{1}{f_p^p} = \max \|A_r\|.$$
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The technical heart of the estimate is the

**Theorem**

\[ \sum \|[A, A_j]\|^2 \leq \|A\|^2 (\sum \|A_j\|^2 + \max \|A_j\|^2) \]

where \( A_1, \ldots, A_m \) are *orthogonal*.

(Chern-do Carmo-Kobayashi gives \( 2\|A\|^2 (\sum \|A_j\|^2) \).)
Let the matrix $F = (\langle A_r, A_s \rangle)$. Let $\lambda_2$ be the second largest eigenvalue of the matrix.

**Theorem**

If

$$0 \leq ||\sigma||^2 + \lambda_2 \leq n$$

then $M$ is totally geodesic, one of the $M_{r,n-r}$, or a Veronese surface.
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Generalization and Unification

$||\sigma||^2$ is the trace of the fundamental matrix.

\[
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Corollary

*If the co-dimension is more than 1, and if*

\[ 0 \leq ||\sigma||^2 + \max ||A_r||^2 \leq n \]

*then M is either totally geodesic, or a Veronese surface.*
When the equality is valid, that is, when

$$||\sigma||^2 + \lambda_2 \equiv n,$$

then the submanifolds must be of the three types mentioned above. A typical result is in the next slide.
There is no minimal immersion $M^n \to S^{2n}$ such that the second fundamental forms are

$$A_1 = c_1 \begin{pmatrix} n-1 & -1 & \ldots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & \ldots & -1 & \end{pmatrix}, \quad A_2 = c_2 \begin{pmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 1 & 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 & \end{pmatrix}$$

$$A_3 = c_2 \begin{pmatrix} 0 & 0 & 1 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & \ldots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & \ddots & \ddots & 0 \end{pmatrix}, \quad A_4 = c_2 \ldots$$
Consider a surface in $R^3$. Principal curvature at $P \in S : k_1, k_2$

2. Gaussian curvature: $K(p) = k_1 \cdot k_2$

3. Mean curvature: $H(p) = \frac{k_1 + k_2}{2}$

Then we have

$$H^2(p) \geq K(p) \quad \text{Extrinsic} \geq \text{Intrinsic}$$
Normal Scalar Curvature Conj.–1st application

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1. Consider a surface in $\mathbb{R}^3$. Principal curvature at $P \in S : k_1, k_2$

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3. Mean curvature: $H(p) = \frac{k_1 + k_2}{2}$

4. Then we have $H^2(p) \geq K(p)$ \quad Extrinsic $\geq$ Intrinsic
Let $M^n$ be an $n$-dimensional manifold isometrically immersed into the space form $N^{n+m}(c)$ of constant sectional curvature $c$. By the same reason,

$$|H|^2 + c \geq \rho,$$

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The Conjecture

The (normalized) scalar curvature of the normal bundle is defined as:

$$\rho^\perp = \frac{1}{n(n-1)} |R^\perp|,$$

where $R^\perp$ is the curvature tensor of the normal bundle. More precisely, let $\xi_1, \cdots, \xi_m$ be a local orthonormal frame of the normal bundle. Then

$$\rho^\perp = \frac{2}{n(n-1)} \left( \sum_{1=i<j}^{n} \sum_{1=r<s}^{m} \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{\frac{1}{2}}.$$
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\( \rho^\perp \) is always nonnegative. In the study of submanifold theory, De Smet, Dillen, Verstraelen, and Vrancken proposed the following *Normal Scalar Curvature Conjecture*:

**Conjecture**

Let \( h \) be the second fundamental form, and let \( H = \frac{1}{n} \text{trace} \, h \) be the mean curvature tensor. Then

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\rho + \rho^\perp \leq |H|^2 + c.
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The algebraic version of the Normal Scalar Curvature Conjecture can be stated as follows:

**Theorem**

Let $A_1, \cdots, A_m$ be symmetric matrices. For $n, m \geq 2$, we have

$$\left(\sum_{r=1}^{m} \|A_r\|^2\right)^2 \geq 2\left(\sum_{r<s} \|[A_r, A_s]\|^2\right).$$
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Erdős-Mordell inequality (1935)–2nd application

\[ PA + PB + PC \geq 2(PP_A + PP_B + PP_C). \]
Erdös-Mordell inequality is the dual version of the Normal Scalar Curvature Conjecture!

1. Is \( a^2 + b^2 + c^2 \geq ab + bc + ca \) a generalization of \( a^2 + b^2 \geq 2ab \)?
2. Not really!
3. The correct generalization is: if \( \alpha + \beta + \gamma = \pi \), then

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a^2 + b^2 + c^2 \geq 2ab \cos \gamma + 2bc \cos \alpha + 2ac \cos \beta,
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which implies the Erdös-Mordell inequality.
The observation

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which implies the Erdös-Mordell inequality.
We consider vectors $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{R}^2$ such that the angles between them are $\pi/2 + \gamma/2, \pi/2 + \beta/2, \pi/2 + \alpha/2$, respectively. Moreover, we assume that that norm of $\vec{x}, \vec{y}, \vec{z}$ are $a, b, c$.

Let $X, Y, Z$ be the corresponding $3 \times 3$ skew-symmetric matrices.

Then the EM inequality is equivalent to

$$||[X, Y]||^2 + ||[Y, Z]||^2 + ||[Z, X]||^2 \leq \frac{1}{8}(||X||^2 + ||Y||^2 + ||Z||^2)^2.$$
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We expect the following generalization of the EM inequality is true.

**Conjecture**

Let $A_1, \ldots, A_m$ be skew-symmetric matrices. Then

\[
\left( \sum_{r=1}^{m} \|A_r\|^2 \right)^2 \geq c(m, n) \left( \sum_{r<s} \|[A_r, A_s]\|^2 \right) \quad n \geq 4
\]

\[
\left( \sum_{r=1}^{m} \|A_r\|^2 \right)^2 \geq 8c(m, n) \left( \sum_{r<s} \|[A_r, A_s]\|^2 \right) \quad n = 3
\]
We have

\[(||X||^2 + ||Y||^2)^2 \geq ||[X, Y]||^2 \quad n \geq 4\]
\[(||X||^2 + ||Y||^2)^2 \geq 8||[X, Y]||^2 \quad n = 3 \quad (This \ implies \ EM)\]

for skew-symmetric matrices.

For proof of the above results we need the Böttcher-Wenzel Theorem!
Proof of the key inequality

First, we prove the following elementary lemma

**Lemma**

*Suppose \( \eta_1, \cdots, \eta_n \) are real numbers and*

\[
\eta_1 + \cdots + \eta_n = 0, \quad \eta_1^2 + \cdots + \eta_n^2 = 1.
\]

*Let \( r_{ij} \geq 0 \) be nonnegative numbers for \( i < j \). Then we have*

\[
\sum_{i<j} (\eta_i - \eta_j)^2 r_{ij} \leq \sum_{i<j} r_{ij} + \text{Max}(r_{ij}). \tag{1}
\]
If $\eta_1 \geq \cdots \geq \eta_n$, and $r_{ij}$ are not simultaneously zero, then the equality in the above holds in one of the following three cases:

1. $r_{ij} = 0$ unless $(i,j) = (1, n), (\eta_1, \cdots, \eta_n) = (1/\sqrt{2}, 0, \cdots, 0, -1/\sqrt{2})$;

2. $r_{ij} = 0$ if $2 \leq i < j$, $r_{12} = \cdots = r_{1n} \neq 0$, and $(\eta_1, \cdots, \eta_n) = (\sqrt{(n-1)/n}, -1/\sqrt{n(n-1)}, \cdots, -1/\sqrt{n(n-1)})$;

3. $r_{ij} = 0$ if $i < j < n$, $r_{1n} = \cdots = r_{(n-1)n} \neq 0$, and $(\eta_1, \cdots, \eta_n) = (1/\sqrt{n(n-1)}, \cdots, 1/\sqrt{n(n-1)}, -\sqrt{(n-1)/n})$;
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   \]

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   \[
   (\eta_1, \cdots, \eta_n) = \left(\frac{1}{\sqrt{n(n-1)}}, \cdots, 1/\sqrt{n(n-1)}, -\sqrt{(n-1)/n}\right);
   \]
The key lemma

**Lemma**

Let $A$ be an $n \times n$ diagonal matrix of norm 1. Let $A_2, \cdots, A_m$ be symmetric matrices such that

1. $\langle A_\alpha, A_\beta \rangle = 0$ if $\alpha \neq \beta$;
2. $\|A_2\| \geq \cdots \geq \|A_m\|$.

Then we have

$$\sum_{\alpha=2}^{m} \| [A, A_\alpha] \|^2 \leq \sum_{\alpha=2}^{m} \| A_\alpha \|^2 + \| A_2 \|^2. \quad (2)$$
The equality holds iff, after an orthonormal base change and up to a sign, we have

- $A_3 = \cdots = A_m = 0$, and

$$
A_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}
$$

$$
A_2 = c \begin{pmatrix}
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& & \ddots & \\
& & & 0
\end{pmatrix},
\]

\[
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0 & \frac{1}{\sqrt{2}} & & \\
\frac{1}{\sqrt{2}} & 0 & & \\
& & \ddots & \\
& & & 0
\end{pmatrix},
\]
Or

- For two real numbers $\lambda = 1/\sqrt{n(n-1)}$ and $\mu$, we have

$$A_1 = \lambda \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & \ddots & & \\ & & -1 \end{pmatrix},$$

and $A_\alpha$ is $\mu$ times the matrix whose only nonzero entries are 1 at the $(1, \alpha)$ and $(\alpha, 1)$ places, where $\alpha = 2, \cdots, n$. 
We assume that each $A_\alpha$ is not zero. Let $A_\alpha = ((a_\alpha)_{ij})$, where $(a_\alpha)_{ij}$ are the entries for $\alpha = 2, \cdots, m$. Let

$$\delta = \max_{i \neq j} \sum_{\alpha=2}^{m} (a_\alpha)_{ij}^2.$$  

Let

$$A = \begin{pmatrix} \eta_1 \\ \cdots \\ \eta_n \end{pmatrix}.$$  

Then by the previous lemma, we have

$$\sum_{\alpha=2}^{m} \| [A, A_\alpha] \|^2 \leq \sum_{\alpha=2}^{m} \| A_\alpha \|^2 + 2\delta.$$
Thus it remains to prove that

$$2\delta \leq \|A_2\|^2.$$

To see this, we identify each $A_\alpha$ with the (column) vector $\vec{A}_\alpha$ in $\mathbb{R}^{\frac{1}{2}n(n+1)}$ as follows:

$$A_\alpha \mapsto (a_{12}, \cdots, a_{1n}, a_{23}, \cdots, a_{2n}, \cdots, a_{(n-1)n}, \frac{1}{\sqrt{2}}a_{11}, \cdots, \frac{1}{\sqrt{2}}a_{nn})^T.$$

Let $\mu_\alpha$ be the norm of the vector $A_\alpha$. Then we have

$$\mu^2_\alpha = \frac{1}{2}\|A_\alpha\|^2$$

for $\alpha = 2, \cdots, m$. 
Extending the set of vectors \( \{ \vec{A}_\alpha / \mu_\alpha \}_{2 \leq \alpha \leq m} \) into an orthonormal basis of \( \mathbb{R}^{\frac{1}{2} n(n+1)} \)

\[ \vec{A}_2 / \mu_2, \ldots, \vec{A}_m / \mu_m, \vec{A}_{m+1}, \ldots, \vec{A}_{\frac{1}{2} n(n+1)+1}, \]

we get an orthogonal matrix. Apparently, each row vector of the matrix is a unit vector. Thus we have

\[
\sum_{\alpha=2}^{m} (\mu_\alpha)^{-2} (a_\alpha)_{ij}^2 \leq 1
\]

for fixed \( i < j \). Since \( \mu_2 \geq \cdots \geq \mu_m \), we get

\[
\sum_{\alpha=2}^{m} (a_\alpha)_{ij}^2 \leq \mu_2^2 \leq \frac{1}{2} ||A_2||^2.
\]

This completes the proof.
In conclusion...

1. many more generalizations in linear algebra
2. the maximum commutator on simple (real or complex) Lie algebras
3. the “correct” quantity for the optimal pinching theorem?
4. the Peng-Terng (彭家贵－滕楚莲)-type gap theorem
5. comass problem in calibrated geometry
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Thank you!